Spectral resolutions in effect algebras

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Dedicated to the memory of David J. Foulis

Effect algebras

An effect algebra is a system $(E, 0, 1, \oplus)$, where $0, 1 \in E$ are constants, $\oplus$ is a partial binary operation on E such that:

(E1) if $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;

(E2) if $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $a \oplus (b \oplus c)$ is defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

(E3) for every $a \in E$ there is unique $a' \in E$ such that $a \oplus a' = 1$;

(E4) if $a \oplus 1 \in E$, then $a = 0$.

Covers many different algebraic structures: MV-effect algebras, OMPs, orthoalgebras, etc.

Foulis & Bennett, 1994

Hilbert space effect algebras

Effect algebras are an algebraic model of Hilbert space effects:

$$E(\mathcal{H}) = \{E \in B(\mathcal{H}), \quad 0 \leq E \leq I\}$$

- measurements on a quantum system in the Hilbert space formalism
- important special property - spectrality:

  for $a \in E(\mathcal{H})$ there is a family $\{p_{a,\lambda}\}_{\lambda \in [0,1]}$ of projections such that

$$a = \int \lambda dp_{a,\lambda}$$

Spectral resolution in Hilbert space effect algebras

Let $a \in E(\mathcal{H})$. The spectral resolution of a is the unique family of projections $\{p_\lambda\}_{\lambda \in [0,1]}$ such that

- $1 = p_1 \geq p_\lambda \geq p_\mu$ for $1 \geq \lambda \geq \mu$ (nondecreasing),
- $\bigwedge_{\lambda > \mu} p_\lambda = p_\mu$ (right continuous),
- $p_\lambda a = a p_\lambda$ (commutativity),
- $p_\lambda a \leq \lambda p_\lambda$, $p_\lambda^\perp a \geq \lambda p_\lambda^\perp$.

Further, a is uniquely determined by $\{p_{a,\lambda}\}$ and a commutes with b if $p_{a,\lambda}$ commutes with $p_{b,\mu}$ for all λ and μ.

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Question

Can we have some type of a spectral resolution for an abstract effect algebra E?

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What are the additional structures and/or properties of E needed to obtain this?

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What are the additional structures and/or properties of E needed to obtain this?

Spectrality in partially ordered unital abelian groups

Let G be a POUAG, with unit u.

- A compression: morphism $J : G \to G$, generalizing

  - the compressions

    $$a \mapsto pap, \qquad a \in B^{sa}(\mathcal{H}),\ p \text{ a projection},$$

  - the projection

    $$a \mapsto a \wedge np,\ a \leq nu,$$

  onto the ideal G_p generated by a sharp element p in an interpolation group.

- A compression base $\{J_p\}_{p \in P}$: a suitable set of compressions

Foulis, 2003-2005

Spectrality in partially ordered unital abelian groups

G with $\{J_p\}_{p\in P}$ is spectral if it has

- comparability property: $g = g_+ - g_-$, $g_+, g_- \in G^+$

$$\exists\ p \in P \text{ such that } J_p(g) = g_+,\ J_{p^\perp}(g) = -g_-$$

- Rickart mapping:

$$g \mapsto g^* \in P \text{ complement of the "support projection".}$$

Rational spectral resolution: for $g \in G$,

$$p_{g,\lambda} := (ng - mu)^*_+, \qquad \lambda = \frac{m}{n}.$$

Spectrality in partially ordered unital abelian groups

Let G be an archimedean spectral POUAG.

The rational spectral resolution of $g \in G$ is the unique family of projections $\{p_\lambda\}_{\lambda \in \mathbb{Q}}$ such that

- for $\lambda < l_g$, $p_\lambda = 0$, $\lambda \geq u_g$, $p_\lambda = 1$ (bounded),
- $p_\lambda \geq p_\mu$ for $\lambda \geq \mu$ (nondecreasing),
- $\bigwedge_{\lambda > \mu} p_\lambda = p_\mu$ (right continuous),
- g compatible with all p_λ,
- $nJ_{p_\lambda}(g) \leq mp_\lambda$, $nJ_{p_\lambda^\perp}(g) \geq mp_\lambda^\perp$, $\lambda = \frac{m}{n}$.

