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Editor:
Radko MESIAR

Co-editor:
Tomáš BACIGÁL

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Contents

Preface

Regular seminar "Uncertainty Modelling" was founded by Prof. B. Riečan and Prof. R. Mesiar in 1990. Since 1995 it is hosted by Department of Mathematics and Descriptive Geometry of the Faculty of Civil Engineering, Slovak University of Technology in Bratislava. Each Wednesday, starting at 8.45 a.m., several original new results, as well as overview talks are presented for about 15 regular participants of the seminar. Speakers are either foreign visitors of Bratislava work-places, or members of the seminar from STU Bratislava or MBU Banská Bystrica. The scientific level of the seminar is guaranteed by Prof. R. Mesiar and Prof. A. Kolesárová, and the seminar is organised by Dr. T. Bacigál.

In the summer term of 2013, there were 12 participants at the seminar devoted to the various topics of uncertainty modelling and related areas. Proceedings *Uncertainty Modelling 2013* bring 10 accepted papers related to the presentations held during the summer term of 2013. All these papers were reviewed by independent reviewers and in Proceedings its final accepted version is published. Our gratitude goes to all authors, as well as to all reviewers whose work has significantly contributed to the high quality of papers included in these Proceedings.

Bratislava, October 4, 2013

T. Bacigál and R. Mesiar
Uncertainty Modelling 2013 editors

Modelling dependence with copulas and R package *acopula* ver. 0.9.2

Tomáš Bacigál *

Abstract: *We introduce* acopula *package (run under R) that aims to support researchers as well as practitioners in the field of modelling stochastic dependence. Description of tools with examples are given, namely several probability related functions, estimation and testing procedures, and two utility functions.*

Keywords: *Archimax copula, R, estimation, GOF test, copula quantile.*

1 Introduction

Copula is a d-dimensional function $C\colon [0,1]^d \to [0,1]$, $d \geq 2$, that can combine any univariate cumulative distribution functions to form a joint distribution function F of a random vector $\mathbf{X} = (X_1, \ldots, X_d)$, such that

$$F(x_1, \ldots, x_d) = C\left(F_{X_1}(x_1), \ldots, F_{X_d}(x_d)\right) \tag{1}$$

with F_{X_i} being distribution function associated with i-th random variable. Copula itself is a joint distribution function with uniform marginals, thus it is d-increasing, has 1 as neutral element and 0 as annihilator (see [12] for an exhaustive introduction).

Since the turn of century when copulas began to attract attention of masses, several software tools arose. The first public yet commercial to mention was EVANESCE library [7] included in FinMetrics extension to S programming environment (predecessor of R), that provided a rich battery of copula classes, though only bivariate. With emergence of R (free software environment for statistical computing and graphics, [13]) there came open-source packages like *copula* [8] (recently incorporating *nacopula*) and *CDVine* [3] with successor *VineCopula*, that are still under vivid development. For further reading about recent copula software see, e.g., [1].

Here we introduce an R package that extends current offerings on the one hand by class of *Archimax* copulas and on the other by several handy tools to test, modify, manipulate and inference from them and *arbitrary* user-defined absolutely continuous copulas, thus making copulas ready for *application*. That explains the initial letter of the package name.

In short, Archimax copula is a copula $C_{\phi,A}$, that can be represented in the form

$$C_{\phi,A}(u_1, \ldots, u_d) = \phi^{-1}\left[\left(\sum_{i=1}^{d} \phi(u_i)\right) A\left(\frac{\phi(u_1)}{\sum_{i=1}^{d}\phi(u_i)}, \ldots, \frac{\phi(u_d)}{\sum_{i=1}^{d}\phi(u_i)}\right)\right] \tag{2}$$

