Monotone relations and effect algebras

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The definition

Definition

Let P, Q be posets. A relation $f \subseteq P \times Q$ is *monotone relation from P to Q* if and only if, for every $p_1, p_2 \in P$ and $q_1, q_2 \in Q$,

$$p_2 \geq p_1 \text{ and } f(p_1, q_2) \text{ and } q_2 \geq q_1 \text{ imply } f(p_2, q_1)$$

We write $f \colon P \twoheadrightarrow Q$ for a monotone relation f from P to Q.

Example

Let P, Q be posets. Both the universal relation $P \times Q \subseteq P \times Q$ and the empty relation $\emptyset \subseteq P \times Q$ are monotone.

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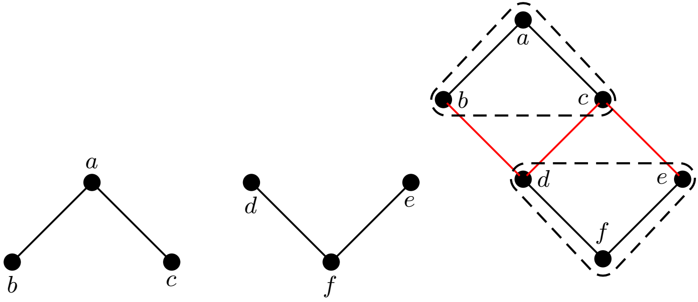
- Draw some additional lines between elements of P and elements of Q,so that the resulting picture is a Hasse diagram of a poset C.
- This poset then determines a monotone relation $f_C \subseteq P \times Q$ given by the rule $f_C(p, q)$ if and only if $q \leq_C p$.



P Q $cog(r)$

For every monotone mapping $f\colon P \to Q$, there is a monotone relation $\widehat{f}\colon P \nrightarrow Q$ given by

$$\widehat{f}(p, q) \Leftrightarrow f(p) \geq q$$

Let P, Q, R be posets, let $f\colon P \to\mapsto Q$ and $g\colon Q \to\mapsto R$ be monotone relations. The composite relation $g \circ f \subseteq |P| \times |R|$ is given by the rule

$$(g \circ f)(p, r) \text{ if and only if } f(p, q) \text{ and } g(q, r) \text{ for some } q \in Q.$$

It is easy to check that the composite relation of two monotone relations is monotone and that the operation of composition is associative.

For a poset P, the *identity monotone relation* is the relation $\mathrm{id}_P \colon P \nrightarrow P$ given by the rule

$$\mathrm{id}_P(x, y) \Leftrightarrow x \geq y$$

It is easy to see that for every monotone relation $f \colon P \nrightarrow Q$, $f = \mathrm{id}_Q \circ f = f \circ \mathrm{id}_P$.

The category of posets and monotone relations, denoted by **RelPos**, is a category whose objects are posets and morphisms are monotone relations.

The direct product $\otimes$ of posets is a bifunctor from **RelPos** $\times$ **RelPos** to **RelPos**. Indeed, for a pair of monotone relations $f\colon A \twoheadrightarrow B$ and $g\colon C \twoheadrightarrow D$ the monotone relation $(f \otimes g)\colon A \otimes C \twoheadrightarrow B \otimes D$ by the rule

$$(f \otimes g)((a, c), (b, d)) \Leftrightarrow f(a, b) \text{ and } g(c, d)$$

- Fix a 1-element poset and call it $\mathbf{1}$.
- $(\mathbf{RelPos}, \otimes, \mathbf{1})$ is a symmetric monoidal category
- For cographs, we have $cog(f \otimes g) = cog(f) \times cog(g)$.

A *dual object* to an object A of a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ is an object A^* such that there are morphisms $\eta_A \colon I \to A^* \otimes A$ and $\epsilon_A \colon A \otimes A^* \to I$ such that the diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\rho_A^{-1}} & A \otimes I \\
 & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \eta_A} \\
{\scriptstyle \mathrm{id}_A}\downarrow & & A \otimes A^* \otimes A \\
 & & \downarrow{\scriptstyle \epsilon_A \otimes \mathrm{id}_A} \\
A & \xleftarrow[\lambda_A]{} & I \otimes A
\end{array}
\qquad
\begin{array}{ccc}
A^* & \xrightarrow{\lambda_{A^*}^{-1}} & I \otimes A^* \\
 & & \downarrow{\scriptstyle \eta_A \otimes \mathrm{id}_{A^*}} \\
{\scriptstyle \mathrm{id}_{A*}}\downarrow & & A^* \otimes A \otimes A^* \\
 & & \downarrow{\scriptstyle \mathrm{id}_{A^*} \otimes \epsilon_A} \\
A & \xleftarrow[\rho_{A^*}]{} & A^* \otimes I
\end{array}
\tag{1}
$$

commute. The morphisms η_A and ϵ_A are called *coevaluation* and *evaluation*, respectively.