Further, g is uniquely determined by $\{p_{g,\lambda}\}$ and g is compatible with $p \in P$ if and only if $p_{g,\lambda}$ is compatible with p for all λ.

Compressions and compression bases in effect algebras

Let E be an effect algebra.

A compression is an additive map $J : E \to E$ such that

$$a \leq J(1) \iff J(a) = a, \qquad a \leq J(1)^{\perp} \iff J(a) = 0.$$

Gudder, 2006; SP, 2006

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Properties:

- J is idempotent.
- J has a supplement: $\mathrm{Im} J = \mathrm{Ker} J'$, $\mathrm{Im} J' = \mathrm{Ker} J$.
- focus of J: $J(1)$, a principal element (sharp).

Gudder, 2006; SP, 2006

Compressions and compression bases in effect algebras

A compression base: $\{J_p\}_{p \in P}$

- $P \subseteq E$ a subalgebra (an OMP)
- $J_p(1) = p$, for all $p \in P$
- if $p \leftrightarrow q$, then $J_p J_q = J_q J_p = J_r$ for some $r \in P$

Elements of P are called projections.

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$$a = J_p(a) \oplus J_{p^\perp}(a) \iff a \leftrightarrow p \iff J_p(a) = a \wedge p.$$

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- bicommutant of a:

$$P(a) = \{p \in P : p \leftrightarrow a, \ \forall q \in P, q \leftrightarrow a \implies q \leftrightarrow p\}.$$

a Boolean subalgebra in P

Examples

- Hilbert space effects: unique (maximal) compression base

$$E(\mathcal{H}),\ \text{with}\ \{U_p\}_{p\in P(\mathcal{H})},\ U_p(a) = pap.$$

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- Central compression bases: $P = \Gamma(E)$ the center of E:

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- Effect algebras with RDP (MV-effect algebras): the central compression base is the unique (maximal) compression base.

Examples

- The horizontal sum of Hilbert space effect algebras

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- Let φ be a faithful state on $E(\mathcal{H})$ ($\varphi(a) = 0$ implies $a = 0$).
- We can construct a compression base with the set of projections $P = P(\mathcal{H}) \hat{\oplus} P(\mathcal{H})$:

$$J_{(p,0)}(a, 0) = (J_p(a), a), \qquad J_{(p,0)}(0, a) = (\varphi(a)p, 0)$$

(similarly for $J_{(0,p)}$).

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(similarly for $J_{(0,p)}$).
- we obtain many different compression bases with the same P.

Spectrality: projection cover property

$(E, \{J_p\}_{p \in P})$ - an effect algebra with a fixed compression base.

Definition (Gudder, 2006)

$(E, \{J_p\}_{p \in P})$ has the projection cover property if for any $a \in E$, there is a projection cover: $a^\circ \in P$ such that

$$a \leq p \iff a^\circ \leq p, \qquad \forall p \in P.$$

Then P is an OML.

Spectrality: b-property

Definition (SP, 2006)

$(E, \{J_p\}_{p\in P})$ has the b-property if for all $a \in E$, $q \in P$,

$$a \leftrightarrow q \iff P(a) \leftrightarrow q.$$

For $a, b \in E$, aCb (a commutes with b) if $P(a) \leftrightarrow P(b)$.

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Under b-property: for $p \in P$,

$$aCp \iff a \leftrightarrow p.$$

Spectrality: b-comparability

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$(E, \{J_p\}_{p \in P})$ has the b-comparability property if

- it has the b-property
- for all $a, b \in E$, aCb, we have

$$\exists p \in P(a,b), \quad J_p(a) \leq J_p(b),\; J_{p^\perp}(b) \leq J_{p^\perp}(a).$$

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Under b-comparability: any $a \in E$ has a splitting projection:

$$p \in P(a):\ J_p(a) \leq J_p(1-a), \qquad J_{p^\perp}(a) \geq J_{p^\perp}(1-a)$$

$$\text{“}J_p(a) \leq 1/2\text{”} \qquad \text{“}J_{p^\perp}(a) \geq 1/2\text{”}$$

Spectrality

Definition (SP, 2006)

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Then

- $P = \{\text{sharp elements in } E\}$, P is an OML,
- E is covered by C-blocks

$$C(B) := \{a \in E, a \leftrightarrow B\} \quad \text{for a block } B \subseteq P$$

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  $\equiv$ maximal sets of mutually commuting elements
- any C-block is a spectral MV-effect algebra
- any $a \in E$ has a largest splitting projection

Binary spectral resolutions

Let $a \in E$.

