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## Foreword

Regular seminar 'Uncertainty Modelling' was founded by Prof. B. Riečan and Prof. R. Mesiar in 1990. Since 1995 it is hosted by Department of Mathematics and Descriptive Geometry of the Faculty of Civil Engineering, Slovak University of Technology in Bratislava. Talks used to be given either weekly or at once in a single day by participants from STU Bratislava, MBU Banská Bystrica and visitors from abroad, which was accompanied by fruitful mutual collaboration.

The present collection of scientific papers has raised from this collaboration and covers many interesting topics from the area of uncertainty modelling including copulas, triangular norms, integrals, fuzzy orders and relations.

All these papers were reviewed by independent reviewers and only its final accepted version is published. Our gratitude goes to all authors, as well as to all reviewers whose work has significantly contributed to the high quality of papers included in this collection.

Bratislava, January 2016
Tomáš Bacigál and Martin Kalina
editors

# Examples of Archimedean generators from the Williamson transform and why to use a linear approximation 

Tomáš Bacigál *i Mária Ždímalová **


#### Abstract

We illustrate a construction method for obtaining additive generators of Archimedean copulas proposed by McNeil and Nešlehová [7], the so-called Williamson n-transform. Then we use weighted sum of Dirac functions to approximate generators of two-dimensional Archimedean copulas by linear splines to circumvent the problem with the non-existence of explicit inverse.


Keywords: Archimedean copula, Williamson transform, approximation

## 1 Introduction

Copulas form an important class of multivariate dependence models. They have a lot of practical applications, including multivariate survival modelling. Recall that copulas aggregate 1-dimensional marginal distribution functions into $n$-dimensional $(n \geq 2)$ joint distribution functions. For more details we recommend [12].

We first define a copula. A function $C:[0,1]^{n} \rightarrow[0,1]$ is called a ( $n$-dimensional) copula whenever it satisfies the boundary conditions ( C 1 ) and it is an $n$-increasing function, see ( C 2 ), where:
(C1) $C\left(x_{1}, \ldots, x_{n}\right)=0$ whenever $0 \in\left\{x_{1}, \ldots, x_{n}\right\}$, i.e., 0 is an annihilator of C , and $C\left(x_{1}, \ldots, x_{n}\right)=$ $x_{i}$ whenever $x_{j}=1$ for each $j \neq i$ (i.e., 1 is a neutral element of $C$ ),
(C2) For any $\mathbf{x}, \mathbf{y} \in[0,1]^{n}, \mathbf{x} \leq \mathbf{y}$, it holds

$$
V_{C}([\mathbf{x}, \mathbf{y}])=\sum_{\varepsilon \in\{-1,1\}^{n}}\left(C\left(\mathbf{z}_{\varepsilon}\right) \prod_{i=1}^{n} \varepsilon_{i}\right) \geq 0
$$

where $\mathbf{z}_{\varepsilon}=\left(z_{1}^{\varepsilon_{1}}, \ldots, z_{n}^{\varepsilon_{n}}\right), z_{i}^{1}=y_{i}, z_{i}^{-1}=x_{i}$.
Note that $V_{C}([\mathbf{x}, \mathbf{y}])$ is called the $C$-volume of the $n$-dimensional interval ( $n$-box) $[\mathbf{x}, \mathbf{y}]$.
Due to Sklar's theorem [15] for a random vector $Z=\left(X_{1}, \ldots, X_{n}\right)$, a function $F_{Z}: R^{n} \rightarrow[0,1]$ is a joint distribution function of $Z$ if and only if there is a copula $C:[0,1]^{n} \rightarrow[0,1]$ so that

$$
\begin{equation*}
F_{Z}\left(x_{1}, \ldots, x_{n}\right)=C\left(F_{X_{1}}\left(x_{1}\right), \ldots, F_{X_{n}}\left(x_{n}\right)\right) \tag{1}
\end{equation*}
$$

where $F_{X_{i}}: R \rightarrow[0,1]$ is a distribution function related to the random variable $X_{i}, i=1, \ldots, n$. The copula $C$ in (1) is unique whenever random variables $X_{1}, \ldots, X_{n}$ are continuous. For some other details on copulas see [4] and [12].

Hereafter we will consider a class of copulas named Archimedean copulas. In the simplest case, Archimedean 2-copulas are characterized by the associativity of $C$ and the diagonal inequality $C(x, x)<$ $x$ for all $x \in] 0,1[$. They are necessarily symmetric, i.e., they can model the stochastic dependence of exchangeable random variables ( $X, Y$ ) only, yet their popularity in practice (hydrology, financial, and other applied areas) is indisputable, mainly due to the representation using one-dimensional functions called generators as shown in the next result, attributed to Moynihan [11].

[^0]Theorem 1 A function $C:[0,1]^{2} \rightarrow[0,1]$ is an Archimedean copula if and only if there is a convex (i.e. a 2-monotone) continuous strictly decreasing function $f:[0,1] \rightarrow[0, \infty], f(1)=0$, so that

$$
\begin{equation*}
C(x, y)=f^{(-1)}(f(x)+f(y)) \tag{2}
\end{equation*}
$$

where the pseudo-inverse $f^{(-1)}:[0, \infty] \rightarrow[0,1]$ is given by

$$
f^{(-1)}(u)=f^{-1}(\min (u, f(0)))
$$

The function $f$ is called an additive generator of the copula $C$, and it is unique up to a positive multiplicative constant.

Let $\mathcal{F}_{2}$ be the class of all additive generators of binary copulas characterized in the above theorem. More details about the generators can be found in $[4,5,12]$ and about construction methods for additive generators in [1, 2, 3, 6, 10].

Before we review several known facts for additive generators of copulas, let us briefly recall a link between copula $C$ and Spearman's correlation coefficient $\rho$,

$$
\begin{equation*}
\rho=12 E[U V]-3=12 \iint_{[0,1]^{2}} u v d C(u, v)-3=12 \iint_{[0,1]^{2}} C(u, v) d u d v-3 \tag{3}
\end{equation*}
$$

as well as Kendall's correlation coefficient $\tau$,

$$
\begin{equation*}
\tau=4 E[C(U, V)]-1=4 \iint_{[0,1]^{2}} C(u, v) d C(u, v)-1 \tag{4}
\end{equation*}
$$

where $U=F_{X}(X)$ and $V=F_{Y}(Y)$ are uniformly distributed random variables, that are connected by the same copula as are $X$ and $Y$. Alternatively, Kendall's tau can be computed directly from copula generator,

$$
\tau=1+4 \int_{0}^{1} \frac{f(t)}{f^{\prime}(t)} d t=1-4 \int_{0}^{\infty} t\left(f^{(-1)^{\prime}}(t)\right)^{2} d t
$$

which is far more convenient.
Any binary Archimedean copula $C:[0,1]^{2} \rightarrow[0,1]$ generated by an additive generator $f:[0,1] \rightarrow$ $[0, \infty]$, is also a triangular norm $[5,14]$ and thus, it can be univocally extended to an $n$-ary function (we keep the original notation also for this extension) $C:[0,1]^{n} \rightarrow[0,1]$ given by

$$
\begin{equation*}
C\left(x_{1}, \ldots, x_{n}\right)=f^{(-1)}\left(\sum_{i=1}^{n} f\left(x_{i}\right)\right) \tag{5}
\end{equation*}
$$

Obviously, for any $n \geq 2, C$ satisfies the boundary conditions (C1). However, for $n>2$, (C2) may fail. For example, the smallest binary copula $W:[0,1]^{2} \rightarrow[0,1]$ given by $W(x, y)=\max (0, x+y-1)$ is generated by the additive generator $f_{W}:[0,1] \rightarrow[0, \infty], f_{W}(x)=1-x$. Its $n$-ary extension is given by

$$
W\left(x_{1}, \ldots, x_{n}\right)=1-\min \left(1, \sum_{i=1}^{n}\left(1-x_{i}\right)\right)=\max \left(0, \sum_{i=1}^{n} x_{i}-(n-1)\right)
$$

Consider $\mathbf{x}, \mathbf{y} \in[0,1]^{n}, \mathbf{x}=\left(\frac{1}{2}, \ldots, \frac{1}{2}\right), \mathbf{y}=(1, \ldots, 1)$. Then $V_{W}([\mathbf{x}, \mathbf{y}])=1-\frac{n}{2}$, i.e., this volume is negative whenever $n>2$, which shows that $W$ is a copula only for $n=2$. A complete description of additive generators of binary copulas such that the corresponding generated $n$-ary function is also an $n$-ary copula, $n>2$, was given by McNeil and Nešlehová in [7] and is recalled in the next theorem.

Theorem 2 Let $f:[0,1] \rightarrow[0, \infty]$ be a continuous strictly decreasing function such that $f(1)=0$ (i.e., $f$ is an additive generator of a continuous Archimedean t-norm, see [5]). Then the n-ary function $C:[0,1]^{n} \rightarrow[0,1]$ given by (5) is an n-ary copula if and only if the function $g:[-\infty, 0] \rightarrow[0,1]$ given by $g(u)=f^{(-1)}(-u)$ is $(n-2)$-times differentiable with non-negative derivatives $g^{\prime}, \ldots, g^{(n-2)}$ on $]-\infty, 0\left[\right.$ (or equivalently, $\left.(-1)^{n}\left(f^{(-1)}\right)^{(n)}(u) \geq 0\right)$, and $g^{(n-2)}$ is a convex function.

We denote by $\mathcal{F}_{n}$ the class of all additive generators that generate $n$-ary copulas as characterized in Theorem 2.

Additive generators, which generate an $n$-ary copula for any $n \geq 2$, are called universal generators. The class of all universal additive generators will be denoted by $\mathcal{F}_{\infty}$. It is not difficult to check that $\mathcal{F}_{2} \supset \mathcal{F}_{3} \supset \ldots \supset \mathcal{F}_{\infty}$.

The $n$-monotone Archimedean copula generators may be characterized using a little known integral transform introduced by Williamson in 1956, see [17]. In McNeil and Nešlehová [7] there is a description of this transform, which, for a fixed $n \geq 2$, will be called the Williamson $n$-transform. In what follows, we discuss the Williamson $n$-transform and illustrate it by examples.

## 2 The Williamson $n$-transform

An interesting link between additive generators of copulas and positive distance functions [8], i.e., distribution functions with support in $] 0, \infty[$, was described in details in [7]. Based on the results of Williamson [17], we recall the next important result.

Theorem 3 (McNeil \& Nešlehová [7], Corollary 3.1.) The following claims are equivalent for an arbitrary $n \in\{2,3, \ldots\}$ :
(i) $f \in \mathcal{F}_{n}$
(ii) Under the notation of Theorem 2, the function $F$ : $]-\infty, \infty[\rightarrow[0,1]$ given by $F(x)=0$ if $x \leq 0$, and for $x>0$,

$$
\begin{equation*}
F(x)=1-\sum_{k=0}^{n-2} \frac{(-1)^{k} x^{k}\left(f^{(-1)}\right)^{(k)}(x)}{k!}-\frac{(-1)^{n-1} x^{n-1}\left(f^{(-1)}\right)_{+}^{(n-1)}(x)}{(n-1)!} \tag{6}
\end{equation*}
$$

is a distribution function of a positive random variable $X$ (i.e., $P(X \leq 0)=0$ ), where $\cdot{ }_{+}^{(n-1)}$ denotes the right-derivative of order $n-1$.

Note that due to [17], if $F$ is a positive distance function, i.e., a distribution function of a positive random variable $X$, then for a fixed $n \in\{2,3, \ldots\}$ the Williamson $n$-transform provides an inverse transformation to (6),

$$
f^{(-1)}(x)=\int_{x}^{\infty}\left(1-\frac{x}{t}\right)^{n-1} d F(t)= \begin{cases}\max \left(0, E\left[1-\frac{x}{X}\right]^{n-1}\right), & x>0  \tag{7}\\ 1-F(0), & x=0\end{cases}
$$

where $x \in\left[0, \infty\left[\right.\right.$ and $f^{(-1)}(\infty)=0$.
Note that a similar relationship can be shown between additive generators from $\mathcal{F}_{\infty}$ and positive distance functions, based on the Laplace transform, i.e

$$
\begin{equation*}
f^{(-1)}(x)=\int_{0}^{\infty} e^{-x t} d F(t) \tag{8}
\end{equation*}
$$

For more and interesting details we recommend [7].
Let $F$ be a distance function related to a positive random variable $X$. For any $c>0$, the random variable $c$. $X$ possesses the distance function $F_{c}$ given by $F_{c}(x)=F\left(\frac{x}{c}\right)$. Then, for any $n \in\{2,3, \ldots\}$,
$f_{c}^{(-1)}(x)=\int_{x}^{\infty}\left(1-\frac{x}{t}\right)^{n-1} d F_{c}(t)=\int_{x}^{\infty}\left(1-\frac{x}{t}\right)^{n-1} d F\left(\frac{t}{c}\right)=\int_{\frac{x}{c}}^{\infty}\left(1-\frac{x}{c u}\right)^{n-1} d F(u)=f^{(-1)}\left(\frac{x}{c}\right)$.
Obviously, for the related additive generators it holds that $f_{c}=c . f$, i.e., they generate the same copula. Vice versa, clearly from (6) it follows that if two generators generate the same ( $n$-ary) Archimedean copula, the corresponding positive random variables differ only in a positive multiplicative constant. The next result follows.

Theorem 4 For each $n \in\{2,3, \ldots\}$, there is an one-to-one correspondence between the class $\mathcal{F}_{n}$ and the class $\mathcal{H}$ of all factor classes of positive distance functions related to the equivalence $F \sim G$ if and only if $G(x)=F\left(\frac{x}{c}\right)$ for some $c>0$.

In the following, we illustrate the construction method by few examples.

Example 1 Let $F$ be equal to a Dirac function ${ }^{1}$ focused at point $x_{0}=1$,

$$
F(x)=\delta_{1}(x)= \begin{cases}0 & x<1 \\ 1 & 1 \leq x\end{cases}
$$

then, as is also shown in [7], by the Williamson $n$-transform we get generator $f_{n}(x)=1-x^{\frac{1}{n-1}}$ of the weakest $n$-dimensional Archimedean copula, i.e., the non-strict Clayton copula with parameter $\lambda=\frac{-1}{n-1}$, see Figure 1. By rescaling generator to $\tilde{f}_{n}(x)=\frac{f(x)}{f(1 / 2)}, x \in[0,1]$, the copula would not change, yet such a generator is fixed to the value $\tilde{f}_{n}\left(\frac{1}{2}\right)=1$, which we will use later to show convergence.


Figure 1: Dirac function $F$, the corresponding generators $f_{n}$ for different $n$ and rescaled generators $\tilde{f}_{n}$.

Example 2 Let $F$ be a uniform probability distribution function

$$
F(x)=\left\{\begin{array}{ll}
0 & x<a \\
\frac{x-a}{b-a} & a \leq x<b \\
1 & b \leq x
\end{array} \quad \text { with } 0 \leq a<b\right.
$$

Then for dimension $n=2$ we get

$$
\begin{aligned}
f_{2}^{(-1)}(x) & =\int_{x}^{\infty}\left(1-\frac{x}{t}\right)^{2-1} F^{\prime}(t) d t= \begin{cases}\int_{a}^{b}\left(1-\frac{x}{t}\right) \frac{1}{b-a} d t & x<a \\
\int_{x}^{b}\left(1-\frac{x}{t}\right) \frac{1}{b-a} d t & a \leq x<b= \\
\int_{x}^{\infty}\left(1-\frac{x}{t}\right) 0 d t & b \leq x\end{cases} \\
& = \begin{cases}\frac{1}{b-a}[t-x \log t]_{a}^{b}=\frac{1}{b-a}(b-x \log b-a+x \log a)=1-\frac{x \log \left(\frac{b}{a}\right)}{b-a} & x<a \\
\frac{1}{b-a}[t-x \log t]_{x}^{b}=\frac{1}{b-a}(b-x \log b-x+x \log x)=\frac{b}{b-a}-\frac{x+x \log \left(\frac{b}{x}\right)}{b-a} & a \leq x<b \\
0 & b \leq x\end{cases}
\end{aligned}
$$

${ }^{1}$ Dirac function is defined as $\delta_{x_{0}}(x)=\left\{\begin{array}{ll}0 & x\end{array}<x_{0}\right.$,
(where $F^{\prime}$ denotes a first derivative of $F$ ) from which the corresponding generator can be obtained only numerically, and so is the case also with the higher dimensions, e.g.,

$$
f_{3}^{(-1)}(x)= \begin{cases}1-\frac{2 x \log \left(\frac{b}{a}\right)}{b-a}+\frac{x^{2}}{a b} & x<a \\ \frac{b}{b-a}-2 x \log \left(\frac{b}{x}\right)-\frac{x^{2}}{(b-a) b} & a \leq x<b \\ 0 & b \leq x\end{cases}
$$

displayed in Figure 2.


Figure 2: Uniform $\mathrm{U}(\mathrm{a}, \mathrm{b})$ probability distribution function $F$ and pseudo-inverses of the corresponding generators $f_{n}$.

Example 3 Consider a positive distance function $F(x)=\min \left(1, x^{2}\right)$ and the corresponding density $F^{\prime}(x)=2 x$ on $[0,1]$. Then
$f_{2}^{(-1)}(x)=\int_{x}^{\infty}\left(1-\frac{x}{t}\right)^{2-1} d F(t)=\left\{\begin{array}{ll}\int_{x}^{1}(t-x) \frac{2 t}{t} d t=\left[(t-x)^{2}\right]_{x}^{1}=(1-x)^{2} & 0 \leq x \leq 1 \\ 0 & 1<x\end{array}=\max (1-x, 0)^{2}\right.$.
Then the generator $f_{2}(x)=1-\sqrt{x}, x \in[0,1]$, is the generator of Clayton copula for parameter $\lambda=-\frac{1}{2}$. Nevertheless, in higher dimensions, $n \geq 3$, the generator has no closed form, e.g., $f_{3}^{(-1)}(x)=$ $1-4 x+x^{2}(3-2 \log x)$ for $x \in[0,1]$ and 0 otherwise.


Figure 3: Illustration of Example 3 with non-invertible case $n=3$.
It is interesting to illustrate also the inverse Williamson $n$-transform.
Example 4 Take a generator of

- the Ali-Mikhail-Haq copula $f(x)=\frac{1}{x}-1$ corresponding to the parameter $\lambda=1$ and denote by $F_{n}, n=2,3, \ldots$, a positive distance function related to $f$ through (6). Then $F_{n}(x)=1-\frac{1}{1+x}-$ $\frac{x}{(1+x)^{2}}-\ldots-\frac{x^{n-1}}{(1+x)^{n}}=\left(\frac{x}{1+x}\right)^{n}$ which can be viewed as a parametric subfamily of all positive valued distribution functions $F_{p}$ with any positive parameter $p$.
- the product copula $f(x)=-\frac{1}{p} \log x$ with constant $p>0$ and inverse $f^{-1}(x)=\exp (-p x)$. From (6) for $n=2$ we get $F(x)=1-\exp (-p x)(1-p x)$. By comparing the density $\frac{\partial F(x)}{\partial x}=$ $p^{2} x \exp (-p x)$ and the convolution of two exponential distribution $\mathcal{D}_{\lambda}$ densities with parameter $\lambda>0, \int_{0}^{x} \lambda \exp (-\lambda t) \lambda \exp (-\lambda(x-t)) d t=\lambda^{2} x \exp (-\lambda x)$ it becomes clear that the resulting distribution is a distribution of the random variable $Y=X_{1}+X_{2}$, where $X_{1}, X_{2} \sim \mathcal{D}_{\lambda}$ are independent (and identically distributed) random variables. The relation holds for any $n \geq 2$, thus (6) yields a cumulative distribution function of the sum of i.i.d. random variables $X_{1}, \ldots, X_{n} \sim$ $\mathcal{D}_{p}, F_{X_{1}+\ldots+X_{n}}(x)=1-\exp (-p x) \sum_{i=1}^{n} \frac{(p x)^{i-1}}{(i-1)!}$ with $p>0$.
To complete the examples, let us illustrate also the Laplace transform.
Example 5 Starting with positive distance function of
- discrete random variable with probability mass concentrated in $\lambda>0$, i.e. Dirac function $F(x)=$ 0 for $x<\lambda$ and 1 otherwise, then the Laplace transform leads through $g(x)=\exp (\lambda x)$ to the product copula $\Pi$.
- exponential distribution $F(x)=1-\exp (-\lambda x), \lambda>0$, we get $f^{-1}(x)=\left(\frac{\lambda}{x+\lambda}\right)$ and $f(x)=$ $\lambda\left(\frac{1}{x}-1\right)$ which generates the same copula (Clayton copula with parameter equal to 1) regardless of the choice of $\lambda$.

Now we focus on the Dirac function since it can be viewed as a building block for distribution functions of a random variable with probability mass concentrated in $l$ discrete points.

## 3 Approximation

In this section we are interested mainly in $(n=2)$-dimensional case, since it is of most benefit in practice. Therefore hereafter the subscript with generator $f$ gains a different meaning: the number of pieces $f$ is approximated by.

Example 6 Let $F(x)=\min \left(1, x^{2}\right)$ be the positive distance function from the Example 3 and function

$$
F_{2}(x)=F\left(\frac{1}{2}\right) \delta_{\frac{1}{2}}(x)+\left(F(1)-F\left(\frac{1}{2}\right)\right) \delta_{1}(x)= \begin{cases}0 & x<\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \leq x<1 \\ 1 & 1 \leq x\end{cases}
$$

approximates $F$ by means of a sum of $m=2$ Dirac functions concentrated in respective points $\left(\frac{1}{2}, \frac{1}{4}\right)$, $\left(1, \frac{3}{4}\right)$. Then the Williamson transform with $n=2$ yields

$$
f_{2}^{(-1)}(x)=\frac{1}{4} \max \left(0,1-\frac{x}{\frac{1}{2}}\right)+\frac{3}{4} \max \left(0,1-\frac{x}{1}\right)= \begin{cases}1-\frac{5}{4} x & x<\frac{1}{2} \\ \frac{3}{4}-\frac{3}{4} x & \frac{1}{2} \leq x<1 \\ 0 & 1 \leq x\end{cases}
$$

From Example 6 illustrated on Figure 4 we see that for $n=2$ the additive generator inverse $f_{2}^{(-1)}$ is piecewise linear and does not coincide with $f^{(-1)}$ in the interval $] 0,1[$.

Dividing an interval $\left[a_{0}, a_{m}\right]$ by points $\left\{a_{i}\right\}_{i=1, \ldots m}, a_{0}<a_{1}<\ldots a_{m}$, with concentration of probability given by some probability mass function $p(x)$, the approximate positive distance function

$$
F_{m}(x)=\sum_{i=1}^{m} p\left(a_{i}\right) \delta_{a_{i}}(x)
$$

is then transformed by (7) to the generator inverse (related to some $n$-dimensional Archimedean copula)

$$
\begin{equation*}
f_{m}^{(-1)}(x)=\sum_{x<a_{i}} p\left(a_{i}\right)\left(1-\frac{x}{a_{i}}\right)^{n-1}=\sum_{i=1}^{m} p\left(a_{i}\right) \max \left(0,1-\frac{x}{a_{i}}\right)^{n-1} \tag{9}
\end{equation*}
$$



Figure 4: Approximation by the sum of $m=2$ Dirac functions
Observe that the function $f_{m}^{(-1)}(9)$ is a $(n-1)$-dimensional spline. For $n=2$, both $f_{m}^{(-1)}$ and the corresponding additive generator $f_{m}$ are linear splines, and the related Archimedean copula $C_{m}$ is piece-wise linear, as shown in Example 8. In the opposite direction, denote $b_{i}=f_{m}^{(-1)}\left(a_{i}\right)$ and $p_{i}=p\left(a_{i}\right)$ for $i=1,2 \ldots m$ with $b_{0}=1$ corresponding to $a_{0}=0$ and, clearly, $b_{m}=0$. Having points $\left\{\left(a_{i}, b_{i}\right)\right\}_{i=1, \ldots m}$, their corresponding probabilities can be found by solving exquations (9) with $x=a_{1}, \ldots, a_{m-1}$ written in the form (for $n=2$ )

$$
\left(\begin{array}{cccc}
1-\frac{a_{1}}{a_{2}} & 1-\frac{a_{1}}{a_{3}} & \cdots & 1-\frac{a_{1}}{a_{m}} \\
0 & 1-\frac{a_{2}}{a_{3}} & \cdots & 1-\frac{a_{2}}{a_{m}} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1-\frac{a_{m-1}}{a_{m}}
\end{array}\right)\left(\begin{array}{c}
p_{2} \\
p_{3} \\
\vdots \\
p_{m}
\end{array}\right)=\left(\begin{array}{c}
b_{1} \\
b_{2} \\
\vdots \\
b_{m-1}
\end{array}\right)
$$

The solution is $p_{1}=1-\left(p_{2}+\ldots+p_{m}\right)$ and

$$
p_{i}=\frac{a_{i}\left[b_{i-1}\left(a_{i+1}-a_{i}\right)-b_{i}\left(a_{i+1}-a_{i-1}+b_{i+1}\left(a_{i}-a_{i-1}\right)\right)\right]}{\left(a_{i+1}-a_{i}\right)\left(a_{i}-a_{i-1}\right)} \quad \text { for } i=2, \ldots m
$$

with auxiliary point $\left(a_{m+1}, b_{m+1}\right)$, where $a_{m+1} \geq a_{m}$ and thus $b_{m+1}=0$.
In the following examples we exercise pointwise convergence and show a piecewise linear copula corresponding to the simplest non-trivial case $n=m=2$.

Example 7 For the simplest case, $n=2, a_{i}=\frac{i}{m}$ and $p\left(a_{i}\right)=\frac{1}{m}, i=1, \ldots m$ (evenly spaced and uniformly distributed), we get $f_{m}^{(-1)}(x)=\sum_{i=1}^{m} \frac{1}{m} \max \left(0,1-\frac{m x}{i}\right)$. If $f_{m}^{(-1)}(x)$ is to converge to $f^{(-1)}(x)=1-x+x \log x$ for $x<1$ and 0 elsewhere, it needs to converge in any point $\left.x \in\right] 0,1[$. Let us examine the convergence, say, in $x=\frac{1}{2}$, where

$$
\begin{aligned}
& f_{m}^{(-1)}\left(\frac{1}{2}\right)=\frac{1}{m} \sum_{i=1}^{m} \max \left(0,1-\frac{m \frac{1}{2}}{i}\right)=\frac{1}{m} \sum_{i=\left\lfloor\frac{m}{2}\right\rfloor+1}^{m}\left(1-\frac{m}{2 i}\right)=\frac{1}{m} \sum_{i=1}^{\frac{m}{2}} \frac{i}{i+\frac{m}{2}}= \\
& \frac{1}{m} \sum_{i=1}^{\frac{m}{2}}\left(1-\frac{\frac{m}{2}}{i+\frac{m}{2}}\right)=\frac{1}{2}-\frac{1}{2} \sum_{i=\left\lfloor\frac{m}{2}\right\rfloor+1}^{m} \frac{1}{i} .
\end{aligned}
$$

Then indeed

$$
\lim _{m \rightarrow \infty} f_{m}^{(-1)}\left(\frac{1}{2}\right)=\frac{1}{2}-\frac{1}{2} \int_{\frac{m}{2}}^{m} \frac{1}{x} d x=\frac{1}{2}-\frac{1}{2}[\ln x]_{\frac{m}{2}}^{m}=\frac{1}{2}-\frac{1}{2} \ln 2=f^{(-1)}\left(\frac{1}{2}\right)
$$

Example 8 Following Example 7, it might help to picture the approximation copula on a simple setting. Due to Example 1 we already know that the trivial case $m=1$ leads to the weakest copula $W$. With $m=2$ we get
$F_{2}(x)=\left\{\begin{array}{ll}0 & x<\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \leq x<1 \\ 1 & 1 \leq x\end{array}\right.$, thus $f_{2}^{(-1)}(x)=\left\{\begin{array}{lll}1-\frac{3}{2} x & x<\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} x & \frac{1}{2} \leq x<1 \\ 0 & 1 \leq x\end{array}\right.$ and $f_{2}(x)= \begin{cases}1-2 x & 0 \leq x \leq \frac{1}{4} \\ \frac{2}{3}(1-x) & \frac{1}{4}<x \leq 1\end{cases}$


Figure 5: a) Distance function, generator (inverse) and b) copula, that correspond to uniform distribution approximated in $m=2$ equally spaced points. c) Probability mass concentrated on copula support.
shown on Figure 5 a), which leads to copula $C_{2}$ expressed on Figure 5 b).
To compute measures of dependence (concordance) such as Spearman's rho and Kendall's tau corresponding to singular copula it is generally a challenge, yet for this simple settings it might be an interesting exercise. Since the copula $C_{2}$ is piecewise linear, the whole probability mass is concentrated on its support, thus to evaluate the expected values (especially in (4)) one need to find out distribution of the probability. In our case, it is depicted on Figure 5 c). By expressing variable $v$ in terms of $u$ the double integral reduces to one-dimensional integral, then

$$
E[U V]=2 \int_{0}^{1 / 4} u(1-3 u) \frac{\frac{1}{4}}{\frac{1}{4}} d u+\int_{1 / 4}^{1} u\left(\frac{5}{4}-u\right) \frac{\frac{1}{2}}{\frac{3}{4}} d u=\frac{2}{64}+\frac{11}{64}=-\frac{13}{64}
$$

and

$$
\begin{aligned}
E[C(U, V)]=2 \int_{0}^{1 / 4} \max (0, u & \left.+\frac{1-3 u-1}{3}\right) \frac{\frac{1}{4}}{\frac{1}{4}} d u+ \\
& +\int_{1 / 4}^{1} \max \left(\frac{1}{3}\left(u+\frac{5}{4}-u-\frac{1}{2}\right), u+\frac{5}{4}-u-1\right) \frac{\frac{1}{2}}{\frac{3}{4}} d u=0+\frac{1}{8}
\end{aligned}
$$

thus $\rho_{2}=12 \frac{13}{64}-3=-\frac{9}{16}$ and $\tau_{2}=4 \frac{1}{8}-1=-\frac{1}{2}$, where the subscript 2 conforms the notation of generator. Although we cannot find explicit form of the original generator $f$ (that corresponds to uniform distribution $U[0,1])$ and analytically calculate $\rho$, we still can get $\tau=1-\int_{0}^{1} t\left((1-t+x \ln t)^{\prime}\right)^{2} d t=$ $1-4 \int_{0}^{1} t \ln ^{2} t d t=0$ to measure accuracy of our $m=2$ approximation.

