Dimensionalities on monoids

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- We are going to introduce a new type of an equivalence on a monoid.
- We call this type of equivalence *dimensionality*.
- We have many natural examples.
- It turns out that a quotient of a monoid by a dimensionality is a hypermonoid.
- Motivated by this, we study dimensionalities on monoids in **Rel**, find even more natural examples.

The definition

Definition

Let $(A, ., e)$ be a monoid. We say that an equivalence relation $\sim$ on A is a *dimensionality* if the following conditions are satisfied.

- If $e \sim e'$, then $e = e'$.
- For all $a, b, c \in A$ such that $a.b \sim c$, there exist $a', b' \in A$ such that $a \sim a'$, $b \sim b'$ and $a'.b' = c$.

Example

Write $F_{\text{mon}}[x]$ for the set of all monic polynomials of one variable over a field F:

$$F_{\text{mon}}[x] = \{x^n + a_{n-1}x^{n-1} + \dots + a_0 : n \in \mathbb{N} \text{ and } a_{n-1}, \dots, a_0 \in F\}$$

Let $\sim$ be an equivalence on $F_{\text{mon}}[x]$ given by the rule $\mathbf{p} \sim \mathbf{q}$ iff $\mathbf{p}$ and $\mathbf{q}$ have the same degree.
Then $\sim$ is a dimensionality on the monoid $(F_{\text{mon}}[x], ., 1)$ iff F is algebraically complete.

- In this example, $\sim$ is a congruence.
- But in general, it does not have to be one.

Example

Consider the monoid $(\mathbb{R}^2, +, (0,0))$. The relation $\sim$ on $\mathbb{R}^2$ given by the rule

$$\vec{x} \sim \vec{y} \text{ iff } ||\vec{x}|| = ||\vec{y}||$$

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- Draw a picture of some board.
- This example is not a congruence.
- However, note that there is an action of $\mathbb{R}$ on the monoid $(\mathbb{R}^2, +, (0,0))$ such that the orbits of the action are exactly the equivalence classes of $\sim$.

Groups of automorphisms induce dimensionalities

Theorem

Let $(A, ., e)$ be a monoid, let Γ be a subgroup of $\mathrm{Aut}(A)$. Write

$$x \sim_\Gamma y \text{ iff } \exists f \in \Gamma : f(x) = y$$

Then $\sim_\Gamma$ is a dimensionality on A.

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Proof.

$$ab \sim_\Gamma c \iff f(ab) = c \iff f(a)f(b) = c$$

$\square$

Not every dimensionality arises from an automorphism group

Example

Recall the monic polynomial example $\mathbb{C}_{\mathrm{mon}}[x]$.

1. $x^2 \sim x.(x+1)$ (same degree).
2. Assume that $f \in \mathrm{Aut}(\mathbb{C}_{\mathrm{mon}}[x])$ is such that $f(x^2) = x.(x+1)$. We have
$$f(x^2) = f(x.x) = f(x).f(x)$$
3. Observe that $f(x).f(x) = x.(x+1)$ is impossible.

The lattice of dimensionalities

- Write $\mathrm{Dim}(A)$ for the set of all dimensionalities on a monoid A.
- $\mathrm{Dim}(A)$ is closed with respect to arbitrary joins taken in the lattice of all equivalences $\mathrm{Eq}(A)$.
- Therefore, $\mathrm{Dim}(A)$ is a complete lattice.
- The dimensionalities arising from automorphism groups form a complete join sub-semilattice of $\mathrm{Dim}(A)$.

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- This hyperoperation is associative, and the singleton class $\{e\}$ is a unit.

The quotient hypermonoid

Theorem

For every dimensionality $\sim$ on a monoid $(A, ., e)$ the hyperalgebra

$$(A/\sim, ., \{e\})$$

is a hypermonoid.

This allows us to construct many hypermonoids for free!

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Define a hyperoperation $+$ on $\mathbb{R}_0^+$ by the rule

$$a + b = [|a - b|, a + b].$$

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This is a hypermonoid $\mathbb{R}^2/\sim$, where $\sim$ is the "same norm" dimensionality.

Conjugacy on a group

- The conjugacy equivalence on a group Γ, given by

$$a \sim b \text{ iff } \exists c : b = c^{-1}ac$$

- is a dimensionality,
- because it arises from an action of an automorphism group.
- In particular, for $\Gamma = S_n$, $\sim$ means "to have the same cyclic type".
- Cyclic types are decompositions of n to a sum of positive integers (integer partitions).
- On S_6, we have

$$[((12)(34)(5)(6))]_\sim = 2 + 2 + 1 + 1$$

- This hypermonoid is strange an I would like to know more about it.

