Dimensionalities on monoids in **Rel**

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The category of sets and relations

...denoted by **Rel**.

- Objects: sets.
- Morphisms: binary relations; $f : A \to B$ in **Rel** is a set of pairs $f \subseteq A \times B$.
- Identities: $\mathrm{id}_A : A \to A$ is the identity relation.
- Composition: if $f : A \to B$ and $g : B \to C$, then $(a, c) \in g \circ f$ iff there exists $b \in B$ such that $(a, b) \in f$ and $(b, c) \in g$.

Additional structure on **Rel**

- In **Rel**, every morphism can be turned around: for each $f\colon A \to B$ we have $f^\dagger\colon B \to A$ with

$$f^\dagger(b,a) \iff f(a,b)$$

- Relations between the same sets can be ordered by inclusion $f \subseteq g$, so **Rel** is enriched over **Pos**.

Additional structure in action

An *equivalence* on A is a relation $\sim\colon A \to A$ such that

- $\mathrm{id}_A \subseteq \sim$
- $\sim = \sim^\dagger$
- $\sim \subseteq \sim \circ \sim$

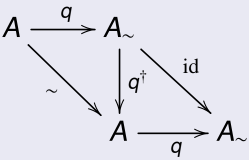
A relation $f\colon A \to B$ is a *mapping* iff

- $\mathrm{id}_A \subseteq f^\dagger \circ f$
- $f \circ f^\dagger \subseteq \mathrm{id}_B$

Equivalences and quotients

Theorem

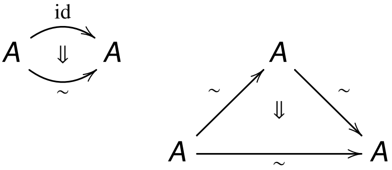
A relation $\sim\colon A \to A$ *is an equivalence if and only if there is a set* $A_\sim$ *and a mapping* $q\colon A \to A_\sim$ *such that*



commutes. The mapping q *is then (essentially) unique and is called* the quotient map of $\sim$.

(Lax) diagrams

Reflexivity and transitivity



Relational magmas

A *pointed magma* in **Rel** is an object S equipped with a binary and a nullary operation

$$\nabla\colon S \times S \to S \qquad e\colon I \to S$$

where I is a singleton set.
We write

- e_*^a for $e(*, a)$, where $I = \{*\}$ and $a \in S$.
- ∇_{ab}^c for $\nabla((a, b), c)$, where $(a, b) \in S \times S$ and $c \in C$.

Relational monoids

A pointed magma in **Rel** (S, ∇, e) is a *monoid* if the diagrams

$$\begin{array}{ccc} S\times S\times S & \xrightarrow{\nabla\times\mathrm{id}} & S\times S \\ {\scriptstyle \mathrm{id}\times\nabla}\downarrow & & \downarrow{\scriptstyle \nabla} \\ S\times S & \xrightarrow[\nabla]{} & S \end{array} \qquad \begin{array}{ccccc} I\times S & \xrightarrow{e\times\mathrm{id}} & S\times S & \xleftarrow{\mathrm{id}\times e} & S\times I \\ & {\scriptstyle\simeq}\searrow & {\scriptstyle \nabla}\downarrow & \swarrow{\scriptstyle\simeq} & \\ & & S & & \end{array}$$

commute.

Examples of relational monoids

- Every monoid in **Set**.
- Every hypermonoid, hypergroup etc.
- Every small category (elements are the arrows).
- Every poset (elements are the comparable pairs).
- Every effect algebra, hence every OML, OA, BA, MV-algebra etc.

Morphisms of monoids in **Rel**

There are at least two meaningful notions of morphisms of monoids in **Rel**.

$$
\begin{array}{ccc}
M \times M & \xrightarrow{h \times h} & M' \times M' \\
\nabla \downarrow & \Nearrow & \downarrow \nabla \\
M & \xrightarrow{h} & M'
\end{array}
\qquad\qquad
\begin{array}{ccc}
M \times M & \xrightarrow{h \times h} & M' \times M' \\
\nabla \downarrow & \Swarrow & \downarrow \nabla \\
M & \xrightarrow{h} & M'
\end{array}
$$

Lax morphism

$$(h \circ \nabla) \subseteq (\nabla \circ (h \times h))$$

$$h \circ e \subseteq e$$

Oplax morphism

$$(h \circ \nabla) \supseteq (\nabla \circ (h \times h))$$

$$e \subseteq h \circ e$$

Homomorphisms, congruences and submonoids are oplax

Let A, B be ordinary monoids (in **Set**).

- A mapping $h\colon A \to B$ is oplax iff it is a homomorphism.
- An equivalence $\sim\colon A \to A$ if oplax iff it is a congruence.
- Let $C \subseteq A$. Write $=_C\colon A \to A$ for the relation

$$=_C (a_1, a_2) \iff a_1 = a_2 \in C$$

Then $=_C$ is an oplax morphism iff C is a submonoid of A.

The definition

Definition

A *dimensionality* on a relational monoid A is a lax equivalence on A.

This gives us a new notion even for ordinary monoids.
If $(A, ., e)$ is a monoid in **Set**, then an equivalence $\sim$ on A is a dimensionality iff

- If $e \sim x$, then $e = x$.
- For all $a, b, c \in A$ such that $a.b \sim c$, there exist $a', b' \in A$ such that $a \sim a'$, $b \sim b'$ and $a'.b' = c$.