## 4 Conclusion

We have discussed a new construction method for obtaining additive generators proposed by McNeil and Nešlehová [7], the so-called Williamson n-transform, and illustrated it by some examples. Some of the generators were shown to not have an explicit form due to non-invertability. Thus a natural approach to utilize any such parametric family is to approximate it by piecewise linear functions with sufficiently dense breakpoints. We showed some simple examples, including calculation of correlation coefficients related to a singular copula.

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# Transitivity of interval-valued fuzzy relations 

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#### Abstract

In this contribution a new relation for the set of interval-valued fuzzy relations is introduced. This relation is more suitable for the epistemic setting of these relations. This is an interval order for the family of intervals and consequences of considering such order are studied in the context of operations on interval-valued fuzzy relations. Moreover, the new transitivity property, namely pos- $T$-transitivity is studied. This transitivity property is connected with the new relation proposed here.


Keywords: partial order, interval order, T-transitivity, pos-T-transitivity, interval-valued fuzzy relation

## 1 Introduction

Interval-valued fuzzy relations were introduced by L. A. Zadeh [17] as a generalization of the concept of a fuzzy relation [16]. Interval valued fuzzy sets and relations have applications in diverse types of areas, for example in classification, image processing and multicriteria decision making.

In [13], a comparative study of the existing definitions of order relations between intervals, analyzing the level of acceptability and shortcomings from different points of view were presented. Orders used for interval-valued fuzzy relations may be connected with ontic and epistemic setting ([5, 6]). Epistemic uncertainty represents the idea of partial or incomplete information. Simply, it may be described by means of a set of possible values of some quantity of interest, one of which is the right one. A fuzzy set represents in such approach incomplete information, so it may be called disjunctive [5]. On the other hand, fuzzy sets may be conjunctive and can be called ontic fuzzy sets [5]. In this situation the fuzzy set is used as representing some precise gradual entity consisting of a collection of items.

The aim of this work is to examine dependencies between the natural (partial) order and the here introduced relation in the set of interval-valued fuzzy relations. Moreover, consequences of considering such relation are studied in the context of operations on interval-valued fuzzy relations, among others the new transitivity property called pos- $T$-transitivity is discussed.

The paper is structured as follows. Firstly, some concepts and results useful in further considerations are recalled (Section 2). Next, results connected with the interval order are presented (Section 3). Moreover, some properties of operations on interval-valued fuzzy relations are studied (Section 4). To finish, in Section 5 pos- $T$-transitivity, based on the definition of the given new relation, is presented and its preservation by basic operations is considered.

## 2 Interval-valued fuzzy relations

First, we recall definition of the lattice operations and the order for interval-valued fuzzy relations. Let $X, Y, Z$ be non-empty sets.
Definition 2.1 (cf. [15, 17]). An interval-valued fuzzy relation $R$ between universes $X, Y$ is a mapping $R: X \times Y \rightarrow L^{I}$ such that

$$
R(x, y)=[\underline{R}(x, y), \bar{R}(x, y)] \in L^{I}
$$

for all couples $(x, y) \in(X \times Y)$, where $L^{I}=\left\{\left[x_{1}, x_{2}\right]: x_{1}, x_{2} \in[0,1], x_{1} \leq x_{2}\right\}$.

[^1]The class of all interval-valued fuzzy relations between universes $X, Y$ will be denoted by $\mathcal{I V} \mathcal{F} \mathcal{R}(X \times$ $Y)$ or $\mathcal{I} \mathcal{V} \mathcal{F} \mathcal{R}(X)$ for $X=Y$.
We use the following partial order for intervals:

$$
\begin{equation*}
\left[x_{1}, y_{1}\right] \leq\left[x_{2}, y_{2}\right] \Leftrightarrow x_{1} \leq x_{2}, y_{1} \leq y_{2} \tag{1}
\end{equation*}
$$

For every $(x, y) \in(X \times Y), P=[\underline{P}, \bar{P}], R=[\underline{R}, \bar{R}] \in I V F R(X)$ we have

$$
P(x, y) \leq R(x, y) \Leftrightarrow \underline{P}(x, y) \leq \underline{R}(x, y), \bar{P}(x, y) \leq \bar{R}(x, y)
$$

The boundary elements in $\mathcal{I} \mathcal{V} \mathcal{F} \mathcal{R}(X \times Y)$ are $\mathbf{1}=[1,1]$ and $\mathbf{0}=[0,0]$.
Let $P, R \in \mathcal{I} \mathcal{V} \mathcal{F} \mathcal{R}(X \times Y)$, then

$$
\begin{aligned}
(P \vee R)(x, y) & =[\max (\underline{P}(x, y), \underline{R}(x, y)), \max (\bar{P}(x, y), \bar{R}(x, y))] \\
(P \wedge R)(x, y) & =[\min (\underline{P}(x, y), \underline{R}(x, y)), \min (\bar{P}(x, y), \bar{R}(x, y))]
\end{aligned}
$$

The structure $(\mathcal{I} \mathcal{V} \mathcal{F} \mathcal{R}(X \times Y), \leq)$ is a partially ordered set, i.e. the relation $\leq$ is: reflexive, $R(x, y) \leq R(x, y)$, antisymmetric, $R(x, y) \leq P(x, y)$ and $P(x, y) \leq R(x, y) \Rightarrow R(x, y)=P(x, y)$, transitive, $R(x, y) \leq P(x, y)$ and $P(x, y) \leq Q(x, y) \Rightarrow R(x, y) \leq Q(x, y)$ for every $(x, y) \in(X \times Y)$ and $P, Q, R \in \mathcal{I} \mathcal{F} \mathcal{R}(X \times Y)$.

For an arbitrary index set $D \neq \emptyset$ it holds that

$$
\begin{aligned}
& \left(\bigvee_{d \in D} R_{d}\right)(x, y)=\left[\sup _{d \in D} \underline{R_{d}}(x, y), \sup _{d \in D} \overline{R_{d}}(x, y)\right] \\
& \left(\bigwedge_{d \in D} R_{d}\right)(x, y)=\left[\inf _{d \in D} \underline{R_{d}}(x, y), \inf _{d \in D} \overline{R_{d}}(x, y)\right]
\end{aligned}
$$

More general classes of operations are triangular norms.
Definition 2.2 ([2]). A triangular norm $\mathbf{T}$ on a bounded poset $\boldsymbol{P}=(\boldsymbol{P}, \leq, 0,1)$ is an increasing, commutative, associative operation $\mathbf{T}: \boldsymbol{P}^{\mathbf{2}} \rightarrow \boldsymbol{P}$ with a neutral element 1.

One construction method for triangular norms is presented below.
An operation $\mathcal{T}:\left(L^{I}\right)^{2} \rightarrow L^{I}$ is called a representable triangular norm if there exist triangular norms $T_{1}, T_{2}:[0,1]^{2} \rightarrow[0,1]$ such that for all $x=\left[x_{1}, x_{2}\right], y=\left[y_{1}, y_{2}\right] \in L^{I}$ and $T_{1} \leq T_{2}:$

$$
\mathcal{T}(x, y)=\left[T_{1}\left(x_{1}, y_{1}\right), T_{2}\left(x_{2}, y_{2}\right)\right]
$$

Many authors, for example in [12, 14], used the following definition of transitivity.
Definition 2.3. Let $\mathcal{T}=\left[T_{1}, T_{2}\right]$ and $T_{1} \leq T_{2}, R \in \mathcal{I V} \mathcal{F} \mathcal{R}(X)$. Relation $R$ is called $\mathcal{T}$-transitive if

$$
T_{1}(\underline{R}(x, y), \underline{R}(y, z)) \leq \underline{R}(x, z)
$$

and

$$
T_{2}(\bar{R}(x, y), \bar{R}(y, z)) \leq \bar{R}(x, z)
$$

In the next part of the paper we will introduce another type of transitivity.

## 3 Interval order in $L^{I}$

To begin with, we recall the definition of an interval order for crisp relations. The name 'interval order' first appeared in print in Fishburn [7, 8, 9].

Definition 3.1 ([10], p. 42). A relation $R \subset X \times X$ is an interval order if it is complete and has the Ferrers property, i.e.:
$R(x, y)$ or $R(y, x)$, for $x, y \in X$,
$R(x, y)$ and $R(z, w) \Rightarrow R(x, w)$ or $R(z, y)$, for $x, y, z, w \in X$,
respectively.
Now we consider the following relation between intervals:

$$
\begin{equation*}
\left[x_{1}, y_{1}\right] \preceq\left[x_{2}, y_{2}\right] \Leftrightarrow x_{1} \leq y_{2} \tag{2}
\end{equation*}
$$

This relation is more adequate in the epistemic setting of the interval-valued fuzzy relations. If $\left[x_{1}, y_{1}\right]$ is an unprecise description of a variable $x$ and $\left[x_{2}, y_{2}\right]$ is an unprecise description of a variable $y$, then $\left[x_{1}, y_{1}\right] \preceq\left[x_{2}, y_{2}\right]$ denotes that it is possible that the true value of $x$ is smaller than or equal to the true value of $y$. The relation $\preceq$ thus has a possibilistic interpretation [4].

Theorem 3.2. In the structure $\left(L^{I}, \preceq\right)$, the relation $\preceq$ is an interval order.
Proof. Let $\left[a_{1}, b_{1}\right] \preceq\left[a_{2}, b_{2}\right]$ and $\left[a_{3}, b_{3}\right] \preceq\left[a_{4}, b_{4}\right]$ for $\left[a_{i}, b_{i}\right] \in L^{I}, i \in\{1,2,3,4\}$, so $a_{1} \leq b_{2}$, $a_{3} \leq b_{4}$.
If $a_{1}>b_{4}$, then $a_{3} \leq a_{1}$, i.e. $a_{3} \leq b_{2}$.
If $a_{3}>b_{2}$, then $a_{1} \leq a_{3}$, i.e. $a_{1} \leq b_{4}$.
So $\preceq$ has the Ferrers property, i.e.

$$
\left[a_{1}, b_{1}\right] \preceq\left[a_{2}, b_{2}\right] \text { and }\left[a_{3}, b_{3}\right] \preceq\left[a_{4}, b_{4}\right] \Rightarrow\left[a_{1}, b_{1}\right] \preceq\left[a_{4}, b_{4}\right] \text { or }\left[a_{3}, b_{3}\right] \preceq\left[a_{2}, b_{2}\right]
$$

If $a_{1} \leq b_{2}$, then $\left[a_{1}, b_{1}\right] \preceq\left[a_{2}, b_{2}\right]$.
If $a_{1} \geq b_{2}$, then $\left[a_{2}, b_{2}\right] \preceq\left[a_{1}, b_{1}\right]$.
So $\preceq$ is complete, i.e.

$$
\left[a_{1}, b_{1}\right] \preceq\left[a_{2}, b_{2}\right] \text { or }\left[a_{2}, b_{2}\right] \preceq\left[a_{1}, b_{1}\right] .
$$

Directly from (1) and (2), we note the following connection between the natural (partial) order $\leq$ and the interval order $\preceq$.

Corollary 3.3. If the natural order (1) holds, then also the interval order holds (2).
The converse implication does not hold, as can be seen from the following example.
Example 3.4. For intervals $A=[0.2,0.8]$ and $B=[0.1,1]$ we observe that $A \preceq B$ but it is not true that $A \leq B$.

We would like to use the new relation on the class $\mathcal{I V} \mathcal{F} \mathcal{R}(X \times Y)$ and examine the consequences of this choice. Thus, for every $(x, y) \in(X \times Y), P=[\underline{P}, \bar{P}], R=[\underline{R}, \bar{R}] \in \mathcal{I} \mathcal{V} \mathcal{F} \mathcal{R}(X \times Y)$ we have

$$
P(x, y) \preceq R(x, y) \Leftrightarrow \underline{P}(x, y) \leq \bar{R}(x, y) .
$$

Let us notice that the relation $\preceq$ in the family $\mathcal{I V} \mathcal{F} \mathcal{R}(X)$ has the reflexivity property only. Thus it is not an order relation in this family.

## 4 Dependencies between operations on $\mathcal{I V} \mathcal{F} \mathcal{R}(X \times Y)$

Firstly, we consider connections between basic operations on $\mathcal{I V} \mathcal{F} \mathcal{R}(X \times Y)$ and the considered relation $\preceq$. For $(x, y) \in(X \times Y), P=[\underline{P}, \bar{P}], R=[\underline{R}, \bar{R}] \in \mathcal{I} \mathcal{V} \mathcal{F} \mathcal{R}(X \times Y)$ we have

$$
\begin{aligned}
& P \wedge R \preceq P \preceq P \vee R, \\
& P \wedge R \preceq R \preceq P \vee R .
\end{aligned}
$$

Moreover, if $\bar{P} \geq \bar{R}$, then $R \preceq P \wedge R$ and if $\bar{P} \leq \bar{R}$, then $P \preceq P \wedge R$.
Interesting differences between the considered relation $\preceq$ and the natural (partial) order present the following conditions

$$
P \preceq R \Leftarrow(P \wedge R=P, \quad P \vee R=R)
$$

and

$$
\text { if } \bar{P} \leq \underline{R}, \text { then } P \preceq R \Rightarrow(P \wedge R=P, \quad P \vee R=R)
$$

Moreover, we have the implication

$$
(R \preceq P, P \preceq R) \Leftarrow(\underline{R}=\underline{P}, \bar{R}=\bar{P}),
$$

but the converse implication we obtain if the relation $\preceq$ is replaced with the natural order $\leq$. If we consider the converse operation $R^{t}(x, y)=R(y, x)$, then it holds

$$
P \preceq R \Leftrightarrow P^{t} \preceq R^{t} .
$$

Another interesting properties for here considered relation $\preceq$, we present in the following result.
Theorem 4.1. Let $(x, y) \in(X \times Y), P=[\underline{P}, \bar{P}], Q=[\underline{Q}, \bar{Q}], R=[\underline{R}, \bar{R}] \in \mathcal{I} \mathcal{V} \mathcal{F} \mathcal{R}(X \times Y)$. Then we have

$$
\begin{gathered}
\bullet P(x, y) \preceq R(x, y), P(x, y) \preceq Q(x, y) \Leftrightarrow P(x, y) \preceq R(x, y) \wedge Q(x, y), \\
\bullet R(x, y) \preceq P(x, y), Q(x, y) \preceq P(x, y) \Leftrightarrow R(x, y) \vee Q(x, y) \preceq P(x, y), \\
\bullet P(x, y) \preceq R(x, y) \text { and } W(x, y) \preceq Q(x, y) \Rightarrow \\
P(x, y) \vee W(x, y) \preceq R(x, y) \vee Q(x, y) \text { and } P(x, y) \wedge W(x, y) \preceq R(x, y) \wedge Q(x, y) .
\end{gathered}
$$

Proof. Let $P(x, y) \preceq R(x, y)$ and $P(x, y) \preceq Q(x, y)$, so $\underline{P} \leq \bar{R}$ and $\underline{P} \leq \bar{Q}$, then $\underline{P} \leq \bar{R} \wedge \bar{Q}$ because $\wedge$ is the infimum in the lattice $([0,1], \wedge, \vee)$. Similarly, we obtain the second condition by the property of supremum $\vee$. Moreover, by isotonicity of these operations we obtain the third condition.

Lets us now recall the notion of the composition for interval-valued fuzzy relations.
Definition 4.2 (cf. [1, 11]). Let $P \in \mathcal{I V F \mathcal { R }}(X \times Y), R \in \mathcal{I V} \mathcal{F} \mathcal{R}(Y \times Z)$. The sup - $\mathbf{T}$ composition of the relations $P$ and $R$ is called the relation $P \circ R \in \mathcal{I V F \mathcal { R }}(X \times Z)$,

$$
(P \circ R)(x, z)=\bigvee_{y \in Y} \mathbf{T}(P(x, y), R(y, z))
$$

Especially, if $\mathbf{T}$ is a representable triangular norm $\mathcal{T}$ we have $\sup -T_{1} T_{2}$ composition,

$$
(P \circ R)(x, z)=\left[\left(\underline{P} \circ_{T_{1}} \underline{R}\right)(x, z),\left(\bar{P} \circ_{T_{2}} \bar{R}\right)(x, z)\right],
$$

where $T_{1} \leq T_{2}$ and

$$
\left(\underline{P} \circ_{T_{1}} \underline{R}\right)(x, z)=\sup _{y \in Y} T_{1}(\underline{P}(x, y), \underline{R}(y, z)),\left(\bar{P} \circ_{T_{2}} \bar{R}\right)(x, z)=\sup _{y \in Y} T_{2}(\bar{P}(x, y), \bar{R}(y, z))
$$

In our further considerations in the whole paper we will use the composition with a representable triangular norm and the symbol $\circ$ will mean $\sup -T_{1} T_{2}$ composition. For simplicity of notations we present the results for composition in the class $\mathcal{I V} \mathcal{F} \mathcal{R}(X)$.

Theorem 4.3. If $T_{1}, T_{2}, T_{1} \leq T_{2}$ are triangular norms, then

$$
\begin{gathered}
P \preceq R \Rightarrow P \circ Q \preceq R \circ Q, Q \circ P \preceq Q \circ R, \\
P \circ(Q \vee R)=P \circ Q \vee P \circ R .
\end{gathered}
$$

Moreover, if $T_{1}, T_{2}, T_{1} \leq T_{2}$ are supremum preserving then

$$
P \circ(Q \circ R)=(P \circ Q) \circ R .
$$

Proof. Let $P \preceq R$, i.e. $\underline{P} \leq \bar{R}$ and $T_{1} \leq T_{2}$, then by Theorem 4.1 we have for $x, y \in X$ $\bigvee_{z \in X} T_{1}(\underline{P}(x, z), \underline{Q}(z, y)) \leq \bigvee_{z \in X} T_{2}(\bar{P}(x, z), \bar{Q}(z, y))$, so $P \circ Q \preceq R \circ Q$. The second inequality in the first condition can be proven similarly. By distributivity of a triangular norm with respect to maximum we obtain the second condition. Moreover, since triangular norms $T_{1}, T_{2}$ are supremum preserving, we have associativity.

In a semigroup $(\mathcal{I} \mathcal{V} \mathcal{F} \mathcal{R}(X), \circ)$ we can consider the powers of its elements, i.e. relations $R^{n}$ for $R \in \mathcal{I V F R}(X), n \in \mathbb{N}$.

Definition 4.4. Let $R \in \mathcal{I} \mathcal{F} \mathcal{F}(X)$. The powers of $R$ are defined in the following way

$$
R^{1}=R, \quad R^{n+1}=R^{n} \circ R, \quad n \in \mathbb{N} .
$$

The upper operation $R^{\vee}$ and the lower operation $R^{\wedge}$ of $R$ are defined in the following way

$$
R^{\vee}=\bigvee_{k=1}^{\infty} R^{k}, \quad R^{\wedge}=\bigwedge_{k=1}^{\infty} R^{k}
$$

where $R^{k}=\left[\underline{R}^{k}, \bar{R}^{k}\right]$. Now we will examine connections between powers and upper (lower) operations and operations $\vee$ and $\wedge$.

Theorem 4.5. Let $T_{1}, T_{2}, T_{1} \leq T_{2}$ be supremum preserving and $P, R \in \mathcal{I V \mathcal { F }}(X)$.

$$
\text { If } R \preceq P, \text { then } R^{n} \preceq P^{n}, R^{\vee} \preceq P^{\vee}, \quad R^{\wedge} \preceq P^{\wedge}, \quad n \in \mathbb{N} .
$$

## Moreover,

$$
\begin{aligned}
& (P \vee R)^{\vee} \succeq P^{\vee} \vee R^{\vee} \\
& (P \wedge R)^{\vee} \preceq P^{\vee} \wedge R^{\vee} \\
& (P \vee R)^{\wedge} \succeq P^{\wedge} \vee R^{\wedge} \\
& (P \wedge R)^{\wedge} \preceq P^{\wedge} \wedge R^{\wedge}
\end{aligned}
$$

Proof. By isotonicity of composition we obtain isotonicity of powers, moreover by isotonicity of supremum and infimum we have dependencies for lower and upper operations. By Theorem 4.1 and the above conditions we obtain the rest of results.

## 5 Possible T-transitivity

Now we will consider the transitivity property connected with the introduced relation $\preceq$ in the epistemic setting. This definition of transitivity naturally follows from the introduced relation $\preceq$, namely replacing the natural order $\leq$ with the relation $\preceq$ we get by Definition 2.3 for a representable triangular norm $\mathcal{T}$ the formula $\mathcal{T}(R(x, y), R(y, z)) \preceq R(x, z)$. As a result, applying definition of the relation $\preceq$ we get the following notion.

Definition 5.1 ([3]). Let $T:[0,1]^{2} \rightarrow[0,1]$ be a triangular norm. A relation $R \in \mathcal{I V} \mathcal{F} \mathcal{R}(X)$ is possibly T-transitive (pos-T-transitive), if

$$
\begin{equation*}
T(\underline{R}(x, y), \underline{R}(y, z)) \leq \bar{R}(x, z) \tag{3}
\end{equation*}
$$

This transitivity property is called possible $T$-transitivity which follows from the interpretation of the relation $\preceq$. Again, if $R(x, y)$ is an imprecise description of the relation between $x$ and $y$, and similarly for $R(y, z)$ and $R(x, z)$, then formula (3) expresses that it is possible to choose values in these intervals such that usual $T$-transitivity holds.

Theorem 5.2. Let $D \neq \emptyset$ and $R_{d} \in \mathcal{I V} \mathcal{F} \mathcal{R}(X), d \in D$. If $\left(R_{d}\right)$ is a family of pos-T-transitive relations, then the fuzzy relation $\bigwedge_{d \in D} R_{d}$ is pos- $T$-transitive.
Proof. If $R_{d}$ are pos- $T$-transitive relations, i.e., $T\left(\underline{R_{d}}(x, y), \underline{R_{d}}(y, z)\right) \leq \overline{R_{d}}(x, z)$, then by isotonicity of triangular norms, we know that min dominates any triangular norm $T$, i.e. $T\left(\bigwedge_{d \in D} \underline{R_{d}}(x, y), \bigwedge_{d \in D} \underline{R_{d}}(y, z)\right) \leq$ $\bigwedge_{d \in D} T\left(\underline{R_{d}}(x, y), \underline{R_{d}}(y, z)\right) \leq \bigwedge_{d \in D} \overline{R_{d}}(x, z)$.

Theorem 5.3. Let $R \in \mathcal{I V} \mathcal{F} \mathcal{R}(X)$. If $R$ is pos- $T$-transitive, then $R^{t}$ is also pos- $T$-transitive.
Proof. For an arbitrary $R \in \mathcal{I V} \mathcal{F} \mathcal{R}(X)$ which is pos- $T$-transitive and by commutativity of a triangular norm we have
$T\left(\underline{R^{t}}(x, y), \underline{R^{t}}(y, z)\right)=T(\underline{R}(y, x), \underline{R}(z, y))=T(\underline{R}(z, y), \underline{R}(y, x)) \leq \bar{R}(z, x)=\overline{R^{t}}(x, z)$.
In the following theorems we use the fact (which follows from definition of pos- $T$-transitivity and definition of composition) that

Theorem 5.5. Let $P, R \in \operatorname{IV} \mathcal{F} \mathcal{R}(X)$. If $P, R$ are pos- $T$-transitive relations and $\underline{R} \circ_{T} \underline{P} \vee \underline{P} \circ_{T} \underline{R} \leq \bar{R} \vee \bar{P}$, then $R \vee S$ is pos- $T$-transitive.

Proof. Let $P, R$ be interval-valued fuzzy pos- $T$-transitive relations. By Lemma 5.4, and the assumption $\underline{R} \circ_{T} \underline{P} \vee \underline{P} \circ_{T} \underline{R} \leq \bar{R} \vee \bar{P}$ and by Theorem 4.3 we have

$$
(\underline{R \vee P})^{2}=(\underline{R \vee P}) \circ_{T}(\underline{R \vee P})=\underline{R}^{2} \vee \underline{R} \circ \underline{P} \vee \underline{P} \circ \underline{R} \vee \underline{P}^{2} \leq \bar{R} \vee \bar{R} \vee \bar{P} \vee \bar{P}=\bar{R} \vee \bar{P}
$$

so $R \vee S$ is pos- $T$-transitive.
Theorem 5.6. Let $T_{1}, T_{2}, T_{1} \leq T_{2}$ be supremum preserving $P, R \in \mathcal{I V} \mathcal{F} \mathcal{R}(X)$. If $P, R$ are pos- $T_{1}$ transitive and $\underline{R} \circ_{T_{1}} \underline{P}=\underline{P} \circ_{T_{1}} \underline{R}$, then $R \circ P$ is pos- $T_{1}$-transitive.

Proof. Let $P, R$ be interval-valued fuzzy pos- $T$-transitive relations. By associativity of composition and the assumption $\underline{R} \circ_{T_{1}} \underline{P}=\underline{P} \circ_{T_{1}} \underline{R}$, by Lemma 5.4 we have
$\left(\underline{R} \circ_{T_{1}} P\right)^{2}=\left(\underline{R} \circ_{T_{1}} \underline{P}\right)^{2}=\underline{R} \circ_{T_{1}}\left(\underline{P} \circ_{T_{1}} \underline{R}\right) \circ_{T_{1}} \underline{P}=\underline{R}^{2} \circ_{T_{1}} \underline{P}^{2} \leq \bar{R} \circ_{T_{1}} \bar{P} \leq \bar{R} \circ_{T_{2}} \bar{P}=\bar{P} \circ_{T_{2}} R$. Thus, $R \circ P$ is a pos- $T_{1}$-transitive relation.