The hypermonoid of partitions

Multiplication by $(2 + 1 + \cdots + 1)$ (a single transposition cyclic type) either merges two cycles or splits one into two:

$$(2{+}3{+}1).(2{+}1{+}1{+}1{+}1) = \{1{+}1{+}3{+}1, 5{+}1, 2{+}2{+}1{+}1, 2{+}4, 3{+}3\}$$

Monoids in a monoidal category

Recall, that a *monoid* is a monoidal category $(\mathcal{C}, \otimes, 1)$ is a triple (M, ∇, e), where M is an object of $\mathcal{C}$, $\nabla\colon M \otimes M \to M$ and $e\colon 1 \to M$ are arrows such that the diagrams

$$\begin{tikzcd}
1 \otimes M \arrow[r, "e\otimes\mathrm{id}_M"] \arrow[dr, "\simeq"'] & M \otimes M \arrow[d, "\nabla"] & M \otimes 1 \arrow[l, "\mathrm{id}_M\otimes e"'] \arrow[dl, "\simeq"] \\
 & M &
\end{tikzcd}$$

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[rr, "\simeq"] \arrow[d, "\nabla\otimes\mathrm{id}_M"'] & & M \otimes (M \otimes M) \arrow[d, "\mathrm{id}_M\otimes\nabla"] \\
M \otimes M \arrow[dr, "\nabla"'] & & M \otimes M \arrow[dl, "\nabla"] \\
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- Monoids in the monoidal category of complete join semilattices $(\textbf{Sup}, \otimes, 2)$ are quantales.
- Monoids in the monoidal category of ordinary monoids $(\textbf{Mon}, \times, 1)$ are commutative monoids.

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- Objects: sets.
- Morphisms: binary relations; $f : A \to B$ in **Rel** is a set of pairs $f \subseteq A \times B$.
- Identities: $\mathrm{id}_A : A \to A$ is the identity relation.
- Composition: if $f : A \to B$ and $g : B \to C$, then $(a, c) \in g \circ f$ iff there exists $b \in B$ such that $(a, b) \in f$ and $(b, c) \in g$.

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- However, $\times$ is *not* the product in **Rel**, because...
- ...disjoint union $\sqcup$ is product and, at the same time, coproduct in **Rel**.

Monoids in **Rel**

- Hypermonoids should be viewed as a particular type of *monoids in the category of sets and relations*, denoted by **Rel**.
- We call these *relational monoids*.
- A mapping $\nabla\colon A \times A \to \mathcal{P}(A)$ in **Set** is the same thing as a relation $\nabla\colon A \times A \to A$.
- A relation $e\colon 1 \to A$ is the same thing as a selection of a subset of A.
- Unlike in a hypermonoid, a relational monoid allows for a set of units, instead of a single one.
- Unlike in a hypermonoid, a relational monoid allows for the operation to be undefined.

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- Small categories are monoids in **Rel**:
  - elements are arrows,
  - the $e : 1 \to M$ is the selection of identity arrows.
  - the operation is the composition of arrows

Rel as a 2-category

- A relation $f \subseteq A \times B$ is a *set* of pairs, so
- every homset $\mathbf{Rel}(A, B)$ is a poset under $\subseteq$.
- That means, that **Rel** is enriched in **Pos**, in other words
- **Rel** a (locally posetal/thin) 2-category.

Rel as a dagger category

For every relation $f\colon A \to B$ there is an opposite relation $f^\dagger\colon B \to A$ given by

$$(b,a) \in f^\dagger \iff (a,b) \in f$$

This makes **Rel** to a *dagger category*.

Morphisms of monoids in **Rel**

There are at least two meaningful notions of morphisms of monoids in **Rel**.

$$\begin{tikzcd} M \times M \arrow[r, "h \times h"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\nearrow\!\!\!\!\nearrow"] & M' \times M' \arrow[d, "\nabla"] \\ M \arrow[r, "h"] & M' \end{tikzcd}$$

Lax morphism

$$(h \circ \nabla) \subseteq (\nabla \circ (h \times h))$$
$$h \circ e \subseteq e$$

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$$e \subseteq h \circ e$$

Equivalences as morphisms in **Rel**

- Mappings, equivalence relations, partial orders are different types of things in **Set**-world.
- In **Rel** all of those things are just morphisms.
- An equivalence on A is just $\sim\colon A \to A$ such that
  - $\mathrm{id}_A \subseteq \sim$
  - $\sim^\dagger = \sim$
  - $\sim \circ \sim = \sim$