Examples

Same degree

Example

Write $F_{\text{mon}}[x]$ for the set of all monic polynomials of one variable over a field F:

$$F_{\text{mon}}[x] = \{x^n + a_{n-1}x^{n-1} + \cdots + a_0 : n \in \mathbb{N} \text{ and } a_{n-1}, \ldots, a_0 \in F\}$$

Let $\sim$ be an equivalence on $F_{\text{mon}}[x]$ given by the rule $\mathbf{p} \sim \mathbf{q}$ iff $\mathbf{p}$ and $\mathbf{q}$ have the same degree.
Then $\sim$ is a dimensionality on the monoid $(F_{\text{mon}}[x], ., 1)$ iff F is algebraically complete.

- In this example, $\sim$ is a congruence.
- But in general, it does not have to be one.

Examples

Same length

Example

Consider the monoid $(\mathbb{R}^2, +, (0,0))$. The relation $\sim$ on $\mathbb{R}^2$ given by the rule

$$\vec{x} \sim \vec{y} \text{ iff } ||\vec{x}|| = ||\vec{y}||$$

is a dimensionality.

- Draw a picture of some board.
- This example is not a congruence.
- However, note that there is an action of $\mathbb{R}$ on the monoid $(\mathbb{R}^2, +, (0,0))$ such that the orbits of the action are exactly the equivalence classes of $\sim$.

Groups of automorphisms induce dimensionalities

Theorem

Let (A, ∇, e) be a monoid in **Rel** *, let Γ be a subgroup of $\mathrm{Aut}(A)$. Write*

$$x \sim_\Gamma y \text{ iff } \exists f \in \Gamma : f(x) = y$$

Then $\sim_\Gamma$ is a dimensionality on A.

- Not all dimensionalities arise in this way:
- see the “monic polynomials” example.

Quotients of magmas in **Rel**

For every equivalence $\sim$ on a pointed magma (S, ∇, e) in **Rel** we may construct a pointed magma $(S_\sim, \nabla_\sim, e_\sim)$, where $\nabla_\sim$ and $e_\sim$ are given by the rules

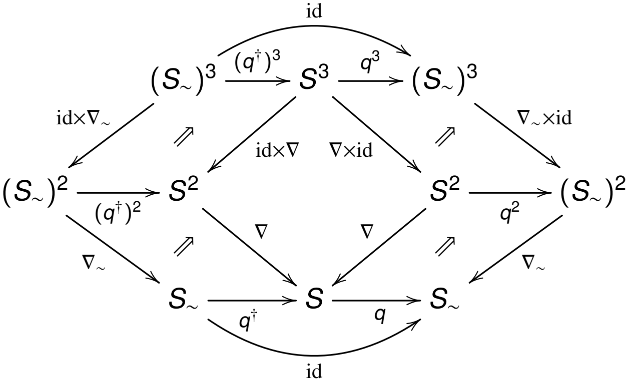
$$\begin{aligned} (\nabla_\sim)_{AB}^{C} &\iff \exists (a,b) \in A \times B, c \in C : \nabla_{ab}^{c} \\ (e_\sim)_{*}^{P} &\iff \exists p \in P : e_{*}^{p} \end{aligned} \tag{1}$$

Main result

Theorem

For every dimensionality on a relational monoid (S, ∇, e)*,* $(S_\sim, \nabla_\sim, e_\sim)$ *is a relational monoid.*

A snippet of the proof



Main result 2

Definition

A monoid A in **Rel** equipped with a mapping $(x \mapsto x^{-1})$ is a *hypergroup* if

$$\nabla^c_{ab} \iff \nabla^b_{(a^{-1})c} \iff \nabla^a_{c(b^{-1})}$$

Theorem

If A is a hypergroup and $\sim$ is a dimensionality that preserves the inverse

$$\begin{array}{ccc} A & \xrightarrow{\sim} & A \\ {\scriptstyle .^{-1}}\downarrow & & \downarrow{\scriptstyle .^{-1}} \\ A & \xrightarrow[\sim]{} & A \end{array}$$

then $A_\sim$ is a hypergroup.

Example

From the “same length” dimensionality on $\mathbb{R}^2$ we obtain the hypergroup of all lengths: $([0,\infty),\nabla,\{0\})$

$$\nabla_{xy}^{z} \Leftrightarrow |x-y| \leq z \leq x+y$$

Example

Pick the subgroup $\Gamma = \{1, 4\}$ of the multiplicative group of the field $\mathbb{Z}_5$, then

$$\mathbb{Z}_5/{\sim_\Gamma} = \{\{0\}, \{1, 4\}, \{2, 3\}\}$$

The additive hypergroup $\mathbb{Z}_5/\sim_\Gamma$ is

$+_\sim$	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]	[1]	[2][0]	[1][2]
[2]	[2]	[1][2]	[1][0]

Inner automorphisms of S_n

- Consider the symmetric group S_n.
- Let Γ be the group of inner automorphisms.
- We have $f \sim_\Gamma g$ iff f, g have the same cyclic type.
- Cyclic types correspond to decomposition of n as the sum of natural numbers (up to commutativity of $+$), i.e. *integer partitions*.
- S_n/Γ is a hypergroup.
- For S_6, we have

$$\nabla^{6}_{(3+3)(2+1+1+1+1)} \quad \nabla^{(3+2+1)}_{(3+3)(2+1+1+1+1)}$$