Corollary 5.7. Let $T_{1}, T_{2}, T_{1} \leq T_{2}$ be supremum preserving and $R \in \mathcal{I V} \mathcal{F} \mathcal{R}(X)$. If $R$ is pos- $T_{1}$ transitive, then $R^{n}$ is also pos- $T_{1}$-transitive.
Proof. By isotonicity of composition and powers, we obtain $\underline{R}^{n} \leq \bar{R}^{n-1}$, so $R^{n}$ preserves pos- $T_{1}$ transitivity.

Theorem 5.8. Let $R \in \mathcal{I} \mathcal{V} \mathcal{F} \mathcal{R}(X)$. If $\underline{R}$ is $T$-transitive, then $R$ is pos- $T$-transitive.
Proof. Let $\underline{R}$ be $T$-transitive. Then $\underline{R}^{2} \leq \underline{R} \leq \bar{R}$. Thus by Lemma 5.4, we obtain pos- $T$-transitivity of $R$.

We also notice the connection between $\mathcal{T}$-transitivity and pos- $T$-transitivity.
Proposition 5.9. Let $R \in \mathcal{I V} \mathcal{F} \mathcal{R}(X)$. If $R$ is $\mathcal{T}$-transitive, then $R$ is pos- $T_{1}$-transitive.
Moreover, we know directly by definitions of $\mathcal{T}$-transitivity and composition, that
Proposition 5.10. Let $R \in \mathcal{I V} \mathcal{F} \mathcal{R}(X), T_{1}, T_{2}, T_{1} \leq T_{2}$ be triangular norms. $R$ is $\mathcal{T}$-transitive if and only if $\underline{R}$ is $T_{1}$-transitive and $\bar{R}$ is $T_{2}$-transitive.

Moreover, we have the following property.
Theorem 5.11. Let $T_{1}, T_{2}$ be triangular norms and $T_{1} \leq T_{2}$. If $R \in \mathcal{I V} \mathcal{F} \mathcal{R}(X)$ is pos- $T_{2}$-transitive, then $R$ is pos- $T_{1}$-transitive.

Proof. Let $R$ be pos- $T_{2}$-transitive. Then $T_{2}(\underline{R}(x, y), \underline{R}(y, z)) \leq \bar{R}(x, z)$ and by the fact that $T_{1} \leq T_{2}$ we have $T_{1}(\underline{R}(x, y), \underline{R}(y, z)) \leq T_{2}(\underline{R}(x, y), \underline{R}(y, z)) \leq \bar{R}(x, z)$ for $x, y, z \in X$. As a result $R$ is pos- $T_{1}$-transitive.

## 6 Conclusion

In future work other operations and some properties for interval-valued fuzzy relations for the relation $\preceq$ may be considered. Next, generalization of the here considered composition, i.e. sup $-A$ composition (where $A$ is an aggregation function), may be discussed. Moreover, other types of transitivity and other relations between interval-valued fuzzy relations may be studied.

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# Fuzzy interval orders and aggregation process 

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#### Abstract

In this contribution, conditions for n-argument functions to preserve fuzzy interval orders during aggregation process are presented. The considered properties of a fuzzy interval order (Ferrers property and connectedness) depend on binary operations including t-norms and t-conorms and, more generally, fuzzy conjunctions and disjunctions. Moreover, some existing results on Ferrers property are generalized and applied for fuzzy interval orders.


Keywords: fuzzy relation, fuzzy connective, aggregation function, Ferrers property, fuzzy interval order

## 1 Introduction

Fuzzy order structures, such as linear orders, semi-orders and interval orders are often used to model preferences in decision making problems. In this contribution we pay attention to fuzzy interval orders which definitions are based on the notions of Ferrers property and total connectedness. Ferrers property is less demanding than transitivity property used in the most of orders, so it is worthy do examine Ferrers property and fuzzy interval orders from the application point of view.

We will consider fuzzy interval orders in the context of their preservation in aggregation process (cf. [ $6,8,11,12,15,17]$ ) which is due to the possible applications, e.g. in fuzzy preference modelling, multicriteria decision making problems and solving other issues related to imprecise and uncertain information. In decision making problems a set $X=\left\{x_{1}, \ldots, x_{m}\right\}$ represents a set of objects, where $m \in \mathbb{N}$. There is also considered a set $K=\left\{k_{1}, \ldots, k_{n}\right\}$ of criteria under which the objects are supposed to be evaluated. Fuzzy relations $R_{1}, \ldots, R_{n}$ reflect judgements of decision makers. The considered aggregation process involves also an $n$-argument function $F$. With the use of given fuzzy relations $R_{1}, \ldots, R_{n}$ and the function $F$, we consider a new fuzzy relation $R_{F}=F\left(R_{1}, \ldots, R_{n}\right)$ representing a final decision on evaluated objects (after considering the involved criteria). Although we focus on aggregation functions, the aim of this paper is to give the results under the weakest assumptions on $F$ used for the aggregation process. Therefore, we start our considerations with an arbitrary $n$-ary function.

The notions of fuzzy relation properties, in their simplest forms, may involve functions min and max. These ones were generalized by the use of a t -norm and t -conorm, respectively [11, Chapter 2.5]. In particular, the following properties were examined: $T$-asymmetry, $T$-antisymmetry, $S$-connectedness, $T$-transitivity, negative $S$-transitivity, $T$ - $S$-semitransitivity, and $T$ - $S$-Ferrers property of fuzzy relations, where $T$ is a t-norm and $S$ a t-conorm, also with regard to their preservation in aggregation process [9]. However, the assumptions put on widely used t-norms are not always necessary or desired. This is why a lot of definitions of binary operations which can play a role of weaker fuzzy connectives were introduced and studied, for example fuzzy conjunctions: weak t-norms, overlap functions, $t$-seminorms (or semicopulas, or conjunctors), and pseudo-t-norms, sometimes along with their dual disjunctions.

In this article, we consider the properties of fuzzy relations which definitions are based on fuzzy conjunctions and disjunctions including t -norms and t -conorms. In order to obtain the most general results we start with binary operations in the unit interval without any additional assumptions. As a result we examine fuzzy interval orders which are totally $B$-connected and fulfil $B_{1}-B_{2}$-Ferrers property, where $B, B_{1}, B_{2}:[0,1]^{2} \rightarrow[0,1]$ are binary operations.

[^2]In Section 2, we provide basic definitions and results concerning $n$-ary functions in $[0,1]$ including fuzzy connectives and dominance between functions. Next, in Section 3, we present basic information about fuzzy relations and some useful results related to preservation of fuzzy relation properties in aggregation process. Finally, in Section 4 we put the main results of this contribution connected with fuzzy interval orders in aggregation process.

## 2 Preliminaries

In this section we present the notions useful in further considerations, i.e. properties of $n$-ary functions in $[0,1]$, fuzzy connectives, dominance between operations.

Definition 2.1 ([5]). Let $n \in \mathbb{N}$. A function $A:[0,1]^{n} \rightarrow[0,1]$ which is increasing, i.e.

$$
A\left(x_{1}, \ldots, x_{n}\right) \leqslant A\left(y_{1}, \ldots, y_{n}\right) \text { for } x_{i}, y_{i} \in[0,1], x_{i} \leqslant y_{i}, i=1, \ldots, n
$$

is called an aggregation function if $A(0, \ldots, 0)=0$ and $A(1, \ldots, 1)=1$.
Example 2.2. Aggregation functions are:

- median

$$
\operatorname{med}\left(t_{1}, \ldots, t_{n}\right)= \begin{cases}\frac{s_{k}+s_{k+1}}{2}, & \text { for } n=2 k \\ s_{k+1}, & \text { for } n=2 k+1\end{cases}
$$

where $\left(s_{1}, \ldots, s_{n}\right)$ is the increasingly ordered sequence of the values $t_{1}, \ldots, t_{n}$, which means that $s_{1} \leqslant \ldots \leqslant s_{n}$.

- a weighted arithmetic mean

$$
A_{w}\left(x_{1}, \ldots, x_{n}\right)=\sum_{k=1}^{n} w_{k} x_{k}, \quad \text { for } \quad w_{k}>0, \sum_{k=1}^{n} w_{k}=1
$$

- a quasi-linear mean

$$
F\left(x_{1}, \ldots, x_{n}\right)=\varphi^{-1}\left(\sum_{k=1}^{n} w_{k} \varphi\left(x_{k}\right)\right), \quad \text { for } \quad w_{k}>0, \sum_{k=1}^{n} w_{k}=1
$$

where $x_{1}, \ldots, x_{n} \in[0,1], \varphi:[0,1] \rightarrow \mathbb{R}$ is a continuous, strictly increasing function.
Definition 2.3. Let $n \in \mathbb{N}$. We say that a function $F:[0,1]^{n} \rightarrow[0,1]:$

- has a zero element $z \in[0,1]$ if for each $k \in\{1, \ldots, n\}$ and each $x_{1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n} \in[0,1]$ one has

$$
F\left(x_{1}, \ldots, x_{k-1}, z, x_{k+1}, \ldots, x_{n}\right)=z
$$

- is without zero divisors if it has a zero element $z$ and

$$
\underset{x_{1}, \ldots, x_{n} \in[0,1]}{\forall}\left(F\left(x_{1}, \ldots, x_{n}\right)=z \Rightarrow\left(\underset{1 \leqslant k \leqslant n}{\exists} x_{k}=z\right)\right) .
$$

Definition 2.4 ([10]). An operation $C:[0,1]^{2} \rightarrow[0,1]$ is called a fuzzy conjunction if it is increasing with respect to each variable and $C(1,1)=1, C(0,0)=C(0,1)=C(1,0)=0$. An operation $D:[0,1]^{2} \rightarrow[0,1]$ is called a fuzzy disjunction if it increasing with respect to each variable and $D(0,0)=0, D(1,1)=D(0,1)=D(1,0)=1$.

Corollary 2.5. A fuzzy conjunction has a zero element 0 . A fuzzy disjunction has a zero element 1.
Definition 2.6. An operation $C:[0,1]^{2} \rightarrow[0,1]$ is called:

- an overlap function [3] if it is a commutative, continuous fuzzy conjunction without zero divisors, fulfilling condition $C(x, y)=1$ if and only if $x y=1$,
- a t-norm [18] if it is a commutative, associative, increasing operation with neutral element 1.

Definition 2.7. An operation $D:[0,1]^{2} \rightarrow[0,1]$ is called:

- a grouping function [4] if it is a commutative, continuous fuzzy disjunction without zero divisors, fulfilling condition $D(x, y)=0$ if and only if $x=y=0$,
- a t-conorm [14] if it is a commutative, associative, increasing operation with neutral element 0 ,
- a strict t-conorm $S:[0,1]^{2} \rightarrow[0,1]$ if it is a t-conorm which is continuous and strictly increasing in $[0,1)^{2}$.

Example 2.8 ([14]). The Łukasiewicz t-norm and $t$-conorm are described in the following way $T_{L}(s, t)=$ $\max (s+t-1,0)$ and $S_{L}(s, t)=\min (s+t, 1)$, respectively.

Definition 2.9. A t-norm $T$ is called nilpotent if it is continuous and each $x \in(0,1)$ is a nilpotent element of $T$, i.e. for each $x \in(0,1)$ there exists $n \in \mathbb{N}$ such that $x_{T}^{(n)}=0$.

Theorem 2.10. Any nilpotent t-norm is isomorphic to the Łukasiewicz t-norm $T_{L}$, i.e.

$$
T(x, y)=\varphi^{-1}\left(T_{L}(\varphi(x), \varphi(y))\right), \quad x, y \in[0,1]
$$

where $\varphi:[0,1] \rightarrow[0,1]$ is an increasing bijection.
Definition 2.11 ([13]). A rotation invariant $t$-norm is a $t$-norm $T$ that verifies for all $x, y, z \in[0,1]$

$$
T(x, y) \leqslant z \Leftrightarrow T(x, 1-z) \leqslant 1-y
$$

Definition 2.12 ([5]). Let $F:[0,1]^{n} \rightarrow[0,1]$. A function $F^{d}$ is called a dual function to $F$, if for all $x_{1}, \ldots, x_{n} \in[0,1]$

$$
F^{d}\left(x_{1}, \ldots, x_{n}\right)=1-F\left(1-x_{1}, \ldots, 1-x_{n}\right)
$$

$F$ is called a self-dual function, if it holds $F=F^{d}$.
Fuzzy disjunctions are dual to fuzzy conjunctions, grouping functions are dual to overlap functions, t -conorms are dual functions to t-norms, in particular $S_{L}$ is dual to $T_{L}$, max is dual to min. Now, we recall the notion of dominance.

Definition 2.13 ([19]). Let $m, n \in \mathbb{N}$. A function $F:[0,1]^{m} \rightarrow[0,1]$ dominates function $G:[0,1]^{n} \rightarrow$ $[0,1](F \gg G)$ if for an arbitrary matrix $\left[a_{i k}\right]=A \in[0,1]^{m \times n}$ the following inequality holds

$$
F\left(G\left(a_{11}, \ldots, a_{1 n}\right), \ldots, G\left(a_{m 1}, \ldots, a_{m n}\right)\right) \geqslant G\left(F\left(a_{11}, \ldots, a_{m 1}\right), \ldots, F\left(a_{1 n}, \ldots, a_{m n}\right)\right)
$$

Example 2.14 ([1]). Any weighted arithmetic mean dominates $t$-norm $T_{L}$ and any weighted arithmetic mean is dominated by $S_{L}$. Minimum dominates any fuzzy conjunction. Fuzzy disjunctions dominate maximum.

## 3 Fuzzy relations

Here we recall the notion of a fuzzy relation, some properties of fuzzy relations and their preservation in aggregation process.

Definition 3.1 ([20]). A fuzzy relation in a set $X \neq \emptyset$ is an arbitrary function $R: X \times X \rightarrow[0,1]$. The family of all fuzzy relations in $X$ is denoted by $F R(X)$.

Definition 3.2 (cf. [11, 16]). Let $B, B_{1}, B_{2}:[0,1]^{2} \rightarrow[0,1]$ be binary operations. Relation $R \in$ $F R(X)$ is:

- reflexive, if $\underset{x \in X}{\forall} R(x, x)=1$,
- totally $B$-connected, if
$\underset{x, y \in X}{\forall} B(R(x, y), R(y, x))=1$,
- $B_{1}-B_{2}$-Ferrers, if
$\underset{x, y, z, w \in X}{\forall} B_{1}(R(x, y), R(z, w)) \leqslant B_{2}(R(x, w), R(z, y))$,
- a $B$ - $B_{1}$ - $B_{2}$-fuzzy interval order, if it is totally $B$-connected and $B_{1}$ - $B_{2}$-Ferrers,
- a $B_{1}-B_{2}$-fuzzy interval order, if it is totally $B_{2}$-connected and $B_{1}-B_{2}$-Ferrers.

We present the notions of the given properties in the most general version, i.e. with operations $B, B_{1}, B_{2}:[0,1]^{2} \rightarrow[0,1]$. However, the natural approach is to consider a fuzzy disjunction $B$ in definition of $B$-connectedness, a fuzzy conjunction $B_{1}$ and a fuzzy disjunction $B_{2}$ in the Ferrers property.

Let $F:[0,1]^{n} \rightarrow[0,1], R_{1}, \ldots, R_{n} \in F R(X)$. An aggregated fuzzy relation $R_{F} \in F R(X)$ is described by the formula $R_{F}(x, y)=F\left(R_{1}(x, y), \ldots, R_{n}(x, y)\right), x, y \in X$. A function $F$ preserves a property of fuzzy relations if for every $R_{1}, \ldots, R_{n} \in F R(X)$ having this property, $R_{F}$ also has this property. Preservation of the properties listed above and also other properties of this kind was considered in [1]. We will recall here only the results for the properties that will be useful in the sequel.
Theorem 3.3. Let $R_{1}, \ldots, R_{n} \in F R(X)$ be reflexive. The relation $R_{F}$ is reflexive, if and only if the function $F$ satisfies the condition $F(1, \ldots, 1)=1$.

Theorem 3.4. Let card $X \geqslant 2, B$ have a zero element 1 and be without zero divisors. A function $F$ preserves total $B$-connectedness ( $B$-connectedness) if and only if it satisfies the following condition for all $s, t \in[0,1]^{n}$

$$
\begin{equation*}
\underset{1 \leqslant k \leqslant n}{\forall} \max \left(s_{k}, t_{k}\right)=1 \Rightarrow \max (F(s), F(t))=1 . \tag{1}
\end{equation*}
$$

Example 3.5. Let $B$ be a fuzzy disjunction without zero divisors (e.g. a strict $t$-conorm or a grouping function). Examples of functions fulfilling (1) for all $s, t \in[0,1]^{n}$ are $F=\max , F=m e d$ or functions $F$ with the zero element $z=1$ with respect to a certain coordinate, i.e.

$$
\underset{1 \leqslant k \leqslant n}{\exists} \underset{i \neq k}{\forall} \underset{t_{i} \in[0,1]}{\forall} F\left(t_{1}, \ldots, t_{k-1}, 1, t_{k+1}, \ldots, t_{n}\right)=1 .
$$

Theorem 3.6. If a function $F:[0,1]^{n} \rightarrow[0,1]$, which is increasing in each of its arguments fulfils $F \gg B_{1}$ and $B_{2} \gg F$, then it preserves $B_{1}-B_{2}$-Ferrers property.
Lemma 3.7. Let $B:[0,1]^{2} \rightarrow[0,1]$ and $B^{d}$ be a corresponding dual operation. If $F:[0,1]^{n} \rightarrow[0,1]$ is a self-dual function, then $F \gg B$ implies $B^{d} \gg F$.

The condition given in Lemma 3.7 is only the sufficient one. Let us consider projections $F=P_{k}$, $B=T$ being a t-norm, $S=T^{d}$. Then $S \gg P_{k}$ and $P_{k} \gg T$, but $F \neq F^{d}$.

Example 3.8. Any weighted arithmetic mean preserves $B_{1}$ - $B_{2}$-Ferrers property for $t$-norm $T_{L}=B_{1}$ and t-conorm $S_{L}=B_{2}$.

Corollary 3.9. Any quasi-linear mean preserves $T$-S-Ferrers property for a nilpotent t-norm $T$ and $S=T^{d}$.

Conditions given in Theorem 3.6 are only the sufficient ones. Let us consider function $F(s, t)=s t$ (so $F=T_{P}$ ) and fuzzy relations presented by the matrices

$$
R_{1}=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right], \quad R_{2}=\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right]
$$

Relations $R_{1}, R_{2}$ are min-max-Ferrers ([11]). Moreover $R=F\left(R_{1}, R_{2}\right)$ is min-max-Ferrers, where $R \equiv 0$. However, it is not true that $F \gg \min$ (the only $t$-norm that dominates minimum is minimum itself).

## 4 Aggregation of fuzzy interval orders

Now, we will consider fuzzy interval orders and their properties in the process of aggregation. In this section $B, B_{1}, B_{2}$ denote binary operations on unit interval, i.e. $B, B_{1}, B_{2}:[0,1]^{2} \rightarrow[0,1]$.

Theorem 4.1. Let $B_{1}$ have an idempotent element 1. A reflexive $B_{1}-B_{2}$-Ferrers relation is totally $B_{2}$ connected.

Proof. If $R$ is a reflexive, $B_{1}$ - $B_{2}$-Ferrers fuzzy relation, then we get

$$
1=B_{1}(1,1)=B_{1}(R(x, x), R(y, y)) \leqslant B_{2}(R(x, y), R(y, x))
$$

which means that $B_{2}(R(x, y), R(y, x))=1$ and $R$ is totally $B_{2}$-connected.
Corollary 4.2. Let $T$ be a t-norm and $S$ a t-conorm. A reflexive $T$ - $S$-Ferrers relation is totally $S$ connected.

Theorem 4.3. Let $B_{1}$ have a zero element 0 , an idempotent element 1 and for each $x, y \in[0,1]$ such that $x+y>1$ fulfil $B_{1}(x, y)=B_{1}(y, x)$ and let $B_{2}$ be dual to $B_{1}$ such that $B_{1} \leqslant B_{2}$. The following assertions are equivalent:
(1) A reflexive $B_{1}-B_{2}$-Ferrers relation is totally $S_{L}$-connected.
(2) Operation $B_{1}:[0,1]^{2} \rightarrow[0,1]$ fulfils $B_{1}(x, y)>0$ for any pair $(x, y) \in[0,1]^{2}$ such that $x+y>1$.

Proof. (1) $\Rightarrow(2)$ Let us consider an operation $B_{1}$ such that there exists a pair $(x, y) \in[0,1]^{2}$ fulfilling $x+y>1$ and $B_{1}(x, y)=0$. Then a reflexive relation that is $B_{1}-B_{2}$-Ferrers but not totally $S_{L}$-connected may be build. For example, let $X=\left\{x_{1}, x_{2}\right\}$ and $R\left(x_{1}, x_{2}\right)=1-x, R\left(x_{2}, x_{1}\right)=1-y$.
$(2) \Rightarrow(1)$ Let $R \in F R(X), x, y \in X$ and $R$ be reflexive and $B_{1}-B_{2}$-Ferrers. Applying these assumptions we obtain

$$
1=B_{1}(R(x, x), R(y, y)) \leqslant B_{2}(R(x, y), R(y, x))=1-B_{1}(1-R(x, y), 1-R(y, x))
$$

which implies that $B_{1}(1-R(x, y), 1-R(y, x))=0$. As a result from (2) it follows that $1-R(x, y)+$ $1-R(y, x) \leqslant 1$, which means that $R$ is totally $S_{L}$-connected.

Corollary 4.4 ([7]). Let us consider a t-norm $T$ and its dual $t$-conorm $S$. The following assertions are equivalent:
(1) A reflexive $T$-S-Ferrers relation is totally $S_{L}$-connected.
(2) The $t$-norm $T$ fulfils $T(x, y)>0$ for any pair $(x, y) \in[0,1]^{2}$ such that $x+y>1$.

In particular, the above corollary applies to all rotation invariant $t$-norms ([7]). The next results concern total max-connectedness. Let us observe that this notion is also named as strong completeness (cf. [11]).

Theorem 4.5. Let $B$ have a zero element 1 and have no zero divisors. Then total $B$-connectedness is equivalent to total max-connectedness.

Proof. Let $R \in F R(X), B$ have a zero element 1 and have no zero divisors. Total $B$-connectedness is equivalent to

$$
B(R(x, y), R(y, x))=1 \Leftrightarrow R(x, y)=1 \vee R(y, x)=1 \Leftrightarrow \max (R(x, y), R(y, x))=1
$$

which is equivalent to the fact that $R$ is totally max-connected.
Corollary 4.6 ([2]). Let $S$ be a $t$-conorm without zero divisors. Then total $S$-connectedness is equivalent to total max-connectedness.

Theorem 4.7. Let a commutative operation $B_{1}$ have a zero element 0 , an idempotent element 1 and let $B_{2}$ be dual to $B_{1}$ such that $B_{1} \leqslant B_{2}$. The following assertions are equivalent:
(1) A reflexive $B_{1}-B_{2}$-Ferrers relation is totally max-connected.
(2) $B_{1}$ has no zero divisors.

Proof. (1) $\Rightarrow(2)$ Let us suppose, to the contrary, that $B_{1}$ is not without zero divisors. Then there exist $x, y \in(0,1]$ such that $B_{1}(x, y)=0$. Let us now consider the relation $R \in F R(X)$, where $X=\left\{x_{1}, x_{2}\right\}$ and $R\left(x_{1}, x_{2}\right)=1-x, R\left(x_{2}, x_{1}\right)=1-y . R$ is $B_{1}-B_{2}$-Ferrers relation but it is not totally max-connected which contradicts to (1).
$(2) \Rightarrow(1)$ Let $R$ be a reflexive, $B_{1}-B_{2}$-Ferrers relation. By Theorem 4.1 it is also totally $B_{2}$-connected. From (2) and the assumption that $B_{2}$ is dual to $B_{1}$ it follows that $B_{2}$ has a zero element 1 and has no zero divisors. By Theorem 4.5 we obtain that $R$ is totally max-connected.

Corollary 4.8 ([7]). Let $T$ be a t-norm and $S$ its dual t-conorm. Then the following conditions are equivalent:
(1) A reflexive $T$-S-Ferrers relation is totally max-connected.
(2) The t-norm T has no zero divisors.

The above results from Section 4 simplify the considerations on aggregation of fuzzy interval orders (condition on $F$ for preservation of reflexivity is much easier than the one for total connectedness). Applying these results and results of Section 3 we get for example the following statements.

Theorem 4.9. Let $T$ be a rotation invariant t-norm, $R_{1}, \ldots, R_{n} \in F R(X)$ be reflexive and $T$ - $S_{L}$-Ferrers. If a function $F:[0,1]^{n} \rightarrow[0,1]$, which is increasing in each of its arguments, fulfils $F(1, \ldots, 1)=1$, $F \gg T$ and $S_{L} \gg F$, then $R_{F}=F\left(R_{1}, \ldots, R_{n}\right)$ is a $T-S_{L}$ fuzzy interval order.

Since $T_{L}$ is an example of a rotation invariant $t$-norm, in particular we get the following results.
Theorem 4.10. Let $R_{1}, \ldots, R_{n} \in F R(X)$ be reflexive and $T_{L}-S_{L}$-Ferrers. If a function $F:[0,1]^{n} \rightarrow$ $[0,1]$, which is increasing in each of its arguments fulfils $F(1, \ldots, 1)=1, F \gg T_{L}$ and $S_{L} \gg F$, then $R_{F}=F\left(R_{1}, \ldots, R_{n}\right)$ is a $T_{L}-S_{L}$ fuzzy interval order.

Corollary 4.11. Let $R_{1}, \ldots, R_{n} \in F R(X)$ be reflexive and $T_{L}$ - $S_{L}$-Ferrers. Then fuzzy relation $R_{F}=$ $F\left(R_{1}, \ldots, R_{n}\right)$ is a $T_{L}-S_{L}$ fuzzy interval order, where $F$ is a weighted arithmetic mean. Moreover, fuzzy relation $R_{F}$ is a $T$-S fuzzy interval order, where $F$ is a quasi-linear mean and $T$ is a nilpotent t-norm, $S=T^{d}$.

Theorem 4.12. Let $R_{1}, \ldots, R_{n} \in F R(X)$ be reflexive and min-max-Ferrers. If a function $F:[0,1]^{n} \rightarrow$ $[0,1]$, which is increasing in each of its arguments fulfils $F(1, \ldots, 1)=1, F \gg \min$ and $\max \gg F$, then $R_{F}=F\left(R_{1}, \ldots, R_{n}\right)$ is a min-max fuzzy interval order.

Examples of increasing functions which dominate minimum and are dominated by maximum are projections ([1]), so they fulfil assumptions on $F$ in the above theorem.

## 5 Conclusion

In this paper fuzzy interval orders were considered in the context of aggregation process. In future work it would be interesting to consider other orders and their preservation in aggregation process, in particular total preorder, total order, strict total order, partial preorder, partial order, strict partial order, or semiorder and their preservation in aggregation process.

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# On Constructing Ordinal Sums of Fuzzy Implications 

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#### Abstract

In this contribution, new ways of constructing of ordinal sum of fuzzy implications are indicated. Sufficient properties of fuzzy implications as summands for obtaining a fuzzy implication as a result are presented.


Keywords: fuzzy implication, ordinal sum, $R$-implication, triangular norm

## 1 Introduction

Fuzzy implications are one of the most important fuzzy connectives in many applications such as fuzzy reasoning and fuzzy control. For that reason new families of these connectives are the subject of investigation. One of the directions of such research is considering an ordinal sum of fuzzy implications on the pattern of the ordinal sum of $t$-norms. Some interesting results connected to representation of the residual implication corresponding to a fuzzy conjunction (for example continuous or at least left-continuous t -norm) given by an ordinal sum were obtained in [2, 3, 6]. In [8] Su et al. introduced a concept of ordinal sum of fuzzy implications similar to the construction of the ordinal sum of $t$-norms.

In this contribution, some of the ideas are recalled and new possibilities of defining ordinal sums of fuzzy implications are proposed. The operations obtained by the presented methods are not necessarily fuzzy implications. Sufficient properties for fuzzy implications as summands for obtaining a fuzzy implication are presented.

