Dacey graphs and posets

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The plan

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- Orthosets

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- Ortholattices associated to orthosets

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- Dacey orthosets and the orthomodular law

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- Orthosets of intervals in a bounded poset
- Incomparability orthosets

Orthosets

Definition

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- O is a set
- $\perp \subseteq O \times O$ is an irreflexive, symmetric relation called orthogonality.

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- Note that this is just a (symmetric, simple, loopless) graph.
- Adjacency is called orthogonality.
- Orthosets were called orthogonality spaces before.

Orthocomplement

- Let $(O, \perp)$ be an orthoset. For $X \subseteq O$, write

$$X^{\perp} = \{y \in O : y \perp x, \text{ for all } x \in X\};$$

we say that $X^{\perp}$ is the orthocomplement of X.

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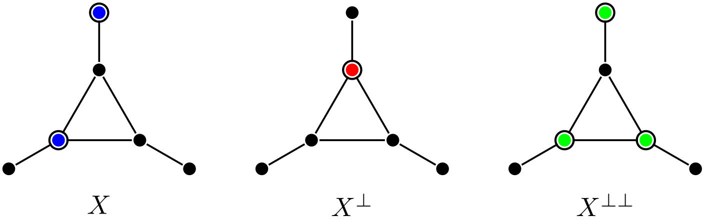
- Graph-theoretical definition: $X^\perp$ is the intersection of neighbourhoods of all vertices in X.

Orthocomplement example



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- $X \mapsto X^{\perp\perp}$ is a closure operator:
  - $X \subseteq Y \implies X^{\perp\perp} \subseteq Y^{\perp\perp}$
  - $X \subseteq X^{\perp\perp}$
  - $(X^{\perp\perp})^{\perp\perp} = X^{\perp\perp}$

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- ...that means, those for which we have $X = X^{\perp\perp}$.

Example

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- The closed subsets form a complete ortholattice $L(O, \perp)$, called the logic of $(O, \perp)$.

The main example (continued)

Recall, that an ortholattice $(L, \vee, \wedge, {}^\perp)$ is an orthomodular lattice if it satisfies the orthomodular law

$$x \leq z^\perp \implies (x \vee y) \wedge z = y \wedge z$$

for all, $x, y, z \in L$.

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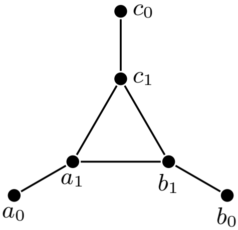
The logic of the orthoset $(V \setminus \{\vec{0}\}, \perp)$ the well-known orthomodular lattice of closed subspaces.

A finite example - Boolean logic



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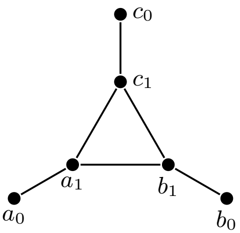
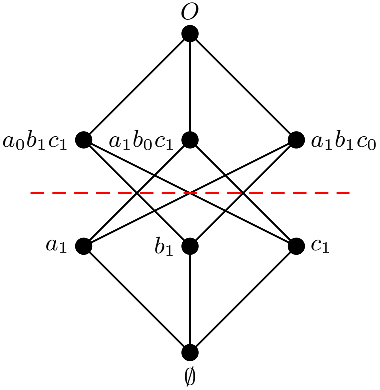
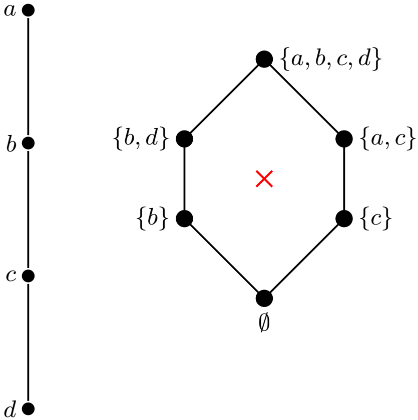




Another finite example - non orthomodular logic



Dacey orthosets

Definition

An orthoset $(O, \perp)$ is called Dacey if $L(O, \perp)$ is orthomodular.

Dacey orthosets

Let $(O, \perp)$ be an orthoset, let X be an closed subset of O. A maximal pairwise orthogonal subset of X is called a basis of X.

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Theorem

Dacey Jr1968) Let $(O, \perp)$ be an orthoset. Then $L(O, \perp)$ is an orthomodular lattice if and only if for every closed subset X of O and every basis B of X, $X = B^{\perp\perp}$.

A well-known corollary

Corollary

Every closed subspace of a Hilbert space has an orthogonal basis.

Cographs

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- So every graph that does not contain P_4 is Dacey.
- The opposite implication is not true.
- The graphs that do not contain P_4 are (nowadays) called cographs.
- They have many characterizations (see Wikipedia).

Orthosets of quotients

- For a poset P, we write $Q^+(P)$ for the set of all pairs $(a, b) \in P \times P$ with $a < b$.
- In lattice theory, the elements of $Q^+(A)$ are called proper quotients. An element $(a, b) \in Q^+(P)$ is denoted by $[a < b]$.

Orthosets of quotients

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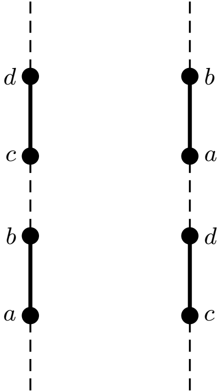

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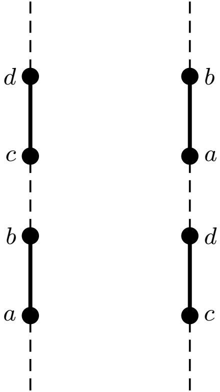


- Clearly, $\perp$ is symmetric and irreflexive, so $(Q^+(P), \perp)$ is an orthoset.

A characterization of finite lattices

Theorem

GJ (2023) Let P be a finite bounded poset. Then P is a lattice if and only if its orthoset of quotients $(Q^+(P), \perp)$ *is Dacey.*

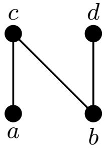
A characterization of bounded chains

Theorem

GJ (2023) Let P be a bounded poset. Then P is a chain if and only if $L(Q^+(P), \perp)$ is a Boolean algebra.

N-free posets

- Let P be a poset.
- For a quadruple of elements $(a, b, c, d) \in P^4$, we say that they form an N if and only if $a \prec c \succ b \prec d$ (note the covering relation here), $b < d$, and all the other distinct pairs of elements of the set $\{a, b, c, d\}$ are incomparable.



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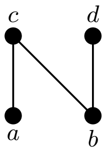

- A poset such that no quadruple of elements forms an N is called N-free.

N-free posets

N-free posets were introduced in Grillet1969). In that paper, the following characterization of N-free posets was proved.

Theorem

A finite poset P is N-free if and only if every maximal chain in P intersects every maximal antichain in P.

Incomparability orthosets

- Let P be a poset.
- For $x, y \in P$ let us now write $x \perp y$ if and only if x, y are incomparable.
- We say that $(P, \perp)$ is the <u>incomparability orthoset</u> of P.

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Theorem

GJ (2024) Let P be a finite poset. Then P is N-free if and only if its incomparability orthoset $(P, \perp)$ *is Dacey.*

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