Equivalences as endomorphisms of monoids in **Rel**

If (M, ∇, e) is an ordinary monoid in **Set**, we obtain the notions of dimensionality and congruence.

$$\begin{tikzcd} M \times M \arrow[r, "\sim\times\sim"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\Nearrow"] & M \times M \arrow[d, "\nabla"] \\ M \arrow[r, "\sim"] & M \end{tikzcd}$$

Dimensionality

$$(\sim \circ \nabla) \subseteq (\nabla \circ (\sim \times \sim))$$

$$\sim \circ e \subseteq e$$

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Congruence

$$(\sim \circ \nabla) \supseteq (\nabla \circ (\sim \times \sim))$$

$$e \subseteq \sim \circ e$$

The quotient relational monoid

Theorem

For every dimensionality $\sim$ on a relational monoid (A, ∇, e),

$$(A/\sim, \nabla, e)$$

is a relational monoid.

- Let us remark that we studied dimensionalities in [1].
- However, at that time we focused on dimensionalities that are *congruences*.
- We did not know that we do not need the congruence property in order for the quatient to be associative.

Dimensionality on lattices

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- On the other hand, lattices are posets, so they are categories, so they are relational monoids:
  - For a lattice L, $I(L)$ is the set of all $a, b \in L$ with $a \leq b$.
  - Define (partial) operation as

$$[b \leq c] \circ [a \leq b] = [a \leq c]$$

  - and select the units as a subset $\{[a \leq a] : a \in L\}$.

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  Then $I(L)$ is a relational monoid.
- Write $[x \leq y] \sim [z \leq w]$ iff the intervals $[x, y]_L$ and $[z, w]_L$ are isomorphic as lattices.
- Then $\sim$ is dimensionality of $I(L)$.
- Except for very simple cases, $I(L)/\sim$ is a genuine hypermonoid.

Dimensionality in rings

- Let A be an involutive ring with unit, in which

$$x^*x + y^*y = 0 \implies x = y = 0.$$

- Let $P(A)$ be the set of all self-adjoint idempotents in A. For $e, f \in P(A)$, write $e \oplus f = e + f$ iff $ef = 0$, otherwise let $e \oplus f$ be undefined. Then $(P(A); \oplus, 0, 1)$ is an effect algebra (a particular type of a partial monoid)
- For e, f in $P(A)$, write $e \sim f$ iff there is $w \in A$ such that $e = w^*w$ and $f = ww^*$.
- Then this is a dimensionality.

Frobenius monoids

A *Frobenius monoid* S is a structure

$$(S, \nabla, \Delta, e, c)$$

such that

- (S, ∇, e) is a monoid
- (S, Δ, c) is a comonoid
- $(\nabla \otimes \mathrm{id}_S) \circ (\mathrm{id}_S \otimes \Delta) = \Delta \circ \nabla = (\mathrm{id}_S \otimes \nabla) \circ (\Delta \otimes \mathrm{id}_S)$

Dagger Frobenius monoids

A *Frobenius monoid S* in a dagger monoidal category is a monoid (S, ∇, e) such that

$$(S, \nabla, \nabla^{\dagger}, e, e^{\dagger})$$

is a Frobenius monoid.

Theorem

For every dagger Frobenius monoid (S, ∇, e) *in* **Rel** *and a dimensionality* $\sim$ *on* S, $S/\sim$ *is a dagger Frobenius monoid.*

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Corollary

For every group Γ *and a dimensionality* $\sim$ *on* Γ, $\Gamma/\sim$ *is a hypergroup.*

Finally, let us mention another definition, from the classical paper [2].

Definition (Loomis 1955)

Let A be a complete orthomodular lattice. A *dimension equivalence* on A is a equivalence relation on A such that

(A) If $a \sim 0$, then $a = 0$.

(B) If $a_1 \perp a_2$ and $a_1 \vee a_2 \sim b$, then there exists an orthogonal decomposition of b, $b = b_1 \vee b_2$, such that $b_1 \sim a_1$ and $b_2 \sim a_2$.

(C) If $\{a_\alpha\}$ and $\{b_\alpha\}$ are pairwise orthogonal families of elements, such that $a_\alpha \sim b_\alpha$ for all α, then $\bigvee_\alpha a_\alpha = \bigvee_\alpha b_\alpha$.

(D) If a and b are not orthogonal in A then there are nonzero a_1, b_1 in A such that $a \geq a_1$, $b \geq b_1$ and $a_1 \sim b_1$.

Note that (A) and (B) match exactly the definition of dimensionality.

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