Firstly, in Section 2, we recall basic definitions and results concerning t-norms and fuzzy implications including constructions of ordinal sums of these fuzzy connectives. Then, in Section 3, we indicate new methods of constructing ordinal sums of fuzzy implications. At the end we suggest further research directions for the ordinal sums of fuzzy implications.

## 2 Preliminaries

Here we recall the notions of a t-norm and a fuzzy implication, as well as some of the constructions of ordinal sums of these fuzzy connectives.

### 2.1 Triangular norms

First, we put some very basic information about triangular norms (t-norms).
Definition 2.1 ([5], p. 4). A triangular norm is an increasing, commutative and associative operation $T:[0,1]^{2} \rightarrow[0,1]$ with neutral element 1.

Definition 2.2 ([5], p. 27). A triangular norm $T$ is called Archimedean, if for each $(x, y) \in(0,1)^{2}$ there is an $n \in \mathbb{N}$ such that $x_{T}^{(n)}<y$.

[^3]Example 2.3 ([5], p. 4, [4], p. 7). Here, we list well-known basic t-norms, from which $T_{M}, T_{P}, T_{L}$ are continuous, and $T_{P}, T_{L}$ are both continuous and Archimedean.

$$
\begin{array}{ll}
T_{M}(x, y)=\min (x, y), & T_{P}(x, y)=x y, \\
T_{L}(x, y)=\max (x+y-1,0), & T_{D}(x, y)= \begin{cases}x, & \text { if } y=1 \\
y, & \text { if } x=1 \\
0, & \text { otherwise }\end{cases} \\
T_{n M}(x, y)= \begin{cases}0, & \text { if } x+y \leq 1 \\
\min (x, y), & \text { otherwise }\end{cases}
\end{array}
$$

Now, let us recall a representation of continuous t-norms by means of ordinal sums.
Theorem 2.4 ([5], p. 128). For an operation $T:[0,1]^{2} \rightarrow[0,1]$ the following statements are equivalent:
(i) $T$ is a continuous t-norm.
(ii) $T$ is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e., there exists a uniquely determined (finite or countably infinite) index set I, a family of uniquely determined pairwise disjoint open subintervals $\left(a_{k}, b_{k}\right)$ of $[0,1]$ and a family of uniquely determined continuous Archimedean t-norms $\left(T_{k}\right)_{k \in A}$ such that

$$
T(x, y)= \begin{cases}a_{k}+\left(b_{k}-a_{k}\right) T_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y-a_{k}}{b_{k}-a_{k}}\right) & \text { if }(x, y) \in\left[a_{k}, b_{k}\right]^{2} \\ \min (x, y) & \text { otherwise }\end{cases}
$$

The above representation is based on the ordinal sum of arbitrary t-norms ([5], p. 82).


Figure 1: The structure of an ordinal sum of $t$-norms

### 2.2 Fuzzy Implications

Now, we focus on fuzzy implications, their possible properties, as well as the class of R-implications.
Definition 2.5 ([1], p. 2, [4], p. 21). A function $I:[0,1]^{2} \rightarrow[0,1]$ is called a fuzzy implication if it satisfies the following conditions:
(I1) decreasing in its first variable,
(I2) increasing in its second variable,
(I3) $I(0,0)=1$,
(I4) $I(1,1)=1$,
$(I 5) I(1,0)=0$.

There are many potential properties of fuzzy implications (see, e.g., [1], p. 9). We recall here only one which will be important in the sequel.

Definition 2.6 ([7]). We say that a fuzzy implication I fulfils the consequent boundary property $(\mathrm{CB})$ if

$$
\begin{equation*}
I(x, y) \geq y, \quad x, y \in[0,1] \tag{CB}
\end{equation*}
$$

Example 2.7 ([1], pp. 4,5). The following are very known examples of fuzzy implications. Almost all of them, except for $I_{R S}$ fulfil property (CB).

$$
\begin{array}{ll}
I_{\mathrm{EK}}(x, y)=\min (1-x+y, 1), & I_{\mathrm{GG}}(x, y)= \begin{cases}1, & \text { if } x \leq y \\
\frac{y}{x}, & \text { if } x>y\end{cases} \\
I_{\mathrm{GD}}(x, y)=\left\{\begin{array}{ll}
1, & \text { if } x \leq y \\
y, & \text { if } x>y
\end{array},\right. & I_{\mathrm{RS}}(x, y)= \begin{cases}1, & \text { if } x \leq y \\
0, & \text { if } x>y\end{cases} \\
I_{\mathrm{RC}}(x, y)=1-x+x y, & I_{\mathrm{YG}}(x, y)= \begin{cases}1, & \text { if } x=0 \text { and } y=0 \\
y, & \text { if else }\end{cases} \\
I_{\mathrm{DN}}(x, y)=\max (1-x, y), & I_{\mathrm{FD}}(x, y)= \begin{cases}1, & \text { if } x \leq y \\
\max (1-x, y), & \text { if } x>y\end{cases} \\
I_{\mathrm{WB}}(x, y)=\left\{\begin{array}{ll}
1, & \text { if } x \leq 1 \\
y, & \text { if } x=1
\end{array},\right. & I_{\mathrm{DP}}(x, y)= \begin{cases}y, & \text { if } x=1 \\
1-x, & \text { if } y=0 \\
1, & \text { if } x<1, y>0\end{cases}
\end{array}
$$

Definition 2.8. A function $I:[0,1]^{2} \rightarrow[0,1]$ is called a residual implication (an $R$-implication) if there exists a t-norm $T$ such that

$$
\begin{equation*}
I(x, y)=I_{T}(x, y)=\sup \{t \in[0,1]: T(x, t) \leq y\}, \quad x, y \in[0,1] \tag{1}
\end{equation*}
$$

Example 2.9. Table 1 shows $R$-implications obtained by formula (1) from basic t-norms presented in Example 2.3.

| t-norm $T$ | R-implication $I_{T}$ |
| :--- | :--- |
| $T_{M}$ | $I_{G D}$ |
| $T_{P}$ | $I_{G G}$ |
| $T_{L}$ | $I_{L K}$ |
| $T_{D}$ | $I_{W B}$ |
| $T_{n} M$ | $I_{F D}$ |

Table 1: Examples of basic R-implications

Theorem 2.10 ([1], p. 83). If T is a continuous t-norm with an ordinal sum structure (see Theorem 2.4), then the corresponding $R$-implication $I_{T}$ is given by

$$
I_{T}(x, y)= \begin{cases}1, & \text { if } x \leq y  \tag{2}\\ a_{k}+\left(b_{k}-a_{k}\right) I_{T_{k}}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y-a_{k}}{b_{k}-a_{k}}\right), & \text { if } x, y \in\left[a_{k}, b_{k}\right], x>y \\ y, & \text { otherwise }\end{cases}
$$

Now, let us recall a recent approach to the construction of ordinal sum of fuzzy implications [8]. This construction method is based on the construction of the ordinal sum of t-norms.


Figure 2: The structures of an ordinal sum of t -norms and R -implication $I_{T}$ given by (2)

Definition 2.11 ([8]). Let $\left\{I_{k}\right\}_{k \in A}$ be a family of implications and $\left\{\left[a_{k}, b_{k}\right]\right\}_{k \in A}$ be a family of pairwise disjoint close subintervals of $[0,1]$ with $0<a_{k}<b_{k}$ for all $k \in A$, where $A$ is a finite or infinite index set. The mapping $I:[0,1]^{2} \rightarrow[0,1]$ given by

$$
I(x, y)= \begin{cases}a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y-a_{k}}{b_{k}-a_{k}}\right), & \text { if } x, y \in\left[a_{k}, b_{k}\right]  \tag{3}\\ I_{G D}(x, y), & \text { otherwise }\end{cases}
$$

we call an ordinal sum of fuzzy implications $\left\{I_{k}\right\}_{k \in A}$.


Figure 3: The structure of an ordinal sum of fuzzy implications given by (3)

It may be that $I$ given by (3) is not an implication.
Example 2.12 ([8]). Let

$$
I(x, y)= \begin{cases}\frac{1}{4}+\left(\frac{1}{2}-\frac{1}{4}\right) I_{R S}\left(\frac{x-\frac{1}{4}}{\frac{1}{2}-\frac{1}{4}}, \frac{x-\frac{1}{4}}{\frac{1}{2}-\frac{1}{4}}\right) & \text { if }(x, y) \in\left[\frac{1}{4}, \frac{1}{2}\right]^{2}, \\ I_{G D}(x, y) & \text { otherwise. }\end{cases}
$$

It is easy to see that $I\left(\frac{1}{2}, \frac{1}{3}\right)=\frac{1}{4}<\frac{1}{3}=I\left(\frac{3}{4}, \frac{1}{3}\right)$, i.e. I does not satisfy (II).
The next theorem gives out the conditions that $I$ given by (3) satisfies (I1).
Theorem 2.13 ([8]). Let $\left\{I_{k}\right\}_{k \in A}$ be a family of implications. Then ordinal sum of implication given by (3) satisfies (I1) if and only if $I_{k}$ satisfies (CB) whenever $k \in A$ and $b_{k}<1$.

Let us notice, that the construction in Definition 2.11 involves intervals $\left[a_{i}, b_{i}\right]$ which are necessarily disjoint. However, in the construction of t -norms the intervals can have a common point. It is still an open problem, whether we can add some additional assumptions on the construction for the intervals do not have to be disjoint.

## 3 Main results

Here, we propose tree ways of generating a new fuzzy implication from given ones. Let start with the first method, which is a kind of generalization of the results obtained e.g. in [6] for residual implications.

Let $\left\{I_{k}\right\}_{k \in A}$ be a family of implications and $\left\{\left(a_{k}, b_{k}\right)\right\}_{k \in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_{k}<b_{k}$ for all $k \in A$, where $A$ is a finite or infinite index set. Let us consider an operation $I:[0,1]^{2} \rightarrow[0,1]$ given by the following formula

$$
I(x, y)= \begin{cases}1, & \text { if } x \leq y  \tag{4}\\ a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y-a_{k}}{b_{k}-a_{k}}\right), & \text { if } x, y \in\left[a_{k}, b_{k}\right], x>y . \\ y, & \text { otherwise }\end{cases}
$$



Figure 4: The structure of an operation given by (4)

Remark 3.1. Let us observe, that the operation I given by (4) can be noted as

$$
I(x, y)=\left\{\begin{array}{ll}
a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y-a_{k}}{b_{k}-a_{k}}\right), & \text { if } x, y \in\left[a_{k}, b_{k}\right], y<x \\
I_{G D}(x, y), & \text { otherwise }
\end{array} .\right.
$$

Lemma 3.2. Let $\left\{I_{k}\right\}_{k \in A}$ be a family of fuzzy implications. Then I given by (4) satisfies (I2), (I3), (I4) and (I5).

Proof. First, let us consider the condition (I2). Let $y_{1}<y_{2}, x, y_{1}, y_{2} \in[0,1]$.
If $x \in\left[a_{k}, b_{k}\right]$ for some $k \in A$, then we obtain the following cases

1. $y_{2}<a_{k}$ or $x \leq y_{1}$ or both $y_{1}<a_{k}$ and $x \leq y_{2}$. Then $I\left(x, y_{1}\right)=I_{G D}\left(x, y_{1}\right) \leq I_{G D}\left(x, y_{2}\right)=$ $I\left(x, y_{2}\right)$.
2. $y_{1}<a_{k} \leq y_{2} \leq x$. Then $I\left(x, y_{1}\right)=y_{1}<a \leq a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y_{2}-a_{k}}{b_{k}-a_{k}}\right)=I\left(x, y_{2}\right)$.
3. $a_{k} \leq y_{1} \leq y_{2} \leq x$. Then using monotonicity of $I_{k}$ we have $I\left(x, y_{1}\right)=a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y_{1}-a_{k}}{b_{k}-a_{k}}\right) \leq$ $a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y_{2}-a_{k}}{b_{k}-a_{k}}\right)=I\left(x, y_{2}\right)$.
4. $a_{k} \leq y_{1}<x \leq y_{2}$. Then $I\left(x, y_{1}\right)=a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y_{1}-a_{k}}{b_{k}-a_{k}}\right) \leq 1=I\left(x, y_{2}\right)$.

In other cases we have similar situation as in 1 .
Directly from (4) we have $I(0,0)=I(1,1)=1$. So $I$ fulfils (I3) and (I4). To prove (I5) let us consider two cases. If there exists $k \in A$ such that $\left[a_{k}, b_{k}\right]=[0,1]$, then $I(1,0)=I_{k}(1,0)=0$. Otherwise $I(1,0)=y=0$.

Example 3.3. Let

$$
I(x, y)= \begin{cases}1, & \text { if } x \leq y \\ 0.5 I_{R S}(2 x, 2 y), & \text { if } x, y \in[0,0.5] \\ y, & \text { otherwise }\end{cases}
$$

## I does not fulfill (Il).

The following result can be proved in a similar way to Theorem 2.13.
Theorem 3.4. The operation I given by (4) satisfies (II) if and only if $I_{k}$ satisfies (CB) whenever $k \in A$ and $b_{k}<1$.

As we can see, not every fuzzy implication can be used in constructions (3) and (4). Below we present a structure in which any fuzzy implications can be used.

Now, let $\left\{I_{k}\right\}_{k \in A}$ be a family of implications and $\left\{\left[a_{k}, b_{k}\right]\right\}_{k \in A}$ be a family of pairwise disjoint close subintervals of $[0,1]$ with $0<a_{k}<b_{k}$ for all $k \in A$, where $A$ is a finite or infinite index set. Let us consider an operation $I:[0,1]^{2} \rightarrow[0,1]$ given by the following formula

$$
I(x, y)=\left\{\begin{array}{ll}
a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y-a_{k}}{b_{k}-a_{k}}\right), & \text { if } x, y \in\left[a_{k}, b_{k}\right]  \tag{5}\\
I_{R S}(x, y), & \text { otherwise }
\end{array} .\right.
$$



Figure 5: The structure of an operation given by (5)

Theorem 3.5. The operation I given by (5) is a fuzzy implication.
Proof. First, let us consider the condition (I1). Let $x_{1}<x_{2}, x_{1}, x_{2}, y \in[0,1]$. If $y \in\left[a_{k}, b_{k}\right]$ for some $k \in A$, then we consider the following cases

1. $x_{1}<a_{k}$. Then $I\left(x_{1}, y\right)=I_{R S}\left(x, y_{1}\right)=1 \geq I\left(x_{2}, y\right)$.
2. $x_{1}, x_{2} \in\left[a_{k}, b_{k}\right]$. Then using monotonicity of $I_{k}$ we have $I\left(x_{1}, y\right)=a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x_{1}-a_{k}}{b_{k}-a_{k}}, \frac{y-a_{k}}{b_{k}-a_{k}}\right) \geq$ $a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x_{2}-a_{k}}{b_{k}-a_{k}}, \frac{y-a_{k}}{b_{k}-a_{k}}\right)=I\left(x_{2}, y\right)$.
3. $b_{k}<x_{2}$. Then $I\left(x_{1}, y\right) \geq 0=I\left(x_{2}, y\right)$.

In other cases values of $I$ are the same as values of $I_{R S}$, which give the condition (I1).
To prove (I2) let us take $y_{1}<y_{2}, x, y_{1}, y_{2} \in[0,1]$.
If $x \in\left[a_{k}, b_{k}\right]$ for some $k \in A$, then we obtain the following cases

1. $y_{1}<a_{k}$. Then $I\left(x, y_{1}\right)=I_{R S}\left(x, y_{1}\right)=0 \leq I\left(x, y_{2}\right)$.
2. $y_{1}, y_{2} \in\left[a_{k}, b_{k}\right]$. Then using monotonicity of $I_{k}$ we have $I\left(x, y_{1}\right)=a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y_{1}-a_{k}}{b_{k}-a_{k}}\right) \leq$ $a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y_{2}-a_{k}}{b_{k}-a_{k}}\right)=I\left(x, y_{2}\right)$.
3. $b_{k}<y_{2}$. Then $I\left(x, y_{1}\right) \leq 1=I\left(x, y_{2}\right)$.

In other cases values of $I$ are the same as values of $I_{R S}$. So, we obtain (I2).
Directly from (6) we have $I(0,0)=I(1,1)=1$. So $I$ fulfils (I3) and (I4). To prove (I5) let us consider two cases. If there exists $k \in A$ such that $\left[a_{k}, b_{k}\right]=[0,1]$, then $I(1,0)=I_{k}(1,0)=0$. Otherwise $I(1,0)=0$. So, operation given by (6) is an implication.

In both constructions (3) and (5) the intervals $\left[a_{k}, b_{k}\right]$ must be separable. This means that we are unable to construct fuzzy implications in which the values $I(x, x)$ for $x \in(0,1)$ depend on the component implications $I_{k}$. Below we present a construction that solves this problem.

Let $\left\{I_{k}\right\}_{k \in A}$ be a family of implications and $\left\{\left(a_{k}, b_{k}\right)\right\}_{k \in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_{k}<b_{k}$ for all $k \in A$, where $A$ is a finite or infinite index set. Let us consider an operation $I:[0,1]^{2} \rightarrow[0,1]$ given by the following formula

$$
I(x, y)= \begin{cases}a_{k}+\left(b_{k}-a_{k}\right) I_{k}\left(\frac{x-a_{k}}{b_{k}-a_{k}}, \frac{y-a_{k}}{b_{k}-a_{k}}\right), & \text { if } x, y \in\left(a_{k}, b_{k}\right]  \tag{6}\\ 1, & \text { if } x \leq y \\ 0, & \text { otherwise }\end{cases}
$$



Figure 6: The structure of an operation given by (6)

Example 3.6. Let

$$
I(x, y)= \begin{cases}1, & \text { if } x \leq y \\ 0.5 I_{R C}(2 x, 2 y), & \text { if } x, y \in(0,0.5] \\ 0.5+0.1 I_{L K}(10 x-5,10 y-5), & \text { if } x, y \in(0.5,0.6] \\ 0, & \text { otherwise }\end{cases}
$$

$I$ is an implication.
The following result can be proved in a similar way to Theorem 3.5.
Theorem 3.7. The operation I given by (6) is a fuzzy implication.

## 4 Conclusion

In this paper we indicate three methods of constructing ordinal sum of fuzzy implications. In future research, it would be useful to examine the properties of these ordinal sums. Another problem is whether the proposed ordinal sums preserve properties of its summands.

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# Invariants of $\varphi$-transformations of uninorms and t-norms 

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#### Abstract

The paper deals with triangular norms and uninorms, and their constructions. Specifically, we study $\varphi$-transformations and their invariants. The work contains selected results of author's work in the student competition SVOČ.


Keywords: $t$-norm, uninorm, $\varphi$-transformation, invariant

## 1 Preliminaries

The main topic of this article is a special type of constructions of triangular norms and uninorms. First we recall some important definitions and statements.

Definition 1.1. [3] A triangular norm $T$ (t-norm for short) is a commutative, associative, monotone binary operator on the unit interval $[0,1]$, fulfilling the boundary condition $T(x, 1)=x$, for all $x \in$ $[0,1]$.

Uninorms were introduced by Yager and Rybalov in 1996 as a generalization of triangular norms and conorms [7].
Definition 1.2. [7] An associative, commutative and increasing operation $U:[0,1]^{2} \rightarrow[0,1]$ is called a uninorm, if there exists $e \in[0,1]$, called the neutral element of $U$, such that

$$
U(e, x)=U(x, e)=x \text { for all } x \in[0,1] .
$$

There exist various constructions of $t$-norms, and we will deal with a method of constructing $t$-norms which gives the new t -norm from a previously known t -norm and a unary function $\varphi$.

Proposition 1.3. [3] Let $\varphi:[0,1] \rightarrow[0,1]$ be a non-decreasing function and $T:[0,1]^{2} \rightarrow[0,1]$ be a $t$-norm. Then the function defined by

$$
T_{\varphi}(x, y)= \begin{cases}\min \{x, y\}, & \text { if } \max \{x, y\}=1 \\ \varphi^{(-1)}[T(\varphi(x), \varphi(y))], & \text { otherwise }\end{cases}
$$

is a t-norm. Note, that $\varphi^{(-1)}$ is a pseudo-inverse, which is a monotone extension of the ordinary inverse function and $\varphi^{(-1)}(x)=\sup \{z \in[0,1] ; \varphi(z)<x\}$.

We can similarly construct uninorms:
Proposition 1.4. [2] Let $\varphi:[0,1] \rightarrow[0,1]$ be a continuous, bijective function, and let there exist $e^{\prime}$ such that $e^{\prime}=\varphi^{-1}(e)$, where $e$ is the neutral element of a given uninorm $U$. Then the function

$$
U_{\varphi}(x, y)=\varphi^{-1}\left[U_{e}(\varphi(x), \varphi(y))\right]
$$

is a uninorm with the neutral element $e^{\prime}$.

[^4]In this paper we will discuss the invariants of $\varphi$-transformation of t-norms and uninorms. It means, we will look for the uninorms and the bijective functions $\varphi$ such that

$$
\varphi(U(x, y))=U(\varphi(x), \varphi(y))
$$

Finally, we include some necessary notions.
Definition 1.5. [3] Let $T:[0,1]^{2} \rightarrow[0,1]$ be a $t$-norm. Then a function $\delta_{n}:[0,1] \rightarrow[0,1]$ defined as

$$
\delta_{1}(x)=x, \quad \delta_{n+1}(x)=T\left(\delta_{n}(x), x\right), \quad \text { for } x \in[0,1], n \in \mathbb{N}
$$

is called the diagonal function of a t-norm $T$. The set of all diagonal functions of given t-norm $T$ is denoted as $\Delta_{T}=\left\{\delta_{n}: n \in \mathbb{N}\right\}$.

Definition 1.6. A t-norm $T$ is called Archimedean if it has the Archimedean property, i.e., if for each $x, y$ in the open interval $(0,1)$ there is a natural number $n$ such that $\delta_{n} \leq y$.

In this paper we deal with a specific class of uninorms, called simple uninorms.
Definition 1.7. [2] A uninorm $U:[0,1]^{2} \rightarrow[0,1]$ is called simple, if there exists left or right neighborhood of $y$ for every $(x, y) \in[0, e) \times(e, 1]$, where uninorm $U$ has constant values, i.e.

$$
\forall(x, y) \in[0, e) \times(e, 1], \forall y_{1}, y_{2} \in U_{\varepsilon}^{+}(y): U\left(x, y_{1}\right)=U\left(x, y_{2}\right) \quad\left(U_{\varepsilon}^{-}(y)\right)
$$

## 2 Invariants of transformation on the set $[0, e) \times(e, 1]$

In our investigation of invariants of uninorm transformations we start with the set $[0, e) \times(e, 1]$.
Definition 2.1. [2] Let $U:[0,1]^{2} \rightarrow[0,1]$ be a uninorm with the neutral element $e$. Then we define $S(U)=\left\{\left(a_{i}, b_{i}\right) \times\left(c_{i}, d_{i}\right) ; i=1, \cdots, n ; n \in \mathbb{N}\right\}$ as a system of the sets, such that

$$
\forall J \in S(U) \text { and } \forall\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right) \in J: U\left(x_{1}, y_{1}\right)=U\left(x_{2}, y_{2}\right)
$$

Moreover for every $J$ must exists $\alpha_{J} \in H(J)$, such that

$$
\forall p \in D(J): U\left(p, \alpha_{J}\right) \neq U\left(x, \alpha_{J}\right), \text { where } x \in[0, e) \backslash D(J)
$$

Definition 2.2. [2] Let $U:[0,1]^{2} \rightarrow[0,1]$ be a uninorm with the neutral element $e$. Then we define the set $M_{x}(U)=\left\{\left(a_{1}, b_{1}\right), \ldots,\left(a_{n}, b_{n}\right)\right\}$ as a set of $x$-coordinate discontinuities of uninorm $U$ on $[0, e) \times(e, 1]$. Similarly we define the set of $y$-coordinate discontinuities as $M_{y}(U)$.

The following theorem deals with the properties of transformation function $\varphi$ in the discontinuity points of given uninorm.

Theorem 2.3. [2] Let $U:[0,1]^{2} \rightarrow[0,1]$ be a simple uninorm and $M_{x}(U)$ be a finite set of $x$-coordinate discontinuities of uninorm $U$. Further we consider nondecreasing bijection $\varphi:[0,1] \rightarrow[0,1]$. Then if the original uninorm is formed by the $\varphi$-transformation, then $\forall\left(a_{i}, b_{i}\right) \in M_{x}(U): \varphi\left(a_{i}\right)=a_{i}$.

The proof is based on an examination of the cases $\varphi\left(a_{i}\right)>a_{i}$ and $\varphi\left(a_{i}\right)<a_{i}$. Note that in a very similar way we can prove this statement for the set $M_{y}(U)$, i.e, that $\forall(x, y) \in M_{y}(U): \varphi(y)=y$. The following example shows the importance of finiteness of the set $M_{x}(U)$ from the previous theorem.

Example 2.4. Let us consider continuous bijective function $f:[0,1] \rightarrow[0,1]$ given by following formula

$$
f(x)= \begin{cases}\sqrt[3]{\frac{x}{4}} & \text { if } x \leq \frac{1}{2} \\ x & \text { otherwise }\end{cases}
$$

Further more consider a uninorm $U^{*}:[0,1]^{2} \rightarrow[0,1]$ with neutral element $e=\frac{1}{2}$ given as:

$$
U^{*}(x, y)= \begin{cases}1 & \text { if } \min \{x, y\}>\frac{1}{2} \\ \min \{x, y\} & \text { if } \max \{x, y\}=\frac{1}{2} \\ \max \{x, y\} & \text { if } \min \{x, y\}=\frac{1}{2} \\ f^{i+1}\left(\frac{1}{4}\right) & \text { if } \max \{x, y\}>\frac{1}{2} \text { and } \\ & \min \{x, y\} \in\left(f^{i}\left(\frac{1}{4}\right), f^{i+1}\left(\frac{1}{4}\right)\right] \text { for } i \in \mathbb{Z} \\ 0 & \text { otherwise }\end{cases}
$$

We study transformation given by the function $\varphi=f$. Here we show only the most interesting case of proving the invariance. Therefore we assume $x \in\left(\varphi^{i}\left(\frac{1}{4}\right), \varphi^{i+1}\left(\frac{1}{4}\right)\right], y \in\left(\frac{1}{2}, 1\right]$. Then

$$
U^{*}(\varphi(x), \varphi(y))=\varphi^{i+2}\left(\frac{1}{4}\right)=\varphi \circ \varphi^{i+1}\left(\frac{1}{4}\right)=\varphi\left(U^{*}(x, y)\right)
$$

Other cases could be proved similarly. The uninorm $U^{*}$ with the function $\varphi$ give us an example of a $\varphi$-transformation, in which the fixed points of the function $\varphi$ in discontinuities of $U^{*}$ are not necessary for invariant.

Theorem 2.5. [2] Let $U:[0,1]^{2} \rightarrow[0,1]$ be a simple uninorm and $\varphi:[0,1] \rightarrow[0,1]$ be a continuous bijective function. If the original uninorm is formed by the $\varphi$-transformation, then

$$
\forall J \in S(U): \varphi\left(\sup J_{x}\right)=\sup J_{x} \text { and } \varphi\left(\inf J_{x}\right)=\inf J_{x}
$$

Proof. The proof is based on generating the set $M(U)$ using an iteration of the function $\varphi$. Since the set $S(U)$ is finite, the set $M(U)$ is finite as well and hence there exists a fixed point of the function $\varphi$ at the points $\inf J_{x}$ and $\sup J_{x}$ for $J \in S(U)$.

Corollary 2.6. [2] Let $U:[0,1]^{2} \rightarrow[0,1]$ be a simple uninorm, $\varphi:[0,1] \rightarrow[0,1]$ be a continuous bijective function and $M_{y}(U)$ be a finite set. If the original uninorm is formed by the $\varphi$-transformation, then the interval $(e, 1]$ can be divided into subintervals $I_{i}=\left(y_{i}, y_{i+1}\right]$ for which $\varphi\left(y_{i}\right)=y_{i}$ holds.