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Definition

A symmetric monoidal category is *compact closed* if every of its objects has a dual.

The category of finite-dimensional vector spaces $\mathbf{FinVect}(K)$ is compact closed.

Theorem

- $(\mathbf{RelPos}, \times, \mathbf{1})$ *is a compact closed category.*
- *The dual object of a poset A is the dual poset of A.*
- *For cographs:* $cog(f^*) \simeq (cog(f))^*$.

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- **FinHilb** (finite dimensional Hilbert spaces) is compact closed as well.
- What do we gain if we equip vector spaces with an inner product?
- We gain the Riesz representation theorem, which means that every object V of **FinHilb** is equipped with a canonical isomorphism $V \to V^*$.
- On categorical level, every morphism $f\colon V \to U$ is equipped with another morphism $f^\dagger\colon U \to V$ such that $f^{\dagger\dagger} = f$.

A *dagger category* is a category $\mathcal{C}$ equipped with an functor $\dagger\colon \mathcal{C} \to \mathcal{C}^{op}$ that is identity on objects and satisfies $f^{\dagger\dagger} = f$ for every morphism f of $\mathcal{C}$. In fact, the $\dagger$ functor can be characterized a mapping on the class of morphisms of $\mathcal{C}$ that has the following properties:

- $(\mathrm{id}_H)^\dagger = \mathrm{id}_H$
- $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$
- $f^{\dagger\dagger} = f$

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- However, there is a partial solution: we can replace **RelPos** with a category of self-dual posets with a fixed isomorphism $': A \to A^*$.

An *involution* on a poset P is mapping $' \colon P \to P$ satisfying the following conditions.

- For all $x, y \in P$, $x \leq y$ if and only if $y' \leq x'$.
- For all $x \in P$, $x'' = x$.

A poset equipped with an involution is called *involutive poset*, or *poset with involution*.

The category **RelPosInv** has posets equipped with involutions for objects and monotone relations for morphisms. Note that the morphism in **RelPosInv** do not interact with the involutive structure at all. However, the involutive structure on objects allows us to flip the morphisms: if $f : A \twoheadrightarrow B$ is a monotone relation, then there is a monotone relation $f^\dagger : B \twoheadrightarrow A$ given by the rule

$$f^\dagger(b, a) = f(a', b').$$

It is easy to check that $f^\dagger$ is a monotone relation. Moreover, **RelPosInv** equipped with $\dagger$ is a dagger category.

Theorem (GJ)

RelPosInv *is a dagger compact category.*

A *Frobenius structure* in a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ is an object A equipped with a monoid structure (A, ∇, e) and a comonoid structure (A, Δ, c) such that the following diagram commutes

$$
\begin{array}{ccccc}
A \otimes A & & \xrightarrow{\mathrm{id}_A \otimes \Delta} & & A \otimes A \otimes A \\
\Big\downarrow{\scriptstyle \Delta \otimes \mathrm{id}_A} & \searrow^{\nabla} & & & \Big\downarrow{\scriptstyle \nabla \otimes \mathrm{id}_A} \\
 & & A & & \\
 & & & \searrow^{\Delta} & \\
A \otimes A \otimes A & & \xrightarrow[\mathrm{id}_A \otimes \nabla]{} & & A \otimes A
\end{array}
\tag{2}
$$

A Frobenius structure is a *dagger Frobenius structure* if $\nabla = \Delta^\dagger$ and $m = c^\dagger$. Clearly, every dagger Frobenius structure is completely determined by its (co)monoid structure.

Theorem (Vicary)

Dagger Frobenius structures in **FinHilb** *are H^*-algebras.*

Problem

What are dagger Frobenius structures in **RelPosInv***?*

I do not know, but I have nice examples!

An *effect algebra* is a partial algebra $(E, \oplus, 0, 1)$ with a binary partial operation $\oplus$ and two nullary operations $0, 1$ satisfying the following conditions.

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a'$ exists and $a \oplus a' = 1$.

(E4) If $a \oplus 1$ is defined, then $a = 0$.

For every effect algebra E, there is a (clearly monotone) relation $\Delta\colon E \nrightarrow E \otimes E$ given by the rule

$$\Delta(x,(a,b)) \Leftrightarrow x \geq a \oplus b$$

Moreover, there is a monotone relation $c\colon E \nrightarrow I$ given by $c = E \to I$ (the total relation).

Theorem (GJ)

For every effect algebra E, (E, Δ, c) is a comonoid that gives rise to a dagger Frobenius structure on E.


That's all Folks!