By repeated applications of splitting, we construct a family $\{p_{a,\lambda(w)}\}_{w\in\{0,1\}^*}$, indexed by binary fractions

$$\lambda(w) = \sum_{i=1}^{n} w_i 2^{-i}, \qquad w \in \{0,1\}^n$$

- the binary spectral resolution of a.

Characterization of the binary spectral resolution

Let E be archimedean and spectral. The binary spectral resolution $\{p_{\lambda(w)}\}_{w\in\{0,1\}^*}$ is the unique family in P

- $1 = p_1 \geq p_\lambda \geq p_\mu$ for $1 \geq \lambda \geq \mu$,
- $\bigwedge_{\lambda>\mu} p_\lambda = p_\mu$,
- $p_\lambda \leftrightarrow a$ for all λ,
- For n, $w \in \{0,1\}^n$, put

$$u_w := (p_{\lambda(w)+2^{-n}}) \wedge p'_{\lambda(w)},$$

then $f_w(J_{u_w}(a))$ exists in $[0, u_w]$, for the partially defined map

$$f_w = f_{w_n} \circ \cdots \circ f_{w_1}, \quad f_0(b) = 2b,\ f_1(b) = (2b')'.$$

Spectral resolution in C-blocks

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- C is the unit interval in a unital ℓ-group G
- C is spectral iff G is spectral $\implies$ there is the rational spectral resolution of a in G (Foulis):

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$$\{p^C_{a,\lambda}\}_{\lambda \in \mathbb{Q}}$$

- this spectral resolution does not depend on the choice of C
- For binary fractions - the previous construction

$$p^C_{a,\lambda(w)} = p_{a,\lambda(w)}, \qquad w \in \{0,1\}^*.$$

Further properties of spectral resolutions

If E has a separating family of states, then for $a \in E$:

- a is uniquely determined by its binary spectral resolution
- a is compatible with $q \in P$ if and only if $p_{a,\lambda}$ is compatible with q for all binary fractions λ.

Further properties of spectral resolutions

If E has a separating family of states, then for $a \in E$:

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Note:

For groups, these properties hold if G is archimedean, which implies that it has an ordering set of states.

Spectrality of interval effect algebras

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$\rightarrow$ (the universal group of E).

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- Question: E is spectral $\overset{?}{\Longleftrightarrow}$ G is spectral?
- True for
  - MV-effect algebras
  - archimedean divisible (convex) effect algebras,
- False in general.

  Counterexample: the horizontal sum $E(\mathcal{H})\hat{\oplus}E(\mathcal{H})$

  (an interval effect algebra which is spectral but its universal group is not spectral).

Convex effect algebras

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Convex effect algebras

- archimedean convex effect algebra $E \simeq$ unit interval in an order unit space (A, A^+, u)
- E is spectral $\iff$ A is spectral (as a POUAG)
- a stronger notion: spectral duality (Alfsen and Schultz, 1976, 2003)

  - A has a predual base normed space V
  - stronger conditions on the compressions

JB-algebras

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Let A be a JB-algebra. Then

- spectral duality holds $\iff$ A is a JBW-algebra
- the following are equivalent:
  - A is spectral
  - A is Rickart
  - any maximal associative subalgebra of A is monotone σ-complete

Generalized spin factors

A generalized spin factor is an order unit space defined from a Banach space $(X, \|\cdot\|)$ (Berdikulov and Odilov, 1994):

$$A = R \times X, \quad A^{+} = \{(a, x),\ \|x\| \leq a\}, \quad u = (1, 0)$$

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Let $(X, \|\cdot\|)$ be reflexive. Then

- A is spectral $\iff$ $(X, \|\cdot\|)$ is strictly convex.

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Let $(X, \|\cdot\|)$ be reflexive. Then

- A is spectral $\iff$ $(X, \|\cdot\|)$ is strictly convex.
- A is is spectral duality $\iff$ $(X, \|\cdot\|)$ is strictly convex and smooth.