Theorem 2.7. [2] Let $U:[0,1]^{2} \rightarrow[0,1]$ be a simple uninorm, $I_{i}=\left(y_{i}, y_{i+1}\right]$ be sub-intervals from Corollary 2.6 and a function $\varphi:[0,1] \rightarrow[0,1]$ be a continuous bijection for which $\varphi\left(y_{i}\right)=y_{i}$ holds. Further we assume a function $\psi_{i}(x)=U\left(x, y_{i}\right)$ for $x \in[0, e)$ and $y \in I_{i}$. Then the original uninorm on the set $[0, e) \times(e, 1]$ is formed by the $\varphi$-transformation iff

$$
\begin{equation*}
\varphi \circ \psi_{i}(x)=\psi_{i} \circ \varphi(x), \quad \forall x \in[0, e), i \leq n \tag{1}
\end{equation*}
$$

where $n$ is the number of intervals.
Proof. We use the definition of a $\varphi$-tranformation and the previous corollary. In short we get

$$
\varphi \circ \psi_{i}(x)=\psi_{i} \circ \varphi(x) \Leftrightarrow \varphi(U(x, y))=U(\varphi(x), y) \Leftrightarrow \varphi(U(x, y))=U(\varphi(x), \varphi(y))
$$

for $x<e, y \in I_{i}$.
If we denote a set of all functions $\varphi$, satisfying equation (1) as $\mathcal{F}_{i}$, then a set of all functions $\mathcal{F}_{\varphi}$ forming the original uninorm by the $\varphi$-transformation on the set $[0, e) \times(e, 1]$ is given as follows

$$
\varphi \in \mathcal{F}_{i} \Leftrightarrow \varphi \circ \psi_{i}(x)=\psi_{i} \circ \varphi(x) \quad \text { and } \quad \mathcal{F}_{\varphi}=\bigcap_{i=0}^{n} \mathcal{F}_{i} .
$$

In the following text we deal with solving the functional equation (1). Functions satisfying this equation are called as permutable functions.

### 2.1 Chebyshev polynomials

The first partial solution of equation (1) is composed of Chebyshev polynomials.
Definition 2.8. [6] Chebyshev polynomials of the first kind $T_{n}$ are defined by

$$
T_{0}(x)=1, T_{1}(x)=x, T_{n+1}(x)=2 x T_{n}(x)-T_{n-1}(x), \text { for } n>0
$$

Chebyshev polynomials of the second kind $U_{n}$ are defined by

$$
U_{0}(x)=1, U_{1}(x)=2 x, U_{n+1}(x)=2 x U_{n}(x)-U_{n-1}(x), \text { for } n>0
$$

Theorem 2.9. [5] Let $T_{n}$ be the Chebyshev polynomial of the first kind, then for $x \geq 0, x \in \mathbb{R}$ and $\alpha=\arccos (x)$ is $T_{n}(\cos \alpha)=\cos (n \alpha)$.

Theorem 2.10. [5] The roots of the polynomial $T_{n}\left(U_{n}\right)$ are given by

$$
x_{k}=\cos \left(\frac{\pi}{2} \frac{2 k-1}{n}\right), \quad\left(x_{k}=\cos \left(\pi \frac{k}{n+1}\right)\right), \quad \text { for } k \in\{1, \ldots, n\} .
$$

Theorem 2.11. [5] Let $T_{n}$ be the Chebyshev polynomial of the first kind. Then its derivative is as follows

$$
T_{n}^{\prime}(x)=n U_{n-1}(x),
$$

where $U_{n-1}$ is Chebyshev polynomial of the second kind .
We will now look for such Chebyshev polynomials which are continuous and increasing on $[0, e]$ and $T_{n}(0)=0, T_{n}(e)=e$, for $e \in(0,1)$ and $T_{n}(x) \geq x$ for $x \in[0, e]$.

Investigation. From $T_{n}(0)=0$ we get $T_{n}(0)=-T_{n-2}(0)=0$. More, $4 \mid(n-1)$. From $T_{n}(e)=$ $e \in(0,1)$ we get

$$
\begin{aligned}
& e=\cos (n \arccos (e)) \Leftrightarrow n \arccos (e)=2 k \pi \pm \arccos (e) \Leftrightarrow \\
& n=\frac{2 k \pi \pm \arccos (e)}{\arccos (e)}=\frac{2 k \pi}{\arccos (e)} \pm 1, \text { for } k \in \mathbb{Z}
\end{aligned}
$$

Therefore

$$
\frac{\pi}{\arccos (e)} \in \mathbb{Q}
$$

For fulfillment of other conditions we will look for $x_{e_{1}}$, which is the smallest positive nonzero point at which the polynomial $T_{n}$ attains its local maximum and $x_{e_{1}}>e$ and $T_{n}\left(x_{e_{1}}\right)=1$. Directly from the previous theorem we get:

Theorem 2.12. Let $T_{n}$ be Chebyshev polynomial of the first kind. Then the local extremes are in the points $x_{e}$ which are given by:

$$
x_{e}=\cos \left(\frac{k \pi}{n}\right), \quad k \in\{1, \ldots, n\} .
$$

Remark: If we consider only polynomials that satisfy the above conditions, then the smallest positive point giving a local maximum is:

$$
x_{e_{0}}=\cos \left(\frac{n-1}{2 n} \pi\right) .
$$

And now we find the smallest point $e \in(0,1)$ such that $T_{n}(e)=e$. Then

$$
e=\cos (n \arccos (e)) \Leftrightarrow n \arccos (e)=2 k \pi \pm \arccos (e) \Leftrightarrow e=\cos \left(\frac{2 k \pi}{n \pm 1}\right)
$$

This equality is satisfied for $k \in\left\{1, \ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$ and for higher $k$ it s the same up to sign. Summarizing the previous we get:

$$
\cos \left(\frac{n-1}{2 n} \pi\right)>\left|\cos \left(\frac{2 k \pi}{n \pm 1}\right)\right| .
$$

Since the cosine function is decreasing in the interval $\left(0, \frac{\pi}{2}\right]$, we get

$$
\left|\frac{\pi}{2}-\frac{n-1}{2 n} \pi\right|>\left|\frac{\pi}{2}-\frac{2 k \pi}{n \pm 1}\right| \Leftrightarrow \frac{n-1}{2 n} \pi<\frac{2 k \pi}{n \pm 1} .
$$

From the previous investigation we have $k=\left\lfloor\frac{n}{4}\right\rfloor$ and

$$
\frac{n-1}{2 n} \pi<\frac{2 \pi}{n+1}\left\lfloor\frac{n}{4}\right\rfloor \Leftrightarrow(n-1)(n+1)<4 n\left\lfloor\frac{n}{4}\right\rfloor=4 n \frac{n-1}{4} .
$$

This inequality is satisfied only for $n=1$. There are no Chebyshev polynomials of the first kind, which would suit our conditions.

### 2.2 Function iteration

Other particular solution of functional equation (1) is closely related to the iteration of functions [4]. In the following text we denote by $\mathcal{F}$ the set of all nondecreasing functions $f:[0, e] \rightarrow[0, e]$ satisfying the conditions $f(x) \geq x, f(0)=0$ and $f(e)=e$.

The following lemma and corollaries explain methods of construction permutable functions.
Lemma 2.13. [2] Let $g$ and $f: X \rightarrow X$ be permutable functions (i.e. $f \circ g(x)=g \circ f(x)$ for all $x \in X$ ). We further assume nondecreasing (nonincreasing) surjective function $\lambda: X \rightarrow X$. Then the functions

$$
\Phi(x)=\lambda^{(-1)} \circ f \circ \lambda(x) \text { and } \Psi(x)=\lambda^{(-1)} \circ g \circ \lambda(x) \text {, }
$$

where $\lambda^{(-1)}$ is pseudoinverse function to $\lambda$, form a pair of permutable functions.
Proof. Since the function $\lambda$ is a nondecreasing (nonincreasing) surjection, the equality $\lambda \circ \lambda^{(-1)}(x)=x$ is satisfied. Which means that

$$
\begin{aligned}
\Phi \circ \Psi(x) & =\lambda^{(-1)} \circ f \circ \lambda \circ \lambda^{(-1)} \circ g \circ \lambda(x)=\lambda^{(-1)} \circ f \circ g \circ \lambda(x) \\
& =\lambda^{(-1)} \circ g \circ f \circ \lambda(x)=\lambda^{(-1)} \circ g \circ \lambda \circ \lambda^{(-1)} \circ f \circ \lambda(x)=\Psi \circ \Phi(x) .
\end{aligned}
$$

Note. Although the function $\lambda$ can be in general nonincreasing as well, in the following text we consider only the nondecreasing case due to our restrictions to permutable functions.

Corollary 2.14. [2] Let $f$ and $g$ be permutable functions and moreover $f, g \in \mathcal{F}$. Further we assume a nondecreasing surjective function $\lambda:[0, e] \rightarrow[0, e]$. Then the functions

$$
\Phi(x)=\lambda^{(-1)} \circ f \circ \lambda(x), \quad \Psi(x)=\lambda^{(-1)} \circ g \circ \lambda(x)
$$

form a pair of permutable functions, and moreover $\Phi, \Psi \in \mathcal{F}$.
Corollary 2.15. [2] Let $f$ be a function such that $f \in \mathcal{F}$. We further assume a nondecreasing surjective function $\lambda:[0, e] \rightarrow[0, e]$, and functions $\Phi_{n}(x)=\lambda^{(-1)} \circ f^{n} \circ \lambda(x)$ for $n \in \mathbb{N}_{0}$. Then the functions $\Phi_{i}$ and $\Phi_{j}$, for $i, j \in \mathbb{N}_{0}$ form a pair of permutable functions and moreover $\Phi_{i}, \Phi_{j} \in \mathcal{F}$.

The proof of the current and previous corollary is based on certain properties of function iteration and on properties of pseudoinverse functions.

In a search for permutable functions we can as well draw from existing functions as it is shown in the following example.

Example 2.16. Consider a $t$-conorm restricted to the set $[0, e]^{2}$, i.e.

$$
S_{e}(x, y)=e S\left(\frac{x}{e}, \frac{y}{e}\right), \quad \text { for }(x, y) \in[0, e]^{2}
$$

and its diagonal functions $\delta_{n}^{*}$. Then $\delta_{m}^{*} \circ \delta_{n}^{*}=\delta_{n}^{*} \circ \delta_{m}^{*}$ for $m, n \in \mathbb{N}$ [1]. More specifically, consider restriction of $t$-conorm probabilistic sum

$$
S_{e}(x, y)=x+y-\frac{x y}{e}, \quad \text { for }(x, y) \in[0, e]^{2}
$$

and the diagonal functions $\delta_{2}^{*}$ and $\delta_{3}^{*}$ given by

$$
\delta_{2}^{*}(x)=x\left(2-\frac{x}{e}\right), \quad \delta_{3}^{*}(x)=x\left(3-\frac{3 x}{e}+\frac{x^{2}}{e^{2}}\right)
$$

Then $\delta_{2}^{*}(x) \circ \delta_{3}^{*}(x)=\delta_{3}^{*}(x) \circ \delta_{2}^{*}(x)$ for all $x \in[0, e]$.

## 3 Invariant transformation of t-norms

As mentioned before, uninorms are generalizations of t-norms. Hence in this section we deal with an invariant transformation of t-norms. Before we introduce the necessary condition for invariant transformations, we demonstrate a $\varphi$-transfomation via the diagonal function $\delta_{n}$ of Frank t-norms.

Example 3.1. Frank t-norms are defined by [3]:

$$
T_{p}^{F}(x, y)= \begin{cases}T_{M}(x, y) & \text { if } p=0 \\ T_{P}(x, y) & \text { if } p=1 \\ T_{L}(x, y) & \text { if } p=+\infty \\ \log _{p}\left(1+\frac{\left(p^{x}-1\right)\left(p^{y}-1\right)}{p-1}\right) & \text { otherwise }\end{cases}
$$

The diagonal function for minimum t-norm is given by $\delta_{n, 0}(x)=x$. Invariance is thus apparent in this case. The diagonal function for product t-norm $T_{P}(x, y)=x y$ is defined by $\delta_{n, 1}(x)=x^{n}$. After transformation we obtain $(x y)^{n}=x^{n} y^{n}$. Invariance is thus again maintained.

The diagonal function for Łukasiewicz t-norm $T_{L}(x, y)=\max \{0, x+y-1\}$ is $\delta_{n, \infty}$ given by $\delta_{n, \infty}(x)=\varphi(x)=\max \{0, n x-n+1\}$. Invariance is again maintained, as can be seen by substitution.

For the other cases the diagonal functions $\delta_{n, p}$ are as follows:

$$
\delta_{n, p}(x)=\varphi(x)=\log _{p}\left(1+\frac{\left(p^{x}-1\right)^{n}}{(p-1)^{n-1}}\right) .
$$

Then the transformation looks as follows

$$
\begin{gathered}
T_{p}^{F}(\varphi(x), \varphi(y))=\log _{p}\left(1+\frac{\left(p^{x}-1\right)^{n}\left(p^{y}-1\right)^{n}}{(p-1)^{2 n-1}}\right) \\
\varphi\left(T_{p}^{F}(x, y)\right)=\log _{p}\left(1+\frac{\left(\frac{\left(p^{x}-1\right)\left(p^{y}-1\right)}{p-1}\right)^{n}}{(p-1)^{n-1}}\right)=\log _{p}\left(1+\frac{\left(p^{x}-1\right)^{n}\left(p^{y}-1\right)^{n}}{(p-1)^{2 n-1}}\right)
\end{gathered}
$$

and thus $T_{p}^{F}(\varphi(x), \varphi(y))=\varphi\left(T_{p}^{F}(x, y)\right)$. This altgother means, that the invariance towards transformation by the diagonal functions, is maintained for the class of Frank t-norms.

Now we can introduce the aforementioned necessary condition of invariance.
Theorem 3.2. [2] (Necessary condition of invariance) Let $T:[0,1]^{2} \rightarrow[0,1]$ be a t-norm, $\delta_{n}$ be diagonal functions of $T$ and $\varphi:[0,1] \rightarrow[0,1]$ be a nondecreasing surjective function. If $\varphi$ is an invariant of the transformation of the $t$-norm $T$, then $\varphi \circ \delta_{n}(x)=\delta_{n} \circ \varphi(x)$ for all $x \in[0,1], n \in \mathbb{N}$.

Proof. Since the function $\varphi$ is a nondecreasing surjection, the original t-norm is formed by the transformation iff $\varphi(T(x, y))=T(\varphi(x), \varphi(y))$. Hence $\varphi \circ \delta_{n}(x)=\delta_{n} \circ \varphi(x)$ for all $x \in[0,1]$ and $n \in \mathbb{N}$.

The following theorems show a further relation between diagonal functions, actually additive generators of t -norms, and invariant transformation.

Theorem 3.3. [2] Let $T:[0,1]^{2} \rightarrow[0,1]$ be a strict $t$-norm. We further assume a function $\varphi:[0,1] \rightarrow$ $[0,1]$. If $\varphi \in \Delta_{T}$, then the original t-norm is formed by the $\varphi$-transformation.
Proof. Since the function $\varphi$ is bijective, equation $\varphi\left(T_{\varphi}(x, y)\right)=T(\varphi(x), \varphi(y))$ is fulfilled. By the assumption $\varphi \in \Delta_{T}$, we will further write only $\delta_{n}\left(T_{\varphi}(x, y)\right)=T\left(\delta_{n}(x), \delta_{n}(y)\right)$, for $n \in \mathbb{N}$. The proof of the equation $T_{\varphi}=T$ will proceed by induction on $n$.

1. For $n=1$, the equation holds trivially. For $n=2$, we assume that there exists some $\left(x_{0}, y_{0}\right) \in$ $[0,1]^{2}$ such that $T\left(x_{0}, y_{0}\right) \neq T_{\varphi}\left(x_{0}, y_{0}\right)$. However, then

$$
T\left(T\left(x_{0}, y_{0}\right), T\left(x_{0}, y_{0}\right)\right)=\delta_{2}\left(T\left(x_{0}, y_{0}\right)\right) \neq T\left(\delta_{2}\left(x_{0}\right), \delta_{2}\left(y_{0}\right)\right)=T\left(T\left(x_{0}, y_{0}\right), T\left(x_{0}, y_{0}\right)\right),
$$

which is a contradiction (in the previous step we use associativity of $T$ and the fact that the function $\delta_{n}^{-1}$ is increasing). Thus $T_{\varphi}=T$ for the transformation by the function $\delta_{2}$.
2. Now we assume that the equation holds for $\delta_{1}, \ldots, \delta_{n}$ and we prove that it holds also for $\delta_{n+1}$. We get

$$
\begin{aligned}
& T_{\varphi}(x, y)=\delta_{n+1}^{-1}\left(T\left(\delta_{n+1}(x), \delta_{n+1}(y)\right)\right) \Rightarrow \delta_{n+1}\left(T_{\varphi}(x, y)\right)=T\left(\delta_{n+1}(x), \delta_{n+1}(y)\right) \Rightarrow \\
& T\left(\delta_{n}\left(T_{\varphi}(x, y)\right), T_{\varphi}(x, y)\right)=T\left(T\left(\delta_{n}(x), x\right), T\left(\delta_{n}(y), y\right)\right) .
\end{aligned}
$$

From the induction assumption and associativity of the t -norm $T$ it follows

$$
T\left(T\left(\delta_{n}(x), \delta_{n}(y)\right), T_{\varphi}(x, y)\right)=T\left(T\left(\delta_{n}(x), \delta_{n}(y)\right), T(x, y)\right) .
$$

Since the t -norm $T$ is strict, equation $T_{\varphi}=T$ holds true.
The original t -norm is thus formed by the transformation via diagonal functions.
Theorem 3.4. [2] Let $T:[0,1]^{2} \rightarrow[0,1]$ be a continuous Archimedean $t$-norm and $f:[0,1] \rightarrow[0, \infty]$ be additive the generator of this $t$-norm. Further let us consider a bijective function $\varphi:[0,1] \rightarrow[0,1]$. Then the original $t$-norm is formed by the $\varphi$-transformation iff there exists $\alpha>0$ such as $\alpha f(x)=$ $f \circ \varphi(x)$ (Schröder's equation).
Proof. $(\Leftarrow)$ The transformed t -norm $T_{\varphi}$ is given by

$$
T_{\varphi}(x, y)=\varphi^{-1}[T(\varphi(x), \varphi(y))]=\varphi^{-1} \circ f^{-1}(\min \{f \circ \varphi(x)+f \circ \varphi(y), f(0)\}) .
$$

Since the t -norm $T_{\varphi}$ is a continuous Archimedean t -norm, its additive generator $g$ is given by $g(x)=$ $f \circ \varphi(x)$. There exists $\alpha>0$, such that $g(x)=\alpha f(x)$, and hence $f$ and $g$ differ only by a positive multiplicative constant. The generator $g$ is thus also a generator of the t -norm $T$, and consequently $T_{\varphi}=T$.
$(\Rightarrow)$ Now we assume $T_{\varphi}(x, y)=T(x, y)$ for all $(x, y) \in[0,1]^{2}$, thus

$$
T_{\varphi}(x, y)=\varphi^{-1}[T(\varphi(x), \varphi(y))]=T(x, y) .
$$

The additive generator of the t -norm $T_{\varphi}$ is given by $g(x)=f \circ \varphi(x)$, but since both the t -norms are equal, there exists some $\alpha>0$ such that $f \circ \varphi(x)=\alpha f(x)$.

All bijective functions $\varphi$ on the unit interval, whose transformation form the original t-norm, determine a group of automorphisms $\operatorname{Aut}(T)$. This group for archimedean $t$-norms is described by Theorem 3.4.

## 4 Conclusion

This paper shows some conditions under which the $\varphi$-transformations of the $t$-norms and uninorms are invariant. Due to restricted space we skip most of the proofs. But we plan to generalize these results and write a more detailed article.

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# Generalization of the discrete Choquet integral 

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#### Abstract

In this paper, we present an approach to generalization of the discrete Choquet integral. We replace the product operator joining capacity $m$ of criteria sets and values of score vector by a fusion function $F$ satisfying some constraints, similarly as was already done for another form of the discrete Choquet integral in [2]. The properties of obtained functional $C_{F}^{m}$ are studied and some examples for particular capacities $m$ are given.


Keywords: Choquet integral, fusion function

## 1 Introduction

The evaluation of a score vector achieved in some set of criteria is a long-term point of interest in multicriteria decision making theory. One of the useful tools used for that purpose is the Choquet integral [1], which was generalized in several ways, see, for instance, [3], [4].

Our generalization was inspired by that of Mesiar et al. in [2], where the authors generalized one of the two usually used discrete forms of the Choquet integral (see below, the formula (1)) replacing the product operator by fusion function satisfying certain conditions. Using the same idea, we generalize the other formula (see the formula (2)) for the discrete Choquet integral. Note that, in general, the resulting functional differs from that obtained in [2].

We recall the definition of the Choquet integral on a general monotone measure space ( $X, \mathfrak{S}, m$ ), where $X$ is a non-empty set $X, \mathfrak{S}$ is a $\sigma$-algebra of its subsets and $m: \mathfrak{S} \rightarrow[0, \infty]$ a monotone measure, i.e., a set function satisfying the properties $m(\emptyset)=0$ and $m(A) \leq m(B)$ for all $A, B \in \mathfrak{S}, A \subseteq B$.

Definition 1.1. Let $(X, \mathfrak{S}, m)$ be a monotone measure space. For any $\mathfrak{S}$-measurable function $f: X \rightarrow$ $[0,1]$ the Choquet integral $C h_{m}(f)$ is given by

$$
C h_{m}(f)=\int_{0}^{1} m(\{x \in X \mid f(x) \geq t\}) \mathrm{d} t
$$

where the integral on the right-hand side is the Riemann integral.
In this paper we will only deal with finite spaces $X=\{1, \ldots, n\}$ for some $n \in N, n \geq 2, \mathfrak{S}=2^{X}$ and normalized monotone measures $m: 2^{X} \rightarrow[0,1]$, i.e., monotone measures with $m(X)=1$, calling them capacities [5]. The set of all capacities $m: 2^{X} \rightarrow[0,1]$ will be denoted by $\mathcal{M}_{n}$. Any $2^{X}$. measurable function $f: X \rightarrow[0,1]$ will be identified with a vector $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right) \in[0,1]^{n}$, where $x_{i}=f(i), i=1, \ldots, n$.

A discrete form of the Choquet integral is of a great importance in decision making theory, regarding a finite set $X=\{1, \ldots, n\}$ as some criteria set, a vector $\mathbf{x} \in[0,1]^{n}$ as a score vector and a capacity $m: 2^{X} \rightarrow[0,1]$ as the weights of particular sets of criteria.

[^5]Proposition 1.2. Let $X=\{1, \ldots, n\}$ and let $m: 2^{X} \rightarrow[0,1]$ be a capacity. Then for any $\mathbf{x} \in[0,1]^{n}$ the discrete Choquet integral is given by

$$
\begin{equation*}
\mathcal{C} h_{m}(\mathbf{x})=\sum_{i=1}^{n}\left(x_{(i)}-x_{(i-1)}\right) \cdot m\left(E_{(i)}\right) \tag{1}
\end{equation*}
$$

where $(\cdot): X \rightarrow X$ is a permutation such that $x_{(1)} \leq \cdots \leq x_{(n)}, E_{(i)}=\{(i), \ldots,(n)\}$ for $i=$ $1, \ldots, n$, and $x_{(0)}=0$,
or, equivalently, by

$$
\begin{equation*}
\mathcal{C} h_{m}(\mathbf{x})=\sum_{i=1}^{n} x_{(i)} \cdot\left(m\left(E_{(i)}\right)-m\left(E_{(i+1)}\right)\right) \tag{2}
\end{equation*}
$$

with $x_{(i)}$ and $E_{(i)}, i=1, \ldots, n$, as above, and $E_{(n+1)}=\emptyset$.
Observe that information contained in a score vector and that in a capacity are joined by the standard product operator. Replacing the product in formulae (1) and (2) by a function $F:[0,1]^{2} \rightarrow[0,1]$ (a binary fusion function), we obtain the formulae:

$$
\begin{equation*}
C_{m}^{F}(\mathbf{x})=\sum_{i=1}^{n} F\left(x_{(i)}-x_{(i-1)}, m\left(E_{(i)}\right)\right) \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{F}^{m}(\mathbf{x})=\sum_{i=1}^{n} F\left(x_{(i)}, m\left(E_{(i)}\right)-m\left(E_{(i+1)}\right)\right) \tag{4}
\end{equation*}
$$

respectively.
The functionals $C_{F}^{m}$ defined by (3) were deeply studied in [2] including a complete characterization of functionals $C_{m}^{F}$ as aggregation functions.

In this paper, we will analyse the functionals defined by (4). The paper is organized as follows. In the next section, we provide the conditions under which a functional $C_{F}^{m}$ is correctly defined for any capacity $m \in \mathcal{M}_{n}$ and any $\mathbf{x} \in[0,1]^{n}$ and, for suitable fusion functions, we exemplify $C_{F}^{m}$ for several particular capacities. In Section 3, we provide several properties of functionals $C_{F}^{m}$ and show their connection with the discrete Choquet integral. Finally, some concluding remarks are added.

## 2 Operators $C_{F}^{m}$

Let us first analyse conditions under which the functionals $C_{F}^{m}$ introduced in (4) are well defined.
Evidently, for a score vector $\mathbf{x} \in[0,1]^{n}$ with card $\left\{x_{1}, \ldots, x_{n}\right\}=n$ there is a unique permutation $(\cdot): X \rightarrow X$ such that $x_{(1)} \leq \cdots \leq x_{(n)}$ (in fact, all inequalities are strict). Thus $C_{F}^{m}$ is correctly defined by formula (4). If some ties occure, i.e., if $\operatorname{card}\left\{x_{1}, \ldots, x_{n}\right\}<n$, we have to analyse the following two cases.
Case 1: Let $n=2$. Consider $\mathbf{x}=\left(x_{1}, x_{2}\right)=(x, x)$, and a capacity $m_{a, b} \in \mathcal{M}_{2}$ defined by $m_{a, b}(\{1\})=$ $a$ and $m_{a, b}(\{2\})=b$, where $a, b \in[0,1]$. Then $C_{F}^{m_{a, b}}(x, x)$ is well defined only if formula (4) gives back the same value for both possible permutations $(1,2)$ and $(2,1)$ ordering the vector $\mathbf{x}$ increasingly, i.e., if it holds

$$
F(x, 1-a)+F(x, a)=F(x, 1-b)+F(x, b)
$$

for all $a, b \in[0,1]$.
Consequently, we obtain the following proposition.
Proposition 2.1. Let $n=2$. Then $C_{F}^{m}:[0,1]^{2} \rightarrow[0,2]$ introduced in (4) is well defined if and only if

$$
F(x, y)+F(x, 1-y)=2 F(x, 1 / 2), \quad \text { for any } x, y \in[0,1]
$$

We can immediately characterize all well defined functionals $C_{F}^{m_{a, b}}$ :

$$
C_{F}^{m_{a, b}}(x, y)= \begin{cases}F(x, 1-b)+F(y, b) & \text { if } x<y \\ 2 F(x, 1 / 2) & \text { if } x=y \\ F(x, a)+F(y, 1-a) & \text { if } x>y\end{cases}
$$

Example 2.2. Consider $F:[0,1]^{2} \rightarrow[0,1]$ defined by $F(x, y)=\frac{x}{2}\left((2 y-1)^{3}+1\right)$, see Fig. 1. Then $F$ satisfies the constraints of Proposition 2.1. and thus $C_{F}^{m}$ is correctly defined for any $m_{a, b} \in \mathcal{M}_{2}$. Note that then

$$
C_{F}^{m_{a, b}}(x, y)= \begin{cases}\frac{x+y}{2}+\frac{(y-x)}{2}(2 b-1)^{3} & \text { if } x \leq y \\ \frac{x+y}{2}+\frac{(x-y)}{2}(2 a-1)^{3} & \text { otherwise }\end{cases}
$$

see Fig. 2.
If $a=b$, i.e., $m_{a, a}$ is a symetric capacity, then

$$
C_{F}^{m_{a, a}}(x, y)=\frac{x+y}{2}+\frac{|x-y|}{2}(2 a-1)^{3} \quad \text { for all } x, y \in[0,1] .
$$



Figure 1: $F(x, y)=\frac{x}{2}\left((2 y-1)^{3}+1\right)$


Figure 2: $C_{F}^{m_{a, b}}, a=0.85, b=0.95$

Case 2: Now, consider $n>2$ and a vector $\mathbf{x}=\left(x_{1}, \cdots, x_{n}\right) \in[0,1]^{n}$ such that $\operatorname{card}\left\{x_{1}, \cdots, x_{n}\right\}<n$. Without loss of generality we can suppose that $\operatorname{card}\left\{x_{1}, \cdots, x_{n}\right\}=n-1$ and $x_{1}=x_{2}=\min \left\{x_{1}, \cdots, x_{n}\right\}=x$. Then, similarly as before, we find out that $C_{F}^{m}(\mathbf{x})$ is well defined only if

$$
\begin{gathered}
F(x, 1-m(\{2,3, \cdots, n\}))+F(x, m(\{2,3, \cdots, n\})-m(\{3, \cdots, n\}))= \\
F(x, 1-m(\{1,3, \cdots, n\}))+F(x, m(\{1,3, \cdots, n\})-m(\{3, \cdots, n\})) .
\end{gathered}
$$

The last equality has to be satisfied for any capacity $m \in \mathcal{M}_{n}$, i.e., for any $\alpha, \beta, \gamma, \delta \in[0,1]$ such that $\alpha+\beta=\gamma+\delta \in[0,1]$ it should hold that

$$
F(x, \alpha)+F(x, \beta)=F(x, \gamma)+F(x, \delta) .
$$

The only solution of this Cauchy's equation is of the form

$$
\begin{equation*}
F(x, y)=f(x) \cdot y \tag{5}
\end{equation*}
$$

where $f:[0,1] \rightarrow[0,1]$ is an arbitrary function. On the other hand, any function $F$ of the form (5) yields a well defined functional $C_{F}^{m}:[0,1]^{n} \rightarrow[0, n]$.