where $\phi\colon [0,1] \searrow [0,\infty]$, $\phi(1) = 0$, is a so-called generator of strict Archimedean copula and $A\colon \Delta_{d-1} \to [0,1]$ is a Pickands dependence function defined on unit simplex $\Delta_{d-1} = \{(w_1, \ldots, w_d) \in [0,1]^d | \sum_i^d w_i = 1\}$ and fulfilling boundary constraint $A(\mathbf{e}_i) = 1$ where $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)$ is the unit vector with 1 at position i. In bold we will denote a d-dimensional vector, e.g., $\mathbf{w} = (w_1, \ldots, w_d)$. Whenever $\phi(t) = -\log(t)$, $\forall t \in [0,1]$, copula $C_{\phi,A}$ belongs also to the class of Extreme-Value (EV) copulas, and equally, with $A \equiv 1$ the Archimax class degenerates to Archimedean class. The only additional constraints put on both functions ϕ, A to generate a bivariate Archimax copula are that they need to be convex and $1 \geq A \geq \max$, as proved by Capéraà et al. [4].

*Slovak University of Technology in Bratislava, *tomas.bacigal@stuba.sk*

However, in more dimensions things get rather complicated. McNeil and Nešlehova [10] showed that ϕ is a generator of an strict Archimedean copula[1] iff its inverse is d-monotone, i.e., (I) $\psi = \phi^{-1}$ has continuous derivatives $\psi^{(k)}$ on $[0, \infty]$ and (II) $(-1)^k \psi^{(k)}(t) \geq 0$ for any $k = 1, \ldots, d-2$, and also (III) $(-1)^{d-2}\psi^{(d-2)}$ is non-negative, non-increasing and convex on $[0, \infty]$. On the other hand, as summarized in [6], a d-variate copula C_l is an EV copula iff there exists a finite Borel measure H on Δ_{d-1}, called spectral measure, such that $C_l(\mathbf{u}) = \exp\left(-l(u_1, \ldots, u_d)\right)$ with tail dependence function $l\colon [0, \infty)^d \to [0, \infty)$ given by $l(\mathbf{x}) = \int_{\Delta_{d-1}} \bigvee_{j=1}^{d} (w_j x_j) dH(\mathbf{w})$ [2] and related to Pickands dependence function (due to its homogeneity) via $l(\mathbf{x}) = (\sum_{i=1}^{d} x_i) A(\mathbf{w})$, $w_j = x_j / \sum_{i=1}^{d} x_i$, $j = 1, \ldots, d$. The spectral measure H is arbitrary except for the d moment constraints $\int_{\Delta_{d-1}} w_j dH(\mathbf{w}) = 1$, $j \in \{1, \ldots, d\}$, that stem from the requirement for the margins of copula to be standard uniform, the constraints imply that $H(\Delta_{d-1}) = d$. The function A is still convex and bounded by $\max(\mathbf{w})$ and 1, however these properties do not characterize the class of Pickands dependence functions any more.

The question whether any ϕ can be combined with any l or A so that $C_{\phi,A}$ is a copula is still an open problem whenever $d > 2$. In [2] several positive examples based on partitions and general convex sum are given.

In the package, Pickands dependence function is implemented to accept $d-1$ dimensional argument since the last element is complementary, formally $A'(w_1, \ldots, w_{d-1}) = A(w_1, \ldots, w_{d-1}, 1 - \sum_{i=1}^{d-1} w_i)$.

Structure of the R package *acopula* is relatively simple, does NOT use object-oriented S4 classes and is comprehensible from the source code accompanied by explanation notes, so that even inexperienced user can, e.g., track erroneous behaviour if any occurs. Also it does not depend on any additional packages, though it suggest to use some. In the next sections particular functions are detailed and demonstrated on examples.

2 Definition lists

Every parametric family of copulas is defined within a list, either by its generator (in case of Archimedean copulas), Pickand's dependence function (Extreme-Value copulas) or directly by cumulative distribution function (CDF) with/or its density. Example of one such definition list follows[3] for generator of Gumbel-Hougaard family of Archimedean copulas

```
> genGumbel()
$parameters
[1] 4

$pcopula
function (t, pars) exp(-sum((-log(t))^pars[1])^(1/pars[1]))

$gen
function (t, pars) (-log(t))^pars[1]

$gen.der
function (t, pars) -pars[1]*(-log(t))^(pars[1]-1)/t

$gen.der2
function (t, pars) pars[1]*(-log(t))^(pars[1]-2)*(pars[1]-1-log(t))/t^2

$gen.inv
function (t, pars) exp(-t^(1/pars[1]))

$gen.inv.der
function (t, pars) -exp(-t^(1/pars[1]))*t^(1/pars[1]-1)/pars[1]

$kendall$coef
function (t) 1 - 1/t
```

[1] In fact, they showed it also for non-strict Archimedean copula when a pseudo-inverse of the generator needs to be used.