Proposition 2.3. Let $n>2$. The functional $C_{F}^{m}:[0,1]^{n} \rightarrow[0, n]$ is well defined for any $m \in \mathcal{M}_{n}$ if and only if $F(x, y)=f(x) \cdot y$ for all $x, y \in[0,1]$ and some function $f:[0,1] \rightarrow[0,1]$. In that case

$$
\begin{equation*}
C_{F}^{m}(\mathbf{x})=\sum_{i=1}^{n} f\left(x_{(i)}\right) \cdot\left(m\left(E_{(i)}\right)-m\left(E_{(i+1)}\right)\right) . \tag{6}
\end{equation*}
$$

Example 2.4. Consider $F:[0,1]^{2} \rightarrow[0,1]$ given by $F(x, y)=(1-x) y$, which satisfies Proposition 2.3. Then for each $m \in \mathcal{M}_{n}$ and $\mathbf{x} \in[0,1]^{n}$ it holds:

$$
C_{F}^{m}(\mathbf{x})=1-\mathcal{C} h_{m}(\mathbf{x})=\mathcal{C} h_{m^{d}}(\mathbf{1}-\mathbf{x})
$$

where $m^{d}$ is a dual capacity to $m$, given by $m^{d}(E)=1-m\left(E^{c}\right)$. Note that $C_{F}^{m}$ is a decreasing operator, $C_{F}^{m}(0, \ldots, 0)=1$ and $C_{F}^{m}(1, \ldots, 1)=0$.

Using (6), for a fixed suitable fusion function $F$ given by (5), we can derive $C_{F}^{m}$ for some particular capacities $m \in \mathcal{M}_{n}$, see the following table.

| $m \in \mathcal{M}_{n}$ | $C_{F}^{m} ; \quad F(x, y)=f(x) \cdot y$ |
| :---: | :---: |
| $m^{*}(E)= \begin{cases}1 & \text { if } E \neq \emptyset \\ 0 & \text { if } E=\emptyset\end{cases}$ | $C_{F}^{m^{*}}(\mathbf{x})=f\left(x_{(n)}\right)=f\left(\max _{1 \leq i \leq n} x_{i}\right)$ |
| $m_{*}(E)= \begin{cases}1 & \text { if } E=\{1, \cdots, n\} \\ 0 & \text { otherwise }\end{cases}$ | $C_{F}^{m_{*}}(\mathbf{x})=f\left(x_{(1)}\right)=f\left(\min _{1 \leq i \leq n} x_{i}\right)$ |
| $\begin{gathered} m_{H}(E)= \begin{cases}1 & \text { if } H \subseteq E, \\ 0 & \text { otherwise }\end{cases} \\ \emptyset \neq H \subseteq X \end{gathered}$ | $\begin{gathered} C_{F}^{m_{H}}(\mathbf{x})=f\left(x_{i}\right), \text { where } \\ \left\{j \in\{1, \cdots, n\} \mid x_{j} \geq x_{i}\right\} \supseteq H \text { but } \\ \left\{j \in\{1, \cdots, n\} \mid x_{j}>x_{i}\right\} \supseteq H \text { does not hold } \end{gathered}$ |
| $\bar{m}(E)=\frac{\operatorname{card} E}{n}$ | $C_{F}^{\bar{m}}(\mathbf{x})=\frac{1}{n} \sum_{i=1}^{n} f\left(x_{i}\right)$ |

Note that $m^{*}$ and $m_{*}$ are the greatest and the smallest elements of $\mathcal{M}_{n}$, respectively, and that $m_{*}=$ $m_{H}$ for $H=X$.

## $3 C_{F}^{m}$ with some particular properties

In this section, we formulate several properties of functionals $C_{F}^{m}$ and also show the connection $C_{F}^{m}$ with the discrete Choquet integral.

At first, we recall that the functional $C_{F}^{m}$ is:

- an aggregation function, if $C_{F}^{m}$ is monotone increasing and $C_{F}^{m}(\mathbf{0})=0, C_{F}^{m}(\mathbf{1})=1$;
- a mean, if for each $\mathbf{x} \in[0,1]^{n}$ it holds $\operatorname{Min}(\mathbf{x}) \leq C_{F}^{m}(\mathbf{x}) \leq \operatorname{Max}(\mathbf{x})$, where $\operatorname{Min}(\mathbf{x})=$ $\min \left\{x_{1}, \ldots, x_{n}\right\}, \operatorname{Max}(\mathbf{x})=\max \left\{x_{1}, \ldots, x_{n}\right\} ;$
- translation invariant, if $C_{F}^{m}\left(x_{1}+c, \ldots, x_{n}+c\right)=c+C_{F}^{m}\left(x_{1}, \ldots, x_{n}\right)$ for all $\left.\left.c \in\right] 0,1\right]$ and $\left(x_{1}, \ldots, x_{n}\right) \in[0,1]^{n}$ such that $\left(x_{1}+c, \ldots, x_{n}+c\right) \in[0,1]^{n}$;
- idempotent, if $C_{F}^{m}(x, \ldots, x)=x$ for each $x \in[0,1]$.

For the functionals $C_{F}^{m}$ of the form (6) with $F$ satisfying (5), the following properties can be directly derived.

Proposition 3.1. Let $F:[0,1]^{2} \rightarrow[0,1], F(x, y)=f(x) \cdot y$, where $f:[0,1] \rightarrow[0,1]$. Then for any fixed $n \geq 2$ it holds that
(i) $C_{F}^{m}$ is an aggregation function for each $m \in \mathcal{M}_{n}$ if and only if $f$ is an increasing function satisfying $f(0)=0$ and $f(1)=1$.
(ii) $C_{F}^{m} \geq \operatorname{Min}$ for each $m \in \mathcal{M}_{n}$ if and only if $f$ is an increasing function satisfying $f(x) \geq x$ for all $x \in[0,1]$.
(iii) $C_{F}^{m} \leq \operatorname{Max}$ for each $m \in \mathcal{M}_{n}$ if and only if $f$ is an increasing function satisfying $f(x) \leq x$ for all $x \in[0,1]$.
(iv) $C_{F}^{m}$ is a mean for each $m \in \mathcal{M}_{n}$ if and only if $F$ is the product operator.
(v) $C_{F}^{m}$ is translantion invariant for each $m \in \mathcal{M}_{n}$ if and only if $F$ is the product operator.
(vi) $C_{F}^{m}$ is idempotent for each $m \in \mathcal{M}_{n}$, if and only if $F$ is the product operator.

Note that for the standard product $F(x, y)=x . y$ the functional $C_{F}^{m}$ coincides with $\mathcal{C} h_{m}$, therefore the properties (iv), (v), (vi) hold only for the Choquet integral itself.

Since an increasing function preserves ordering of an input vector and a decreasing one inverts it, we obtain the following propositions that show the connection between $C_{F}^{m}$ and the discrete Choquet integral.

Proposition 3.2. Let $F:[0,1]^{2} \rightarrow[0,1], F(x, y)=f(x) \cdot y$, where $f:[0,1] \rightarrow[0,1]$ is an increasing function. Then, for each $m \in \mathcal{M}_{n}$ and $\mathbf{x} \in[0,1]^{n}$,

$$
C_{F}^{m}(\mathbf{x})=\mathcal{C} h_{m}(f(\mathbf{x})),
$$

where $f(\mathbf{x})=\left(f\left(x_{1}\right), \cdots, f\left(x_{n}\right)\right)$.
Proposition 3.3. Let $F:[0,1]^{2} \rightarrow[0,1], F(x, y)=f(x) \cdot y$, where $f:[0,1] \rightarrow[0,1]$ is a decreasing function. Then, for each $m \in \mathcal{M}_{n}$ and $\mathbf{x} \in[0,1]^{n}$,

$$
C_{F}^{m}(\mathbf{x})=1-\mathcal{C} h_{m}(1-f(\mathbf{x}))=\mathcal{C} h_{m^{d}}(f(\mathbf{x}))
$$

where $m^{d} \in \mathcal{M}_{n}$ is a capacity dual to $m$.
Note that the last property was already ilustrated for a special function $F$ in Example 2.4.

## 4 Concluding remarks

We have generalized the formula (2) for the discrete Choquet integral, replacing the standard product operator by a function $F:[0,1]^{2} \rightarrow[0,1]$. Several particular operators $C_{F}^{m}$ were discussed, based either on a fixed capacity $m \in \mathcal{M}_{n}$ or on a fixed function $F$. We expect applications of our results in all domains where the generalizations of the discrete Choquet integral are considered, for example in medicine.

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# On a preorder relation induced by uninorms 

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#### Abstract

In this paper we study a pre-order $\preceq_{U}$ induced by uninorms $U$. We will be interested especially in cases when $\preceq_{U}$ is not an ordering. We will also study algebraic properties of equivalence classes $\sim_{U}$. We will show examples of uninorms when the uninorm restricted to an equivalence class $A$ is an Abelian subgroup of the monoid ( $[0,1], U, e$ ), examples when this is an Abelian group but not a subgroup of $([0,1], U, e)$, as well as examples when an equivalence class $A$ cannot be organized into a group. We will further see that the Abelian groups occurring in the monoid $([0,1], U, e)$, may have non-trivial subgroups.


Keywords: uninorm, pre-order, partition

## 1 Introduction and known facts

Uninorms since their introduction by Dombi [2] under the name aggregative operator, and later reintroduction by Yager and Rybalov [10], have found broad interest among researchers, and also broad applicability in many areas, such as decision making, fuzzy control, etc. Uninorms were proposed by Yager and Rybalov as a natural generalization of both, t -norms and t -conorms. Because of lack of space, for basic properties on $t$-norms and $t$-conorms we refer readers to $[7,9]$.

### 1.1 Uninorms

The definition of uninorm proposed by Yager and Rybalov [10] is the following.
Definition 1.1. A uninorm $U$ is a function $U:[0,1]^{2} \rightarrow[0,1]$ that is increasing, commutative, associative and has a neutral element $e \in[0,1]$.

An overview of basic properties of uninorms is in [1]. For an overview of known classes of uninorms see [8].

A uninorm $U$ is said to be conjunctive if $U(x, 0)=0$, and $U$ is said to be disjunctive if $U(1, x)=1$, for all $x \in[0,1]$.

A uninorm $U$ is called representable if it can be written in the form

$$
U(x, y)=g^{-1}(g(x)+g(y)),
$$

where $g:[0,1] \rightarrow[-\infty, \infty]$ is a continuous strictly increasing function with $g(0)=-\infty$ and $g(1)=\infty$. Note yet that for each generator $g$ there exist two different uninorms depending on convention we take $\infty-\infty=\infty$, or $\infty-\infty=-\infty$. In the former case we get a disjunctive uninorm, in the latter case a conjunctive uninorm.

Representable uninorms are "almost continuous", i.e., they are continuous everywhere on $[0,1]^{2}$ except of points $(0,1)$ and $(1,0)$.

Conjunctive and disjunctive uninorms are dual in the following way

$$
U_{d}(x, y)=1-U_{c}(1-x, 1-y),
$$

[^6]where $U_{c}$ is an arbitrary conjunctive uninorm and $U_{d}$ its dual disjunctive uninorm. Assuming $U_{c}$ has a neutral element $e$, the neutral element of $U_{d}$ is $1-e$.

For an arbitrary uninorm $U$ and arbitrary $(x, y) \in] 0, e[\times] e, 1] \cup] e, 1] \times] 0, e[$ we have

$$
\begin{equation*}
\min \{x, y\} \leq U(x, y) \leq \max \{x, y\} \tag{1}
\end{equation*}
$$

We say that a uninorm $U$ contains a homomorphic image of a representable uninorm in $] a, b\left[^{2}\right.$ for $0 \leq a<e<b \leq 1$ (where $a \neq 0$ and/or $b \neq 1$ ), if there exists a continuous strictly increasing function $\tilde{g}:[a, b] \rightarrow[-\infty, \infty]$ such that $\tilde{g}(a)=-\infty, \tilde{g}(b)=\infty, \tilde{g}(e)=0$ and

$$
\begin{equation*}
\left.U(x, y)=\tilde{g}^{-1}(\tilde{g}(x)+\tilde{g}(y)) \quad \text { for } x, y \in\right] a, b[. \tag{2}
\end{equation*}
$$

### 1.2 Orders induced by t-norms

In [6] t-norms on bounded lattices were introduced.
Definition 1.2. Let $L$ be a bounded lattice. A function $T: L^{2} \rightarrow L$ is said to be a t-norm if $T$ is commutative, associative, monotone and $\mathbf{1}_{L}$ is its neutral element.

Each uninorm $U$ with a neutral element $0<e<1$, when restricted to the square $[0, e]^{2}$, is a $t$-norm (on the lattice $L=[0, e]$ equipped with meet and join) and when restricted to the square $[e, 1]^{2}$, is a t -conorm (on the lattice $L=[e, 1]$ equipped with meet and join). We will denote this t-norm by $T_{U}$ and the t-conorm by $S_{U}$.

In [5], for a given t-norm $T$ on a bounded lattice $L$ a relation $\preceq_{T}$ generated by $T$ was introduced. The definition is as follows

Definition 1.3. Let $T: L^{2} \rightarrow L$ be a given t-norm. For arbitrary $x, y \in L$ we denote $x \preceq_{T} y$ if there exists $\ell \in L$ such that $T(y, \ell)=x$.

Proposition 1.4. ([5]) Let $T$ be an arbitrary t-norm. The relation $\preceq_{T}$ is a partial order. Moreover, if $x \preceq_{T} y$ holds for $x, y \in L$ then $x \leq y$, where $\leq i$ is the order generated by lattice operations.

Dually, we can introduce a partial order $\preceq_{S}$ for arbitrary t-conorm $S$ by

$$
x \preceq_{S} y \quad \text { if there exists } \ell \in[0,1] \text { such that } S(y, \ell)=x
$$

However, in this case we have

$$
x \preceq_{S} y \quad \Rightarrow \quad x \geq y
$$

### 1.3 Relation $\preceq_{U}$

As a generalization of the relation $\preceq_{T}$, Hliněná, Kalina and Král' in [3] introduced relation $\preceq_{U}$.
Definition 1.5 ([3]). Let $U$ be arbitrary uninorm. $B y \preceq_{U}$ we denote the following relation

$$
x \preceq_{U} y \quad \text { if there exists } \ell \in[0,1] \text { such that } U(y, \ell)=x .
$$

Associativity of $U$ implies transitivity of $\preceq_{U}$. Existence of a neutral element $e$ implies reflexivity of $\preceq_{U}$. However, anti-symmetry of $\preceq_{U}$ is rather problematic.

Since for representable uninorm $U$ and for arbitrary $x \in] 0,1\left[\right.$ and $y \in[0,1]$ there exists $\ell_{y}$ such that $U\left(x, \ell_{y}\right)=y$, the relation $\preceq_{U}$ is not necessarily anti-symmetric.
Lemma 1.6 ([3]). Let $U$ be arbitrary uninorm. The relation $\preceq_{U}$ is a pre-order.
We introduce a relation $\sim_{U}$.
Definition 1.7 ([3]). Let $U$ be arbitrary uninorm. We say that $x, y \in[0,1]$ are $U$-indifferent if

$$
x \preceq_{U} y \quad \text { and } \quad y \preceq_{U} x .
$$

If $x, y$ are $U$-indifferent, we write $x \sim_{U} y$.
Lemma 1.8 ([3]). For arbitrary uninorm $U$ the relation $\sim_{U}$ is an equivalence relation.

## 2 Properties of uninorms induced by the relation $\preceq_{U}$

For arbitrary uninorm $U$ with neutral element $e$ the uninorm can be considered to be a binary operation on $[0,1]$. Thus ( $[0,1], U, e$ ) becomes a commutative (i.e., Abelian) monoid which is moreover isotone with respect to the standard ordering of reals.

Lemma 2.1. Let $U:[0,1]^{2} \rightarrow[0,1]$ be an arbitrary uninorm and $x_{1}, x_{2} \in[0,1]$ such that $x_{1} \sim_{U} x_{2}$. Then

$$
\begin{equation*}
U\left(x_{1}, x_{1}\right) \sim_{U} U\left(x_{2}, x_{2}\right) \sim_{U} U\left(x_{1}, x_{2}\right) . \tag{3}
\end{equation*}
$$

A direct corollary to Lemma 2.1 is the next assertion.
Proposition 2.2. Let $U:[0,1]^{2} \rightarrow[0,1]$ be an arbitrary uninorm and $\left.e \in\right] 0,1[$ be its neutral element. Assume there exists $x \in[0,1], x \neq e$ and $y \in[0,1]$ such that $U(x, y)=e$. Then there exists a set $A_{e} \subset[0,1]$ such that for all $x \in A_{e}, x \sim_{U} e$. Moreover, if we denote by $\odot$ the binary operation on $A_{e}$ defined by $x \odot y=U(x, y)$ for all $x, y \in A_{e}$, then $\left(A_{e}, \odot, e\right)$ is a non-trivial Abelian subgroup of $([0,1], U, e)$.

Lemma 2.1 can be further generalized.
Proposition 2.3. Let $U:[0,1]^{2} \rightarrow[0,1]$ be an arbitrary uninorm and $x_{1}, x_{2}, y \in[0,1]$. Then the following holds

$$
\begin{equation*}
\left(x_{1} \sim_{U} x_{2}\right) \quad \Rightarrow \quad\left(U\left(x_{1}, y\right) \sim_{U} U\left(x_{2}, y\right)\right) . \tag{4}
\end{equation*}
$$

As Proposition 2.3 shows, the set $A_{e}$ of all elements of $[0,1]$ which are indifferent from $e$ generates classes of indifferent elements. In general, we have the following possibilities.

Proposition 2.4. Let $U$ be a fixed uninorm and $e$ its neutral element. For arbitrary $x \in[0,1]$ denote the set $A_{x}=\left\{y \in[0,1] ; x \sim_{U} y\right\}$. Then $A_{x}$ is either a singleton or an infinite set.
Further, assume that for a fixed $x$ the set $A_{x}$ is infinite and denote by $\odot_{x}$ the binary operation on $A_{x}$ defined by $x \odot y=U(x, y)$ for all $x, y \in A_{x}$. Then there are the following possibilities.

- $A_{x}=A_{e}$ and $\left(A_{x}, \odot_{x}, e\right)$ is a non-trivial Abelian subgroup of $([0,1], U, e)$.
- $A_{x} \neq A_{e}$ and $\left(A_{x}, \odot_{x}, \tilde{e}\right)$ is a non-trivial Abelian group with a neutral element $\tilde{e} \neq e$. In this case $\left(A_{x}, \odot_{x}, \tilde{e}\right)$ is not a subgroup of $([0,1], U, e)$.
- $x \not \chi_{U} U(x, x)$, i.e., $\odot_{x}$ is an operation on $A_{x}$ but into $[0,1] \backslash A_{x}$. Moreover, there exists $y \in[0,1]$ such that $\left(A_{y}, \odot_{y}, \tilde{e}\right)$ is a non-trivial Abelian group and for all $z \in A_{y}$ we have $U(x, z) \sim_{U} x$.


## 3 Illustrative examples

In this section we provide illustrative examples. The first example shows uninorm $U_{1}$ which generates two infinite indifference classes with respect to $\sim_{U_{1}}$. One indifference class is $A_{e}$ such that ( $\left.A_{e}, \odot_{e}, e\right)$ is a non-trivial subgroup of $\left([0,1], U_{1}, e\right)$. The other indifference class $A_{\frac{1}{8}}$ is such that for all $x \in A_{\frac{1}{8}}$ and all $y \in A_{e}$ we have $U(x, y) \in A_{\frac{1}{8}}$. The set $A_{\frac{1}{8}}$ cannot be organized into a group.

Example 3.1 ([3]). We recall the construction of a conjunctive uninorm which contains a homomorphic image of a representable uninorm $U_{r}$ on $] \frac{1}{4}, \frac{3}{4}\left[{ }^{2}\right.$. Further, on the rectangle $\left[0, \frac{1}{4}\left[\times\left[\frac{1}{4}, \frac{3}{4}\right]\right.\right.$ the values of $U_{1}$
are given by the partial function $U_{\frac{1}{8}}(z)=\frac{z-\frac{1}{4}}{2}$. The explicit formula for the uninorm $U_{1}$ is the following

$$
U_{1}(x, y)= \begin{cases}0 & \text { if } \min \{x, y\}=0 \\ \text { or if } \max \{x, y\} \leq \frac{1}{4}, \\ 1 & \text { if } \min \{x, y\} \geq \frac{3}{4}, \\ \frac{1}{4} & \text { if } 0<\min \{x, y\} \leq \frac{1}{4} \\ \text { and if } \max \{x, y\} \geq \frac{3}{4}, \\ & \text { or if } \min \{x, y\}=\frac{1}{4} \\ \text { and } \max \{x, y\}>\frac{1}{4}, \\ & \text { if }(x, y) \in] \frac{1}{4},\left.\frac{3}{4}\right|^{2}, \\ U_{r}(x, y) \\ \max \{x, y\} & \text { if } \frac{1}{4}<\min \{x, y\}<\frac{3}{4} \\ & \text { and } \max \{x, y\} \geq \frac{3}{4},\end{cases}
$$

and values on $] 0, \frac{1}{4}[\times] \frac{1}{4}, \frac{3}{4}[$ and $] \frac{1}{4}, \frac{3}{4}[\times] 0, \frac{1}{4}\left[\right.$ are given by the partial function $U_{\frac{1}{8}}$ by formula (5) showing the computation of the value at a point $\left.\left(x_{2}, y_{2}\right) \in\right] 0, \frac{1}{4}[\times] \frac{1}{4}, \frac{3}{4}\left[\right.$. Assume that $x_{2}=U\left(\frac{1}{8}, y_{1}\right)$. Then

$$
\begin{equation*}
U\left(x_{2}, y_{2}\right)=U\left(\frac{1}{8}, U\left(y_{1}, y_{2}\right)\right) . \tag{5}
\end{equation*}
$$

The uninorm $U_{1}$ and its level-set functions of levels $\frac{1}{16}, \frac{1}{8}, \frac{3}{16}$ are sketched on Fig. 1 .


Figure 1: Uninorm $U_{1}$

The next example shows a uninorm $U_{2}$ which generates two infinite indifference classes with respect to $\sim_{U_{2}}$. One indifference class is $A_{e}$ such that $\left(A_{e}, \odot_{e}, e\right)$ is a non-trivial subgroup of the monoid ( $\left.[0,1], U_{2}, e\right)$. The other indifference class $A$ equipped with operation $\odot_{A}=U_{2} \upharpoonright A$ is also an Abelian group, but not a subgroup of the monoid $\left([0,1], U_{2}, e\right)$.

Example 3.2. The construction of the uninorm $U_{2}$ we are going to present in this example, is based on the idea of paving that was introduced in [4]. The idea is the following. We split the unit interval into countably many disjoint subintervals $\left\{I_{i}\right\}_{i \in \mathbb{Z}}$ (in such a way we split the unit square into countably many disjoint sub-rectangles $I_{i} \times I_{j}$ ). Then we choose an operation $\otimes:[0,1]^{2} \rightarrow[0,1]$ we want to use for paving, choose increasing bijective transformations $\varphi_{i}: I_{i} \rightarrow[0,1]$ and we "pave" the whole unit square (see Fig. 2 for a graphical schema of paving) by

$$
\begin{equation*}
\varphi_{i+j}^{-1}\left(\varphi_{i}(x) \otimes \varphi_{j}(y)\right) . \tag{6}
\end{equation*}
$$

| $I_{i+2}$ | $I_{2 i}$ | $I_{2 i+1}$ | $I_{2 i+2}$ | $I_{2 i+3}$ |
| :---: | :---: | :---: | :---: | :---: |
| $a_{i+1}$ | $I_{2 i+4}$ |  |  |  |
| $I_{i+1}$ | $I_{2 i-1}$ | $I_{2 i}$ | $I_{2 i+1}$ | $I_{2 i+2}$ |
| $a_{i}$ | $I_{2 i+3}$ |  |  |  |
| $I_{i}$ | $I_{2 i-2}$ | $I_{2 i-1}$ | $I_{2 i}$ | $I_{2 i+1}$ |
| $a_{i-1}$ | $I_{2 i+2}$ |  |  |  |
| $I_{i-1}$ | $I_{2 i-3}$ | $I_{2 i-2}$ | $I_{2 i-1}$ | $I_{2 i}$ |
| $a_{i-2}$ | $I_{2 i+1}$ |  |  |  |
| $I_{i-2}$ | $I_{2 i-4}$ | $I_{2 i-3}$ | $I_{2 i-2}$ | $I_{2 i-1}$ |
| $a_{2 i-}^{*}$ | $I_{2 i}$ |  |  |  |
| $a_{i-3} \downarrow$ | $I_{i-2^{a_{i-2}}} I_{i-1} I_{i-1}^{a_{i-1}}$ | $I_{i}$ | $a_{i}$ | $I_{i+1^{a_{i+1}}} I_{i+2}$ |

Figure 2: Graphical schema of paving
To make each point $(x, y) \in[0,1]^{2}$ uniquely identifiable with a rectangle $I_{i} \times I_{j}$, we will use leftopen intervals. Then the operation $\otimes$ used for paving must be without zero-divisors. In this case we choose a representable uninorm $U_{r}$ as the operation $\otimes$, and the following partition of $] 0,1[$ :

$$
I_{i}= \begin{cases}] \frac{1}{4}, \frac{3}{4}\right] & \text { if } i=0  \tag{7}\\ ] \frac{2^{i+1}-1}{2^{i+1}}, \frac{2^{i+2}-1}{2^{i+2}}\right] & \text { if } i>0, \\ ] \frac{1}{2^{2-i}}, \frac{1}{2^{1-i}}\right] & \text { if } i<0\end{cases}
$$

The uninorm $U_{2}$ is defined by

$$
U_{2}(x, y)= \begin{cases}\varphi_{i+j}^{-1}\left(U_{r}\left(\varphi_{i}(x), \varphi_{j}(y)\right)\right) & \text { if } x \in I_{i}, y \in I_{j} \\ 0 & \text { if } \min \{x, y\}=0 \\ 1 & \text { if } \max \{x, y\}=1 \text { and } \min \{x, y\} \neq 0\end{cases}
$$

Let us remark that the increasing bijective transformations $\varphi_{i}: I_{i} \rightarrow[0,1]$ are chosen arbitrarily (and for every choice we get a different uninorm). The relation $\preceq_{U_{2}}$ generates two indifference classes -$A_{\frac{3}{4}}=\left\{\frac{2^{i+1}-1}{2^{i+1}} ; i \in \mathbb{N}\right\} \cup\left\{\frac{1}{2^{1+i}} ; i \in \mathbb{N}\right\}$, where $\mathbb{N}$ is the set of positive integers, and $\left.A_{e}=\right] 0,1\left[\backslash A_{\frac{3}{4}}\right.$. Then $\left(A_{e}, \odot_{e}, e\right)$ is a non-trivial subgroup of $\left([0,1], U_{2}, e\right)$ and $\left(A_{\frac{3}{4}}, \odot_{\frac{3}{4}}, \frac{3}{4}\right)$ is an Abelian group that is not a subgroup of $\left([0,1], U_{2}, e\right)$.

Remark 3.3. If we look at the two Abelian groups induced by the uninorm $U_{2}$ (Example 3.2), they are in some sense different. While $\left(A_{\frac{3}{4}}, \odot_{\frac{3}{4}}, \frac{3}{4}\right)$ has no non-trivial subgroups, $\left(A_{e}, \odot_{e}, e\right)$ has a non-trivial subgroup, namely (]$\frac{1}{4}, \frac{3}{4}\left[,\left(\odot_{e} \upharpoonright\right] \frac{1}{4}, \frac{3}{4}[), e\right)$.

In the last example we modify the uninorm $U_{2}$ from Example 3.2 in two ways.
Example 3.4. We take the product t -norm $T_{\Pi}$ for the operation $\otimes$ used for paving. We split the interval $] 0,1$ [in the same way as in Example 3.2, i.e., the partition is given by formula (7). As the result of paving we get uninorm $U_{3}$ defined by the following

$$
U_{3}(x, y)= \begin{cases}\varphi_{i+j}^{-1}\left(T_{\Pi}\left(\varphi_{i}(x), \varphi_{j}(y)\right)\right) & \text { if } x \in I_{i}, y \in I_{j}  \tag{8}\\ 0 & \text { if } \min \{x, y\}=0 \\ 1 & \text { if } \max \{x, y\}=1 \text { and } \min \{x, y\} \neq 0\end{cases}
$$

Also in this case we can choose arbitrarily the increasing bijective transformations $\varphi_{i}: I_{i} \rightarrow[0,1]$. I.e., correctly speaking, we have got a system of uninorms. But they all induce the same pre-order $\preceq_{U_{3}}$.

Denote $a_{i}$ and $b_{i}$ the left- and right-end-points of the interval $I_{i}$, respectively. Then for $x_{i} \in I_{i}$ and $x_{j} \in I_{j}$ we have $x_{i} \sim_{U_{3}} x_{j}$ if and only if $\frac{x_{i}-a_{i}}{b_{i}-a_{i}}=\frac{x_{j}-a_{j}}{b_{j}-a_{j}}$. The set $A_{x_{0}}$ for $x_{0} \in I_{0}, x_{0} \neq \frac{3}{4}$, cannot be organized into a group, and $\left(A_{\frac{3}{4}}, \odot_{\frac{3}{4}}, \frac{3}{4}\right)$ is a subgroup of $\left([0,1], U_{3}, \frac{3}{4}\right)$.

If we choose the minimum t-norm, $T_{M}$, instead of the product in the formula (8), and use the same partition given by formula (7), we get again the same system of equivalence classes. But in this case for all $x_{0} \in I_{0}$ the algebraic system $\left(A_{x_{0}}, \odot_{x_{0}}, x_{0}\right)$ is an Abelian group, and for $x_{0}=\frac{3}{4}$ it is a subgroup of $\left([0,1], U_{3}, \frac{3}{4}\right)$.

Remark 3.5. We have seen in Example 3.4 that uninorms $U_{3}, U_{4}$ induce the same pre-order, i.e., $\preceq_{U_{3}}=\preceq_{U_{4}}$. If we look at algebraic properties of equivalence classes got by the pre-orders $\preceq_{U_{3}}$ and $\preceq_{U_{4}}$, they are different. This means, in some cases, when different uninorms induce the same pre-order the underlying algebraic properties of equivalence classes may help to distinguish types of uninorms in question.

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# Fuzzy Bags 

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#### Abstract

This study introduces a revised definition for fuzzy bags. It is based on the definition of bags given by Delgado et al. 2009 in which each bag has two parts, function and summary information. Furthermore, the concept of $\alpha$-cuts and related theorems is given. By some examples, the new concepts are illustrated.


Keywords: $\alpha$-cut of fuzzy bags; bags; fuzzy bags

## 1 Introduction

The initial notion of bags was introduced by Yager [1] as an algebraic set-like structure where an element can appear more than once. So far, several works have been done using this new concept. Also, bags have been employed in practice, for instance; in flexible querying, representation of relational information, decision problem analysis, criminal career analysis, and even in fields such as biology.

However, due to some existing drawbacks in the first definition of bags [1], the necessity of a revision of this notion reveals. The proposed definition by Delgado et al. [2] has improved these drawbacks. By some examples, they showed that Yager's definition for bags has some deficiencies and it was not well suited for representing and reasoning with real-world information. Then, they proposed new definitions for bags and fuzzy bags.

As it is shown in [3], the lattice of all fuzzy bags defined by Delgado et al. [2] is a complete Boolean algebra which is not compatible with the nature of fuzziness. Improving this incompatibility, in this paper, we introduce a revised definition for fuzzy bags based on the proposed definition of bags in [2].

## 2 Preliminaries

In this section, some basic concepts which are needed in the sequel are given. For more details, see [2].
Definition 1. [2] Let $P$ and $O$ be two universes (sets) called "properties" and "objects", respectively. A (crisp) bag $\mathcal{B}^{f}$ is a pair $\left(f, B^{f}\right)$, where $f: P \rightarrow \mathcal{P}(O)$ is a function and $B^{f}$ is the following subset of $P \times \mathcal{N}$

$$
B^{f}=\{(p, \operatorname{card}(f(p))) \mid p \in P\} .
$$

Here, $\mathcal{N}$ is the set of natural numbers, $\mathcal{P}(O)$ is the power set of $O, \operatorname{card}(X)$ is the cardinality of set $X$.
We will use the convention here that $\operatorname{card}(\emptyset)=0$.
In this characterization, a bag $\mathcal{B}^{f}$ consists of two parts. The first one is the function $f$ that can be seen as an information source about the relation between objects and properties. The second part $B^{f}$ is a summary of the information in $f$ obtained by means of the count operation $\operatorname{card}($.$) . This summary$ corresponds to the classical view of bags in the sense of [1].

Notation 1. We set $\mathbf{B}(P, O)$ as the set of all bags $\mathcal{B}^{f}=\left(f, B^{f}\right)$ defined in Definition 1.

[^7]Definition 2. Define $\mathcal{B}^{0}=\left(0, B^{0}\right)$ and $\mathcal{B}^{1}=\left(1, B^{1}\right)$ where, $0(p)=\emptyset, 1(p)=O$ for all $p \in P$, $B^{0}=\{(p, 0), p \in P\}$ and $B^{1}=\{(p, \operatorname{card}(O)), p \in P\}$. Clearly, $\mathcal{B}^{0}, \mathcal{B}^{1} \in \mathbf{B}(P, O)$.
Example 1. [2] Let $O=\{$ John, Ana, Bill, Tom, Sue, Stan, Ben $\}$ and $P=\{17,21,27,35\}$ be the set of objects and the set of properties, respectively. Let $f: P \rightarrow \mathcal{P}(O)$ be the function in Table 1 with $f(p) \subseteq O$ for all $p \in P$.

Table 1: Function: age-people.

| p | 17 | 21 | 27 | 35 |
| :---: | :---: | :---: | :---: | :---: |
| $f(p)$ | $\{$ Bill, Sue $\}$ | $\{$ John, Tom, Stan $\}$ | $\emptyset$ | $\{$ Ben $\}$ |

So, we can define bag $\mathcal{B}^{f}=\left(f, B^{f}\right)$ where, $B^{f}=\{(17,2),(21,3),(27,0),(35,1)\}$.
In the next section, we introduce the concept of fuzzy bags and give some results about them.

## 3 Fuzzy Bags

In what follows, $O$ is the set of all objects, and $\mathcal{F}(O)=\{A \mid A: O \rightarrow[0,1]\}$ is the set of all fuzzy subsets of $O$. Also, $i \in I_{n}=\{1,2, \ldots, n\}$, where $n \in \mathcal{N}$ and $\mathcal{N}$ is the set of natural numbers.
Definition 3. A fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}}$ is a pair $\left(\tilde{f}, B^{\tilde{f}}\right)$, where $\tilde{f}: P \rightarrow \mathcal{F}(O)$ is a function and $B^{f}$ is the following subset of $P \times[0,1] \times \mathcal{N}$

$$
B^{\tilde{f}}=\left\{\left(p, \delta, \operatorname{card}\left(O_{\delta}^{p}\right)\right) \mid p \in P, \delta \in[0,1]\right\}
$$

Where, $O_{\delta}^{p}=\{o \in O \mid \tilde{f}(p)(o)=\delta\}$.
Clearly, a crisp bag is a particular case of fuzzy bag where, for all $p \in P, \tilde{f}(p)$ is a crisp subset of $O$. Here, the concept of fuzzy bag is illustrated by an example.
Example 2. Let $O=\{$ Ben, Sue, Tom, John, Stan, Bill, Kim, Ana, Sara $\}$ and $P=\{$ young, middle age, old $\}$ is the set of some linguistic descriptions of age. Let the degrees of membership of all $o \in O$ in the set of each property $p \in P$ are given as in Table 2.

Table 2: The degrees of memberships for Example 2

| p | Ben | Sue | Tom | John | Stan | Bill | Kim | Ana | Sara |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| young | 0.7 | 0.2 | 0.4 | 0.0 | 0.7 | 0.4 | 0.2 | 0.7 | 0.1 |
| middle age | 0.3 | 0.8 | 0.7 | 0.3 | 0.3 | 0.7 | 0.8 | 0.3 | 0.5 |
| old | 0.1 | 0.2 | 0.1 | 0.9 | 0.1 | 0.1 | 0.2 | 0.1 | 0.5 |

So, by Definition 3, we can define fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}}=\left(\tilde{f}, B^{\tilde{f}}\right)$ where,

$$
\begin{aligned}
& \tilde{f}(\text { young })=\left\{\frac{0.7}{\text { Ben }}, \frac{0.2}{\text { Sue }}, \frac{0.4}{\text { Tom }}, \frac{0.0}{\text { John }}, \frac{0.7}{\text { Stan }}, \frac{0.4}{\text { Bill }}, \frac{0.2}{\text { Kim }}, \frac{0.7}{\text { Ana }}, \frac{0.1}{\text { Sara }}\right\}, \\
& \tilde{f}(\text { middle age })=\left\{\frac{0.3}{\text { Ben }}, \frac{0.8}{\text { Sue }}, \frac{0.7}{\text { Tom }}, \frac{0.3}{\text { John }}, \frac{0.3}{\text { Stan }}, \frac{0.7}{\text { Bill }}, \frac{0.8}{\text { Kim }}, \frac{0.3}{\text { Ana }}, \frac{0.5}{\text { Sara }}\right\}, \\
& \tilde{f}(\text { old })=\left\{\frac{0.1}{\text { Ben }}, \frac{0.2}{\text { Sue }}, \frac{0.1}{\text { Tom }}, \frac{0.9}{\text { John }}, \frac{0.1}{\text { Stan }}, \frac{0.1}{\text { Bill }}, \frac{0.2}{\text { Kim }}, \frac{0.1}{\text { Ana }}, \frac{0.5}{\text { Sara }}\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
B^{\tilde{f}}=\{ & (\text { young }, 0.7,3),(\text { young }, 0.4,2),(\text { young }, 0.2,2),(\text { young }, 0.1,1),(\text { young }, 0.0,1) \\
& (\text { middle age }, 0.8,2),(\text { middle age }, 0.7,2),(\text { middle age }, 0.5,1),(\text { middle age }, 0.3,4) \\
& (\text { old }, 0.9,1),(\text { old }, 0.5,1),(\text { old, } 0.2,2),(\text { old, } 0.1,5)\}
\end{aligned}
$$

Remark 1. As it can be seen, the more important part of an fuzzy bag is information function $\tilde{f}$. Therefore, it is possible to study the properties of fuzzy bags just by considering their information functions.

Notation 2. We set $\tilde{\mathbf{B}}(P, O)$ as the set of all fuzzy bags $\tilde{\mathcal{B}} \tilde{f}=\left(\tilde{f}, B^{\tilde{f}}\right)$. Where, $\tilde{f}: P \rightarrow \mathcal{F}(O)$ and $B^{\tilde{f}}$ are as defined in Definition 3. Clearly, $\mathbf{B}(P, O) \subseteq \tilde{\mathbf{B}}(P, O)$.

Here, we can define intersection and union of fuzzy bags.
Definition 4. Let $\tilde{\mathcal{B}}^{\tilde{f}_{i}} \in \tilde{\mathbf{B}}\left(P_{i}, O_{i}\right)$ for all $i \in I_{n}$ be given fuzzy bags and $\bar{O}=\cup_{i \in I_{n}} O_{i}$. Then, their intersection is fuzzy bag

$$
\cap_{i \in I_{n}} \tilde{\mathcal{B}}^{\tilde{f}_{i}}=\left(\cap_{i \in I_{n}} \tilde{f}_{i}, B^{\cap_{i \in I_{n}} \tilde{f}_{i}}\right)
$$

Where, $\cap_{i \in I_{n}} \tilde{f}_{i}: \Pi_{i \in I_{n}} P_{i} \rightarrow \mathcal{F}(\bar{O})$ such that $\left(\cap_{i \in I_{n}} \tilde{f}_{i}\right)\left(p_{1}, p_{2}, \ldots, p_{n}\right)=\cap_{i \in I_{n}} \tilde{f}_{i}\left(p_{i}\right)$ for all $p_{i} \in P_{i}$.
Note that by Definition 3, $\cap_{i \in I_{n}} \tilde{\mathcal{B}}^{\tilde{f}_{i}}=\tilde{\mathcal{B}}^{\cap_{i \in I_{n}} \tilde{f}_{i}}$, where

$$
B^{\cap_{i \in I_{n}} \tilde{f}_{i}}=\left\{\left(\left(p_{1}, p_{2}, \ldots, p_{n}\right), \delta, \operatorname{card}\left(O_{\delta}^{p_{1}, p_{2}, \ldots, p_{n}}\right)\right) \mid p_{i} \in P_{i}, \delta \in[0,1]\right\},
$$

where $O_{\delta}^{p_{1}, p_{2}, \ldots, p_{n}}=\left\{o \in \bar{O} \mid\left(\cap_{i \in I_{n}} \tilde{f}_{i}\right)\left(p_{1}, p_{2}, \ldots, p_{n}\right)(o)=\delta\right\}$.
Definition 5. Let $\tilde{\mathcal{B}}^{\tilde{f}_{i}} \in \tilde{\mathbf{B}}\left(P_{i}, O_{i}\right)$ for all $i \in I_{n}$ be given fuzzy bags and $\bar{O}=\cup_{i \in I_{n}} O_{i}$. Then, their union is fuzzy bag

$$
\cup_{i \in I_{n}} \tilde{\mathcal{B}}^{\tilde{f}_{i}}=\left(\cup_{i \in I_{n}} \tilde{f}_{i}, B^{\cup_{i \in I_{n}} \tilde{f}_{i}}\right)
$$

where $\cup_{i \in I_{n}} \tilde{f}_{i}: \Pi_{i \in I_{n}} P_{i} \rightarrow \mathcal{F}(\bar{O})$ such that $\left(\cup_{i \in I_{n}} \tilde{f}_{i}\right)\left(p_{1}, p_{2}, \ldots, p_{n}\right)=\cup_{i \in I_{n}} \tilde{f}_{i}\left(p_{i}\right)$ for all $p_{i} \in P_{i}$.
Note that by Definition $3, \cup_{i \in I_{n}} \tilde{\mathcal{B}}^{\tilde{f}_{i}}=\tilde{\mathcal{B}}^{\cup_{i \in I_{n}} \tilde{f}_{i}}$, where

$$
B^{\cup_{i \in I_{n}} \tilde{f}_{i}}=\left\{\left(\left(p_{1}, p_{2}, \ldots, p_{n}\right), \delta, \operatorname{card}\left(O_{\delta}^{p_{1}, p_{2}, \ldots, p_{n}}\right)\right) \mid p_{i} \in P_{i}, \delta \in[0,1]\right\},
$$

where $O_{\delta}^{p_{1}, p_{2}, \ldots, p_{n}}=\left\{o \in \bar{O} \mid\left(\cup_{i \in I_{n}} \tilde{f}_{i}\right)\left(p_{1}, p_{2}, \ldots, p_{n}\right)(o)=\delta\right\}$.
Definition 6. A fuzzy bag $\tilde{\mathcal{B}} \tilde{f}$ is a fuzzy sub bag of $\tilde{\mathcal{B}}^{\tilde{g}}$, denoted by $\tilde{\mathcal{B}} \tilde{\underline{f}} \tilde{\underline{\mathcal{B}}} \tilde{\mathcal{G}}^{\tilde{g}}$ if and only if $\tilde{f}(p) \tilde{\subseteq} \tilde{g}(p)$ for all $p \in P$. That means $\tilde{\mathcal{B}} \tilde{f}^{\tilde{G}} \tilde{\mathcal{B}} \tilde{\mathcal{B}}^{\tilde{g}}$ if and only if for all $p \in P, \tilde{f}(p)$ be a fuzzy subset of $\tilde{g}(p)$.
 means if $\tilde{f}=\tilde{g}$.

The next theorem gives some useful results about fuzzy bags.
Theorem 1. Operations $\cup$ and $\cap$ in $\tilde{\boldsymbol{B}}(P, O)$ satisfy the laws of idempotency, commutativity, associativity and distributivity.

In the following definition, we introduce the concept of complement of a fuzzy bag.
Definition 8. Let $\eta:[0,1] \rightarrow[0,1]$ be a fixed strong negation [4], this means an involutive decreasing bijection. Consider $\tilde{\mathcal{B}} \tilde{f}=\left(\tilde{f}, B^{\tilde{f}}\right)$. Then, the $\eta$-complement of $\tilde{\mathcal{B}}^{\tilde{f}}$ is fuzzy bag $\left(\tilde{\mathcal{B}}^{\tilde{f}}\right)^{c}=\left(\tilde{f}^{c}, B^{\tilde{f}^{c}}\right)$, where $\tilde{f}^{c}: P \rightarrow \mathcal{F}(O)$ such that $\tilde{f}^{c}(p)(o)=\eta(\tilde{f}(p)(o))$ for all $p \in P$ and $o \in O$.

Note that by Definition 3, $\left(\tilde{\mathcal{B}}^{\tilde{f}}\right)^{c}=\tilde{\mathcal{B}}^{f^{c}}$.
Note 1. In Definition 8, if $\eta$ is the standard negation, $\eta(x)=1-x$ for all $x \in[0,1][4]$, then $\tilde{\mathcal{B}}^{\tilde{c}}$ is called complement of $\tilde{\mathcal{B}} \tilde{f}$.

Example 3. Consider the fuzzy bag of Example 2. The complement of this fuzzy bag is $\tilde{\mathcal{B}}^{\tilde{f}^{c}}=\left(\tilde{f}^{c}, B^{\tilde{f}^{c}}\right)$ where,

$$
\begin{aligned}
& \tilde{f}^{c}(\text { young })=\left\{\frac{0.3}{\text { Ben }}, \frac{0.8}{\text { Sue }}, \frac{0.6}{\text { Tom }}, \frac{1.0}{\text { John }}, \frac{0.3}{\text { Stan }}, \frac{0.6}{\text { Bill }}, \frac{0.8}{\text { Kim }}, \frac{0.3}{\text { Ana }}, \frac{0.9}{\text { Sara }}\right\}, \\
& \tilde{f}^{c}(\text { middle age })=\left\{\frac{0.7}{\text { Ben }}, \frac{0.2}{\text { Sue }}, \frac{0.3}{\text { Tom }}, \frac{0.7}{\text { John }}, \frac{0.7}{\text { Stan }}, \frac{0.3}{\text { Bill }}, \frac{0.2}{\text { Kim }}, \frac{0.7}{\text { Ana }}, \frac{0.5}{\text { Sara }}\right\}, \\
& \tilde{f}^{c}(\text { old })=\left\{\frac{0.9}{\text { Ben }}, \frac{0.8}{\text { Sue }}, \frac{0.9}{\text { Tom }}, \frac{0.1}{\text { John }}, \frac{0.9}{\text { Stan }}, \frac{0.9}{\text { Bill }}, \frac{0.8}{\text { Kim }}, \frac{0.9}{\text { Ana }}, \frac{0.5}{\text { Sara }}\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
B^{\tilde{f}^{c}}= & \{(\text { young }, 1.0,1),(\text { young }, 0.9,1),(\text { young }, 0.8,2),(\text { young }, 0.6,2),(\text { young }, 0.3,3) \\
& (\text { middle age }, 0.7,4),(\text { middle age }, 0.5,1),(\text { middle age }, 0.3,2),(\text { middle age }, 0.2,2) \\
& (\text { old }, 0.9,5),(\text { old }, 0.8,2),(\text { old }, 0.5,1),(\text { old }, 0.1,1)\}
\end{aligned}
$$

## 4 Alpha-Cuts of Fuzzy Bags

The notion of $\alpha$-cut plays a fairly big role in the fuzzy theory. So, here, we define this notion for the fuzzy bags.
Definition 9. Let $\alpha \in[0,1]$. Then, $\alpha$-cut of fuzzy bag $\tilde{\mathcal{B}}^{\tilde{f}} \in \tilde{\mathbf{B}}(P, O)$ is the crisp $\underset{\sim}{\text { bag }}\left(\tilde{\mathcal{B}}^{\tilde{f}}\right)_{\alpha}=\left(\tilde{f}_{\alpha}, B^{\tilde{f}_{\alpha}}\right)$ where, $\tilde{f}_{\alpha}: P \rightarrow \mathcal{P}(O)$ is a function in which for all $p \in P, \tilde{f}_{\alpha}(p)=\{o \in O \mid \tilde{f}(p)(o) \geqslant \alpha\}$ and

$$
B^{\tilde{f}_{\alpha}}=\left\{\left(p, \operatorname{card}\left(\tilde{f}_{\alpha}(p)\right)\right) \mid p \in P\right\}
$$

Definition 10. Let $\alpha \in[0,1]$. Then, strong $\alpha$-cut of fuzzy bag $\tilde{\mathcal{B}} \tilde{f} \in \tilde{\mathbf{B}}(P, O)$ is the crisp bag $\left(\tilde{\mathcal{B}}^{\tilde{f}}\right)_{\alpha}$. $=$ $\left(\tilde{f}_{\alpha^{*}}, B^{\tilde{f}_{\alpha^{*}}}\right)$ where, $\tilde{f}_{\alpha^{*}}: P \rightarrow \mathcal{P}(O)$ is a function which for all $p \in P, \tilde{f}_{\alpha^{*}}(p)=\{o \in O \mid \tilde{f}(p)(o)>\alpha\}$ and

$$
B^{\tilde{f}_{\alpha}}=\left\{\left(p, \operatorname{card}\left(\tilde{f}_{\alpha \cdot}(p)\right)\right) \mid p \in P\right\}
$$

Note that by Definition 1, we have $\mathcal{B}^{\tilde{f}_{\alpha}}=\left(\tilde{\mathcal{B}}^{\tilde{f}}\right)_{\alpha}$ and $\mathcal{B}^{\tilde{f}_{\alpha}}=\left(\tilde{\mathcal{B}}^{\tilde{f}}\right)_{\alpha}$.
Notation 3. We set $\tilde{f}_{[\alpha, \beta)}(p)=\{o \in O \mid \alpha \leq \tilde{f}(p)(o)<\beta\}$ and $\tilde{f}_{(\alpha, \beta]}(p)=\{o \in O \mid \alpha<\tilde{f}(p)(o) \leq \beta\}$ for all $p \in P$.

Some useful results for the fuzzy bags are given in the next theorem.
Theorem 2. Let $\tilde{\mathcal{B}}^{\tilde{f}}, \tilde{\mathcal{B}}^{\tilde{g}} \in \tilde{\boldsymbol{B}}(P, O), \alpha, \beta \in[0,1]$ and $\alpha \leqslant \beta$. Then,
i) $\mathcal{B}^{\tilde{f}_{\beta}} \cdot \tilde{\sqsubseteq} \mathcal{B}^{\tilde{f}_{\beta}} \sqsubseteq \mathcal{B}^{\tilde{f}_{\alpha}} \check{\sqsubseteq} \mathcal{B}^{\tilde{f}_{\alpha}}$,
ii) $\mathcal{B}^{\tilde{f}_{\alpha}}=\mathcal{B}^{\tilde{f}_{\beta}}$ if and only if $\mathcal{B}^{\tilde{f}_{[\alpha, \beta)}}=\mathcal{B}^{0}$,
iii) $\mathcal{B}^{\tilde{f}_{\alpha}}=\mathcal{B}^{\tilde{f}_{\beta}}$. if and only if $\mathcal{B}^{\tilde{f}_{(\alpha, \beta]}}=\mathcal{B}^{0}$,
iv) $\left(\tilde{\mathcal{B}}^{\tilde{f}} \cup \tilde{\mathcal{B}}^{\tilde{g}}\right)_{\alpha}=\mathcal{B}^{\tilde{f}_{\alpha}} \cup \mathcal{B}^{\tilde{g}_{\alpha}}$ and $\left(\tilde{\mathcal{B}}^{\tilde{f}} \cup \tilde{\mathcal{B}}^{\tilde{g}}\right)_{\alpha}=\mathcal{B}^{\tilde{f}_{\alpha}} \cup \mathcal{B}^{\tilde{g}_{\alpha}}$,
v) $\left(\tilde{\mathcal{B}}^{\tilde{f}} \cap \tilde{\mathcal{B}}^{\tilde{g}}\right)_{\alpha}=\mathcal{B}^{\tilde{f}_{\alpha}} \cap \mathcal{B}^{\tilde{g}_{\alpha}}$ and $\left(\tilde{\mathcal{B}}^{\tilde{f}} \cap \tilde{\mathcal{B}}^{\tilde{g}}\right)_{\alpha}=\mathcal{B}^{\tilde{f}_{\alpha}} \cap \mathcal{B}^{\tilde{g}_{\alpha}}$.

In the following example, we compute $\alpha$-cuts of a fuzzy bag.
Example 4. Consider the fuzzy bag of Example 2. We compute $\alpha$-cuts, $\mathcal{B}^{\tilde{f}_{\alpha}}=\left(\tilde{f}_{\alpha}, B^{\tilde{f}_{\alpha}}\right)$. Where, $\tilde{f}_{\alpha}(p)$ is presented in Table 3.
and $B^{\tilde{f}_{\alpha}}$ is as follows
$B^{\tilde{f}_{0}}=\{($ young, 9$),($ middle age, 9$),($ old, 9$)\}, \quad B^{\tilde{f}_{0.1}}=\{($ young, 8$),($ middle age, 9$),($ old, 9$)\}$
$B^{\tilde{f}_{0.2}}=\{($ young, 7$),($ middle age, 9$),($ old, 4$)\}, \quad B^{\tilde{f}_{0.3}}=\{($ young, 5$),($ middle age, 9$),($ old, 2$)\}$
$B^{\tilde{f}_{0.4}}=\{($ young, 5$),($ middle age, 5$),($ old, 2$)\}, \quad B^{\tilde{f}_{0.5}}=\{($ young, 3$),($ middle age, 5$),($ old, 2$)\}$
$B^{\tilde{f}_{0.7}}=\{($ young, 3$),($ middle age, 4$),($ old, 1$)\}, \quad B^{\tilde{f}_{0.8}}=\{($ middle age, 2$),($ old, 1$)\}, B^{\tilde{f}_{0.9}}=\{($ old, 1$)\}$.