[2] $\bigvee$ denotes maximum (join).

[3] Output printing is simplified here whenever contains irrelevant parts.

```
$kendall$icoef
function (t) 1/(1 - t)

$kendall$bounds
[1] 0 1

$spearman$coef
function (t) pPareto(t, c(1.41917, 2.14723, 1, 1))

$spearman$icoef
function (t) qPareto(t, c(1.41917, 2.14723, 1, 1))

$spearman$bounds
[1] 0 1

$lower
[1] 1

$upper
[1] Inf

$id
[1] "Gumbel"
```

where, although some items may be fully optional (here `$pcopula`, `$kendall` and `$spearman`), they can contribute to better performance. The user is encouraged to define new parametric families of Archimedean copula generator (similarly of Pickands dependence function or copula in general) according to his/her needs, bounded only by this convention and allowed to add `pcopula` (stands for probability distribution function or CDF), `dcopula` (density) and `rcopula` (random sample generator) items, however compatibility with desired dimension has to be kept in mind. Currently implemented generators can be listed,

```
> ls("package:acopula",pattern="gen")
[1] "genAMH" "genClayton" "generator" "genFrank" "genGumbel" "genJoe" "genLog"
```

notice the generic function `generator` which points to specified definition lists.

Similarly, Pickands dependence functions are defined, namely Gumbel-Hougaard, Tawn, Galambos, Hüsler-Reiss (last three form only bivariate EV), extremal dep. functions and generalized convex combination of arbitrary valid dep. functions (see [11]). So are definition lists available for generic (i.e., not necessarily Archimax) copula, e.g. normal, Farlie-Gumbel-Morgenstern, Plackett and Gumbel-Hougaard parametric family. Their corresponding function names starts with `dep` and `cop`, respectively.

Since the class of Archimax copulas contains Archimedean and EV class as its special cases, the setting `depfu = dep1()` and `generator = genLog()` can distinguish them, respectively.

There are not many known dependence function parametric families capable of producing more-than-2-dimensional EV copula, much less Archimax copula, for that reason the (generalized) convex combination may come useful, used for instance in partition-based approach introduced by [2]. More specifically, having m dependence functions A_j, $j = 1, \ldots, m$, the function $A_{gcc}\colon \Delta_{d-1} \to [\frac{1}{d}, 1]$ given as

$$A_{gcc}(\mathbf{w}) = \sum_{j=1}^{m} s_j A_j \left(\frac{\alpha_{j1} w_1}{s_j}, \ldots, \frac{\alpha_{jd} w_d}{s_j} \right), \quad \text{with } s_j = \left(\sum_{i=1}^{d} \alpha_{ji} w_i \right) \text{ and } \alpha_{ji} > 0. \tag{3}$$

Note that $\sum_{j=1}^{m} \alpha_{ji} = 1$, and if $\alpha_{ji} = \lambda_j$ for all i, j, then $A_{gcc} = \sum_{j=1}^{m} \lambda_j A_j$ is standard convex combination (thus symmetric). Furthermore consider a partition $\mathcal{P} = \{B_1, \ldots, B_k\}$ of $\{1, \ldots, d\}$. Then the function $l_{\mathcal{P}}(x_1, \ldots, x_d) = \sum_{j=1}^{k} \left(\bigvee_{i \in B_j} x_i \right)$ is a tail dependence function based on spectral measure $H(\mathbf{w}) = \sum_{j=1}^{k} (\operatorname{card} B_j)\, \delta_{B_j} \left((\operatorname{card} B_j) w_1, \ldots, (\operatorname{card} B_j) w_d \right)$ where $\delta_{B_j}\colon [0, \infty)^d \to [0, \infty)$ is the generalized Dirac measure given by $\delta_{B_j}(1_{B_j}) = 1$ and $\delta_{B_j}(x) = 0$ whenever $x \neq 1_{B_j}$. For fixed