Table 3: The values of $\tilde{f}_{\alpha}(p)$ for Example 4

|  | young | middle age | old |
| :---: | :---: | :---: | :---: |
| 0.0 | O | O | O |
| 0.1 | $\mathrm{O} \backslash\{$ John, Sara $\}$ | O | O |
| 0.2 | $\mathrm{O} \backslash\{$ John, Sara $\}$ | O | O |
| 0.3 | $\mathrm{O} \backslash\{$ Sue, John, Kim, Sara $\}$ | O | $\mathrm{O} \backslash\{$ Sue, John, Kim, Sara $\}$ |
| 0.4 | $\mathrm{O} \backslash\{$ Sue, John, Kim, Sara $\}$ | \{Sue, Tom, Bill, Kim, Sara $\}$ | $\{$ John, Sara $\}$ |
| 0.5 | $\{$ Ben, Stan, Ana $\}$ | $\{$ Sue, Tom, Bill, Kim, Sara | $\{$ John, Sara $\}$ |
| 0.7 | $\{$ Ben, Stan, Ana $\}$ | $\{$ Sue, Tom, Bill, Kim $\}$ | $\{$ John, Sara $\}$ |
| 0.8 | $\emptyset$ | $\{$ Sue, Kim $\}$ | $\{$ John $\}$ |
| 0.9 | $\emptyset$ | $\emptyset$ | $\{$ John $\}$ |

Definition 11. Let $\mathcal{B}^{f} \in \mathbf{B}(P, O)$ and $\alpha \in[0,1]$. We define fuzzy bag $\widetilde{\alpha \mathcal{B}^{f}}=\widetilde{\mathcal{B}^{\alpha f}}=\left(\widetilde{\alpha f}, \widetilde{B^{\alpha f}}\right)$ where,

$$
\widetilde{\alpha f}(p)(o)=\min \left(\alpha, \chi_{f_{(p)}}(o)\right)=\alpha \chi_{f_{(p)}}(o),
$$

for all $o \in O$ and $p \in P$.
Theorem 3. i) Let $\tilde{\mathcal{B}} \tilde{f}^{\tilde{f}}$ be a fuzzy bag and let $\mathcal{B}^{\tilde{f}_{\alpha}}$ be $\alpha$-cut of $\tilde{\mathcal{B}} \tilde{f}$. Then,

$$
\tilde{\mathcal{B}^{\tilde{f}}}=\bigcup_{\alpha \in[0,1]} \widetilde{\alpha \mathcal{B}}^{\tilde{f}_{\alpha}} .
$$

i) Let $\tilde{\mathcal{B}} \tilde{f}$ be a fuzzy bag and let $\mathcal{\mathcal { B }}^{\tilde{f}_{\alpha}}$. be the strong $\alpha$-cut of $\tilde{\mathcal{B}} \tilde{f}$. Then,

$$
\tilde{\mathcal{B}} \tilde{f}=\bigcup_{\alpha \in[0,1]} \widetilde{\alpha \mathcal{B}}^{\tilde{f}_{\alpha}} .
$$

Theorem 4. Let $\tilde{\mathcal{B}} \tilde{f} \in \tilde{\boldsymbol{B}}(P, O)$ and $\left\{\mathcal{B}^{\tilde{g}_{\alpha}} \mid \alpha \in[0,1]\right\}$ is a class of elements of $\boldsymbol{B}(P, O)$ such that $\mathcal{B}^{\tilde{f}_{\alpha}} \sqsubseteq \mathcal{B}^{\tilde{g}_{\alpha}} \sqsubseteq \mathcal{B}^{\tilde{f}_{\alpha}}$. Then,

$$
\tilde{\mathcal{B}} \tilde{\mathcal{f}}_{\alpha \in[0,1]} \widetilde{\alpha \mathcal{B}}^{\tilde{g}_{\alpha}} .
$$

Theorem 5. Let $\left\{\mathcal{B}^{g_{\alpha}} \mid \alpha \in[0,1]\right\}$ is a class of elements of $\boldsymbol{B}(P, O)$. There exists $\tilde{\mathcal{B}} \tilde{f} \in \tilde{\boldsymbol{B}}(P, O)$ such that for all $\alpha \in[0,1], \mathcal{B}^{\tilde{f}_{\alpha}}=\mathcal{B}^{g_{\alpha}}$ if and only if for all $\alpha, \beta \in[0,1]$ such that $\alpha \leq \beta, \mathcal{B}^{g_{\beta}} \sqsubseteq \mathcal{B}^{g_{\alpha}}$ and $\mathcal{B}^{g_{0}}=\mathcal{B}^{1}$.

## 5 Conclusion

Using Delgado et al.'s definition of bags, which is improved version of Yager's one, a new definition for fuzzy bags has been introduced. Also, a concept of the $\alpha-$ cut of fuzzy bags has been studied.

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# OWA operators for fuzzy truth values 

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#### Abstract

In this work we deal with OWA operators for normal, convex fuzzy truth values (fuzzy sets in $[0,1]$ ). Our approach is as follows. We adapt a more general notion of OWA operators on any complete lattice endowed with a t-norm and a t-conorm [4, 6], and study this notion on a particular case of complete lattice, namely the set of all normal, convex fuzzy truth values. We focus on a specific nature of the operators in this settings.


Keywords: OWA operator, type-2 fuzzy set, fuzzy truth value, aggregation, distributive weighting vector

## 1 Introduction

In $[4,6]$ the concept of an ordered weighted averaging (OWA for short) operator is extended to any complete lattice endowed with a t-norm and a t-conorm. The intention of authors was to avoid the need of a linear order in environments in which it is available only partial order. Our aim is to study a specific nature of ideas from [4, 6] in one particular case of complete lattice, namely the set of all normal, convex fuzzy truth values (fuzzy sets in $[0,1]$ ). It is well-known that this set is not linearly ordered. We discuss the notion of (distributive) weighting vector, formulate a sufficient and necessary condition under which given elements constitute a distributive weighting vector and investigate bounds of the proposed operators.

The aggregation of fuzzy truth values is essential in the type-2 fuzzy sets settings [8, 9]. Also, the need of aggregation of fuzzy truth values arises in decision making problems when the alternatives are assessed by fuzzy truth values. Recall that Yager's OWA operators are of special significance in solving decision making problems. This leads to growing interest of scholars to investigate OWA operators for various kinds of elements, e.g., for intervals [1, 13], fuzzy intervals [15, 16], gradual intervals [10], i.e., also for fuzzy truth values.

The paper is organized as follows. Section 2 contains basic definitions and notations that are used in the remaining parts of the paper. In Section 3, we study OWA operator on the set of normal, convex fuzzy truth values and some its properties. The conclusions are discussed in Section 4

## 2 Preliminaries

In this section we present some basic concepts and terminology that will be used throughout the paper.
Let $X$ be a set. A fuzzy set in $X$ is a mapping from $X$ to $[0,1]$. Let $\mathcal{F}(X)$ denote the class of all fuzzy sets in $X$, and let $\mathcal{F}$ denote the class of all fuzzy sets in [0, 1]. A type- 2 fuzzy set in $X$ is a fuzzy set whose membership grades are fuzzy sets in $[0,1]$. Hence, type-2 fuzzy set in $X$ is a mapping

$$
\tilde{f}: X \rightarrow \mathcal{F}
$$

and the elements of $\mathcal{F}$ are called fuzzy truth values.
A fuzzy set $f$ in $X$ is normal if there exists $x \in X$ such that $f(x)=1$. Let $X$ be a linear space, a fuzzy set $f$ in $X$ is convex if it is satisfied $f\left(\lambda x_{1}+(1-\lambda) x_{2}\right) \geq \min \left(f\left(x_{1}\right), f\left(x_{2}\right)\right)$ for all $\lambda \in[0,1]$,

[^8]for each $x_{1}, x_{2} \in X$. We denote by $\mathcal{F}_{N C}$ the class of all normal, convex fuzzy truth values. We will use operations $\sqcup, \sqcap$, relations $\sqsubseteq, \preceq$ and special elements $\mathbf{0}, \mathbf{1}$ on $\mathcal{F}$ given by:
\[

$$
\begin{array}{ll}
(f \sqcup g)(z)=\sup _{x \vee y=z}(f(x) \wedge g(y)), & f \sqsubseteq g \quad \text { iff } \quad f \sqcap g=f, \\
(f \sqcap g)(z)=\sup _{x \wedge y=z}(f(x) \wedge g(y)), & f \preceq g \quad \text { iff } \quad f \sqcup g=g,  \tag{1}\\
\mathbf{0}(x)= \begin{cases}1 & , \text { if } x=0, \\
0 & , \text { otherwise },\end{cases} & \mathbf{1}(x)= \begin{cases}1 & , \text { if } x=1, \\
0 & , \text { otherwise }\end{cases}
\end{array}
$$
\]

The algebra of fuzzy truth values $(\mathcal{F}, \sqcup, \sqcap, \mathbf{0}, \mathbf{1}, \sqsubseteq, \preceq)$ is closely described in [5] and [11]. In [11] it is showed that $\left(\mathcal{F}_{N C}, \sqcup, \sqcap, \mathbf{0}, \mathbf{1}, \sqsubseteq\right)$ is a bounded, distributive lattice, and in [2] the authors showed that the lattice is complete. Recall that the two orders $\sqsubseteq$ and $\preceq$ coincide on the set of normal, convex fuzzy truth values.

In 1988 Yager [14] introduced OWA operator which is one of the most widely used aggregation methods for real numbers.

Definition 2.1. Let $\mathbf{w}=\left(w_{1}, \ldots, w_{n}\right) \in[0,1]^{n}$ with $w_{1}+\ldots+w_{n}=1$ be a weighting vector. An OWA operator associated with $\mathbf{w}$ is a mapping $O W A_{\mathbf{w}}:[0,1]^{n} \rightarrow[0,1]$ defined by

$$
O W A_{\mathbf{w}}\left(x_{1}, \ldots, x_{n}\right)=\sum_{i=1}^{n} w_{i} x_{(i)}
$$

where $x_{(i)}$ denotes the $i$ th largest number among $x_{1}, \ldots, x_{n}$.

## 3 OWA operators defined on the set of convex normal fuzzy truth values

In this section we apply the ideas of $[4,6]$ to the settings of type- 2 fuzzy sets. In other words, we will study distributive weighting vectors and consequently OWA operators on the set of fuzzy truth values $\mathcal{F}$. Let us start with the notion of a t -norm and a t -conorm on $\mathcal{F}$.

Definition 3.1. A mapping $T: \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ is said to be a $t$-norm on $(\mathcal{F}, \sqsubseteq)$ if it is commutative, associative, increasing in each component and has a neutral element 1.

A mapping $S: \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ is said to be a t-conorm on $(\mathcal{F}, \sqsubseteq)$ if it is commutative, associative, increasing in each component and has a neutral element $\mathbf{0}$.

The operations $\sqcap$ and $\sqcup$ given by (1) are t-norm and t-conorm on $\mathcal{F}$, respectively. The following propositions are easy to check, see [3], [7] and [12].

Proposition 3.2. The operation $\sqcap$ given by (1) is a t-norm on $(\mathcal{F}, \sqsubseteq)$.
Proposition 3.3. The operation $\sqcup$ given by $(1)$ is a $t$-conorm on $(\mathcal{F}, \sqsubseteq)$.
According to the following lemma, it is possible to construct linearly ordered vector from any given vector in $\mathcal{F}_{N C}^{n}$.

Lemma 3.4 ([4]). Let $\left(f_{1}, \ldots, f_{n}\right) \in \mathcal{F}_{N C}^{n}$, and let

$$
\begin{aligned}
g_{1} & =f_{1} \sqcup \ldots \sqcup f_{n}, \\
g_{2} & =\left(\left(f_{1} \sqcap f_{2}\right) \sqcup \ldots \sqcup\left(f_{1} \sqcap f_{n}\right)\right) \sqcup\left(\left(f_{2} \sqcap f_{3}\right) \sqcup \ldots \sqcup\left(f_{2} \sqcap f_{n}\right)\right) \sqcup \ldots \sqcup\left(\left(f_{n-1} \sqcap f_{n}\right)\right), \\
& \vdots \\
g_{k} & =\sqcup\left\{f_{j_{1}} \sqcap \ldots \sqcap f_{j_{k}} \mid\left\{j_{1}, \ldots, j_{k}\right\} \subseteq\{1, \ldots, n\}\right\}, \\
& \vdots \\
g_{n} & =f_{1} \sqcap \ldots \sqcap f_{n} .
\end{aligned}
$$

Then

$$
g_{n} \sqsubseteq g_{n-1} \sqsubseteq \ldots \sqsubseteq g_{1}
$$

Moreover, if the set $\left\{f_{1}, \ldots, f_{n}\right\}$ is linearly ordered, then the vector $\left(g_{1}, \ldots, g_{n}\right)$ coincides with $\left(f_{\sigma(1)}, \ldots, f_{\sigma(n)}\right)$ for some permutation $\sigma$ of $\{1, \ldots, n\}$.

We proceed with the study of distributive weighting vector in $\mathcal{F}_{N C}^{n}$ and some of its properties.
Definition 3.5. Let $w_{1}, \ldots, w_{n} \in \mathcal{F}$. A vector $\left(w_{1}, \ldots, w_{n}\right)$ is said to be a weighting vector in $(\mathcal{F}, \sqsubseteq)$ if $w_{1} \sqcup \ldots \sqcup w_{n}=\mathbf{1}$, and it is said to be a distributive weighting vector if it also satisfies

$$
f \sqcap\left(w_{1} \sqcup \ldots \sqcup w_{n}\right)=\left(f \sqcap w_{1}\right) \sqcup \ldots \sqcup\left(f \sqcap w_{n}\right)
$$

for all $f \in \mathcal{F}$.
The following theorem gives a necessary condition under which elements $w_{1}, \ldots, w_{n}$ constitute a weighting vector in $\left(\mathcal{F}_{N C}, \sqsubseteq\right)$.

Theorem 3.6. Let $w_{1}, \ldots, w_{n} \in \mathcal{F}_{N C}$. If $w_{1} \sqcup \ldots \sqcup w_{n}=1$, then $w_{i}=\mathbf{1}$ for some $i \in\{1, \ldots, n\}$.
Proof. 1. We show that $w_{i}(1)=1$ for some $i \in\{1, \ldots, n\}$. From $w_{1} \sqcup \ldots \sqcup w_{n}=\mathbf{1}$ it follows $\left(w_{1} \sqcup \ldots \sqcup w_{n}\right)(1)=1$, hence there exist $a_{1}, \ldots a_{n} \in[0,1]$ such that $\max \left(a_{1}, \ldots, a_{n}\right)=1$ and $\min \left(w_{1}\left(a_{1}\right), \ldots, w_{n}\left(a_{n}\right)\right)=1$, and consequently there exist $a_{1}, \ldots a_{n} \in[0,1]$ such that $\max \left(a_{1}, \ldots, a_{n}\right)=$ 1 and $w_{1}\left(a_{1}\right)=\ldots=w_{n}\left(a_{n}\right)=1$. It means that, for some $i \in\{1, \ldots, n\}$, it holds $a_{i}=1$ and $w_{i}\left(a_{1}\right)=1$, thus $w_{i}(1)=1$ for some $i \in\{1, \ldots, n\}$. Let us write $w_{k_{0}}(1)=1$.
2. Now we are going to show that $w_{k_{0}}(x)=0$ for all $x \in\left[0,1\left[\right.\right.$ if $w_{i} \neq \mathbf{1}$ for all $i \in\{1, \ldots, n\}-$ $\left\{k_{0}\right\}$. Let there exist $x_{0} \in\left[0,1\left[\right.\right.$ such that $w_{k_{0}}\left(x_{0}\right)>0$. Then there exist $b_{1}, \ldots, b_{k_{0}-1}, b_{k_{0}+1}, \ldots, b_{n} \in$ [ $0,1[$ such that

$$
w_{1}\left(b_{1}\right), \ldots, w_{k_{0}-1}\left(b_{k_{0}-1}\right), w_{k_{0}+1}\left(b_{k_{0}+1}\right), \ldots, w_{n}\left(b_{n}\right)>0
$$

hence

$$
\begin{gathered}
\left(w_{1} \sqcup \ldots \sqcup w_{n}\right)\left(\max \left(b_{1}, \ldots, b_{k_{0}-1}, x_{0}, b_{k_{0}+1}, \ldots, b_{n}\right)\right)= \\
=\min \left(w_{1}\left(b_{1}\right), \ldots, w_{k_{0}-1}\left(b_{k_{0}-1}\right), w_{k_{0}}\left(x_{0}\right), w_{k_{0}+1}\left(b_{k_{0}+1}\right), \ldots, w_{n}\left(b_{n}\right)\right)>0
\end{gathered}
$$

which contradicts our assumption $w_{1} \sqcup \ldots \sqcup w_{n}=\mathbf{1}$.
The following corollary states a simple necessary and sufficient condition under which $\left(w_{1}, \ldots, w_{n}\right) \in$ $\mathcal{F}_{N C}^{n}$ is a distributive weighting vector.

Corollary 3.7. A vector $\left(w_{1}, \ldots, w_{n}\right) \in \mathcal{F}_{N C}^{n}$ is a distributive weighting vector in $\left(\mathcal{F}_{N C}, \sqsubseteq\right)$ if and only if there exists $i \in\{1, \ldots, n\}$ such that $w_{i}=1$.

Proof. 1. Necessity: Let $\left(w_{1}, \ldots, w_{n}\right) \in \mathcal{F}_{N C}^{n}$ be a distributive weighting vector. Then, according to Definition 3.5, $w_{1} \sqcup \ldots \sqcup w_{n}=\mathbf{1}$; and from Theorem 3.6 it follows $w_{i}=\mathbf{1}$ for some $i \in\{1, \ldots, n\}$.
2. Sufficiency: Let us first observe that $\left(\mathcal{F}_{N C}, \sqsubseteq\right)$ is a distributive lattice, thus it is sufficient to show that $\left(w_{1}, \ldots, w_{n}\right)$ is a weighting vector in $\left(\mathcal{F}_{N C}, \sqsubseteq\right)$. Let $w_{i}=\mathbf{1}$ for some $i \in\{1, \ldots, n\}$. The proof follows from the observation that $\mathbf{1} \sqcup f=\mathbf{1}$ for all $f \in \mathcal{F}_{N C}$.

Now we can use the notion of distributive weighting vector and define an OWA operator on the set of normal, convex fuzzy truth values $\mathcal{F}_{N C}$.

Definition 3.8. Let $\mathbf{w}=\left(w_{1}, \ldots, w_{n}\right) \in \mathcal{F}_{N C}^{n}$ be a distributive weighting vector in $\left(\mathcal{F}_{N C}, \sqsubseteq\right)$. The mapping $F_{\mathbf{w}}: \mathcal{F}_{N C}^{n} \rightarrow \mathcal{F}_{N C}$ given, for all $\left(f_{1}, \ldots, f_{n}\right) \in \mathcal{F}_{N C}^{n}$, by

$$
F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right)=\left(w_{1} \sqcap g_{1}\right) \sqcup \ldots \sqcup\left(w_{n} \sqcap g_{n}\right),
$$

where $\left(g_{1}, \ldots, g_{n}\right)$ is a linearly ordered vector constructed from $\left(f_{1}, \ldots, f_{n}\right)$ according to Lemma 3.4, is called an n-ary OWA operator on $\mathcal{F}_{N C}$.

Example 3.9. Let weighting vector be $\mathbf{w}=\left(w_{1}, \mathbf{1}\right)$ and $w_{1}, f_{1}, f_{2}$ be fuzzy truth values given by Figure 1. Then $g_{1}=f_{1} \sqcup f_{2}, g_{2}=f_{1} \sqcap f_{2}$, and

$$
F_{\mathbf{w}}\left(f_{1}, f_{2}\right)=\left(w_{1} \sqcap g_{1}\right) \sqcup\left(\mathbf{1} \sqcap g_{2}\right)=\left(w_{1} \sqcap g_{1}\right) \sqcup g_{2}
$$

The results are depicted in Figure 1 (for simplicity, fuzzy truth values $g_{1}$ and $g_{2}$ are not depicted - they can be found in Figure 2).


Figure 1: See Example 3.9.

From our point of view, the most important property of operator $F_{\mathbf{w}}$ is given by the following theorem.

Theorem 3.10. Let $F_{\mathbf{w}}$ be an n-ary $O W A$ operator on $\mathcal{F}_{N C}$. Then

$$
f_{1} \sqcap \ldots \sqcap f_{n} \sqsubseteq F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right) \sqsubseteq f_{1} \sqcup \ldots \sqcup f_{n}
$$

for all $f_{1}, \ldots, f_{n} \in \mathcal{F}_{N C}$.
Proof. We prove the right inequality, the left one can be checked in a similar way.

$$
\begin{gathered}
F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right)=\left(w_{1} \sqcap g_{1}\right) \sqcup \ldots \sqcup\left(w_{n} \sqcap g_{n}\right) \sqsubseteq\left(w_{1} \sqcap g_{1}\right) \sqcup \ldots \sqcup\left(w_{n} \sqcap g_{1}\right)= \\
=\left(w_{1} \sqcup \ldots \sqcup w_{n}\right) \sqcap g_{1}=\mathbf{1} \sqcap g_{1}=g_{1}=f_{1} \sqcup \ldots \sqcup f_{n} .
\end{gathered}
$$

The theorem says that the results of $F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right)$ are bounded by $f_{1} \sqcap \ldots \sqcap f_{n}$ and $f_{1} \sqcup \ldots \sqcup f_{n}$. It is worth pointing out that for standard OWA operators for real numbers from $\min \left(x_{1}, \ldots, x_{n}\right) \leq$ $O W A_{w}\left(x_{1}, \ldots, x_{n}\right) \leq \max \left(x_{1}, \ldots, x_{n}\right)$ it follows that $O W A_{w}\left(x_{1}, \ldots, x_{n}\right) \geq x_{i}$ for some $i \in$ $\{1, \ldots, n\}$, and $O W A_{w}\left(x_{1}, \ldots, x_{n}\right) \leq x_{j}$ for some $j \in\{1, \ldots, n\}$. However, the similar property does not hold for $F_{\mathrm{w}}$, i.e., it is possible that

$$
F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right) \sqsubset f_{i}, \quad \text { for all } i \in\{1, \ldots, n\}
$$

or

$$
F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right) \sqsupset f_{j}, \quad \text { for all } j \in\{1, \ldots, n\}
$$

See Example 3.9) where $F_{\mathbf{w}}\left(f_{1}, f_{n}\right) \sqsupset f_{1}$ and $F_{\mathbf{w}}\left(f_{1}, f_{n}\right) \sqsupset f_{2}$.
Example 3.11. Let weighting vector be $\mathbf{w}=\left(1, w_{2}\right)$ and $w_{2}, f_{1}, f_{2}$ be fuzzy truth values given by Figure 2. Then $g_{1}=f_{1} \sqcup f_{2}, g_{2}=f_{1} \sqcap f_{2}$, and (see Lemma 3.12 for the last equality)

$$
F_{\mathbf{w}}\left(f_{1}, f_{2}\right)=\left(\mathbf{1} \sqcap g_{1}\right) \sqcup\left(w_{2} \sqcap g_{2}\right)=g_{1} \sqcup\left(w_{2} \sqcap g_{2}\right)=g_{1}
$$

It is easy to check that $F_{\mathbf{w}}\left(f_{1}, f_{2}\right)=g_{1} \sqsupset f_{i}$, for $i=1,2$.


Figure 2: See Example 3.11.

The following lemma was used in previous example and will also be needed in proof of Theorem 3.13.

Lemma 3.12. Let $f, g, h \in \mathcal{F}_{N C}$ with $f \sqsubseteq g$. Then $g \sqcup(f \sqcap h)=g$.
Proof. Applying the distributive laws ([11], Proposition 36) and absorption laws ([11], Proposition 37) we have:

$$
g \sqcup(f \sqcap h)=(g \sqcup f) \sqcap(g \sqcup h)=g \sqcap(g \sqcup h)=g .
$$

Note that the property of Lemma 3.12 does not hold in $\mathcal{F}$. The reason is that the absorption laws fail if $g$ is not convex or $h$ is not normal.

We can now strengthen Proposition 3.8 from [4] in the settings of fuzzy truth values. Our result is that if $\mathbf{1}$ is on the first position of a weighting vector $\mathbf{w}$, then our OWA operator is simply maximum, no matter what are the other weights - see item 1 of the following theorem. Note that there are much stronger assumptions for a similar assertion on minimum - see item 2 of the theorem.
Theorem 3.13. Let $\mathbf{w}=\left(w_{1}, \ldots, w_{n}\right)$ be a distributive weighting vector in $\mathcal{F}_{N C}$.

1. If $w_{1}=\mathbf{1}$, then $F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right)=f_{1} \sqcup \ldots \sqcup f_{n}$.
2. If $w_{n}=1$ and $w_{i} \sqsubseteq f_{1} \sqcap \ldots \sqcap f_{n}$ for all $i \in\{1, \ldots, n-1\}$, then $F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right)=f_{1} \sqcap \ldots \sqcap f_{n}$.
3. If $w_{k}=\mathbf{1}$ for some $k \in\{1, \ldots, n\}$ and $w_{i}=\mathbf{0}$ for all $i \in\{1, \ldots, n\}-\{k\}$, then $F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right)=$ $g_{k}$.

Proof. 1. Let $\mathbf{w}=\left(\mathbf{1}, w_{2}, \ldots, w_{n}\right)$. Then

$$
\begin{gathered}
F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right)=\left(\mathbf{1} \sqcap g_{1}\right) \sqcup\left(w_{2} \sqcap g_{2}\right) \sqcup \ldots \sqcup\left(w_{n} \sqcap g_{n}\right)= \\
=g_{1} \sqcup\left(w_{2} \sqcap g_{2}\right) \sqcup \ldots \sqcup\left(w_{n} \sqcap g_{n}\right)
\end{gathered}
$$

and applying Lemma $3.12(n-1)$ times we conclude

$$
F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right)=g_{1}=f_{1} \sqcup \ldots \sqcup f_{n}
$$

2. Let $\mathbf{w}=\left(w_{1}, \ldots, w_{n-1}, \mathbf{1}\right)$ with $w_{i} \sqsubseteq g_{n}$ for all $i \in\{1, \ldots, n-1\}$. Then $w_{i} \sqsubseteq g_{i}$ for all $i \in\{1, \ldots, n-1\}$, and we have

$$
\begin{aligned}
& F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right)=\left(w_{1} \sqcap g_{1}\right) \sqcup \ldots \sqcup\left(w_{n-1} \sqcap g_{n-1}\right) \sqcup\left(\mathbf{1} \sqcap g_{n}\right)=w_{1} \sqcup \ldots \sqcup w_{n-1} \sqcup g_{n}= \\
&=g_{n}=f_{1} \sqcap \ldots \sqcap f_{n} .
\end{aligned}
$$

3. Let $\mathbf{w}=\left(\mathbf{0}, \ldots, \mathbf{0}, w_{k}=\mathbf{1}, \mathbf{0} \ldots, \mathbf{0}\right)$. Then

$$
\begin{gathered}
F_{\mathbf{w}}\left(f_{1}, \ldots, f_{n}\right)=\left(\mathbf{0} \sqcap g_{1}\right) \sqcup \ldots \sqcup\left(\mathbf{0} \sqcap g_{k-1}\right) \sqcup\left(\mathbf{1} \sqcap g_{k}\right) \sqcup\left(\mathbf{0} \sqcap g_{k+1}\right) \ldots \sqcup\left(\mathbf{0} \sqcap g_{n}\right)= \\
=\mathbf{0} \sqcup \ldots \sqcup \mathbf{0} \sqcup g_{k} \sqcup \mathbf{0} \ldots \sqcup \mathbf{0}=g_{k} .
\end{gathered}
$$

## 4 Conclusion

In $[4,6]$ an OWA operator on any complete lattice endowed with a $t$-norm and a $t$-conorm was introduced. In this paper we focused on OWA operators on one particular case of complete lattice, namely the set of all normal, convex fuzzy truth values. We have restricted our attention on operations $\square$ and $\sqcup$. Our next intention is to apply some other t-norms and t-conorms on the set of fuzzy truth values and study properties of corresponding OWA operators. Another line of our investigation is a relationship of the proposed OWA operators to existing operators, namely type-1 OWA operators [15, 16] and OWA operators for gradual intervals [10].

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