$d = 3$ there are 5 partitions of $\{1, 2, 3\}$ with corresponding Pickands dependence functions [4]

$$\begin{aligned} A^*(w_1, w_2, w_3) &= w_1 + w_2 + w_3 \quad \text{when } \mathcal{P}^* = \{\{1\}, \{2\}, \{3\}\} \\ A_*(w_1, w_2, w_3) &= w_1 \vee w_2 \vee w_3 \qquad \mathcal{P}_* = \{\{1, 2, 3\}\} \\ A_1(w_1, w_2, w_3) &= w_1 + w_2 \vee w_3 \qquad \mathcal{P}_1 = \{\{1\}, \{2, 3\}\} \\ A_2(w_1, w_2, w_3) &= w_2 + w_1 \vee w_3 \qquad \mathcal{P}_2 = \{\{2\}, \{1, 3\}\} \\ A_3(w_1, w_2, w_3) &= w_3 + w_1 \vee w_2 \qquad \mathcal{P}_3 = \{\{3\}, \{1, 2\}\}. \end{aligned}$$

Thus we get special parametric class `depGCC(ldepPartition3D(),dim=3)` with 3×5 parameters leading to 3-dimensional copula.

Any definition list item can be replaced already during the function call as shown in the next subsections. Thus one can set starting value of parameter(s) and their range in estimation routine, for instance.

3 Probability functions

First thing one would expect from a copula package is to obtain a value of desired copula in some specific point. To show variability in typing commands, consider again Gumbel-Hougaard copula with parameter equal to 3.5 in point (0.2,0.3). Then the following commands give the same result.

```
> pCopula(data=c(0.2,0.3),generator=genGumbel(),gpars=3.5)
> pCopula(data=c(0.2,0.3),generator=genGumbel(parameters=3.5))
> pCopula(data=c(0.2,0.3),generator=generator("Gumbel"),gpars=3.5)
> pCopula(data=c(0.2,0.3),generator=generator("Gumbel",parameters=3.5))
> pCopula(data=c(0.2,0.3),copula=copGumbel(),pars=3.5)
> pCopula(data=c(0.2,0.3),copula=copGumbel(parameters=3.5))
> pCopula(data=c(0.2,0.3),generator=genLog(),depfun=depGumbel(),dpars=3.5)
> pCopula(data=c(0.2,0.3),generator=genLog(),depfun=depGumbel(parameters=3.5))
[1] 0.1723903
```

If we need probabilities that a random vector would not exceed several points, those can be supplied to `data` in rows of matrix or data frame.

Conversely, given an incomplete point and a probability, the corresponding quantile emerge.

```
> pCopula(c(0.1723903,0.3),gen=genGumbel(),gpar=3.5,quantile=1)
> pCopula(c(NA,0.3),gen=genGumbel(),gpar=3.5,quan=1,prob=0.1723903)
> qCopula(c(0.3),quan=1,prob=0.1723903,gen=genGumbel(),gpar=3.5)
[1] 0.1999985
```

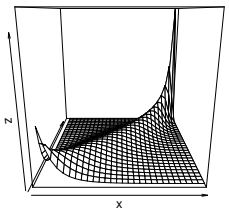
Conditional probability $P(X < x|Y = y)$ of a random vector (X, Y) has similar syntax.

```
> cCopula(c(0.2,0.3),conditional.on=2,gen=genGumbel(),gpar=3.5)
[1] 0.2230437
> qCopula(c(0.3),quan=1,prob=0.2230437,cond=c(2),gen=genGumbel(),gpar=3.5)
[1] 0.200005
```

Sometimes the density of a copula is of interest, perhaps for visualisation purposes, such as in the following example

```
x <- seq(0,1,length.out=30)
y <- seq(0,1,length.out=30)
z <- dCopula(expand.grid(x,y),generator=genGumbel(),gpars=3.5)
dim(z) <- c(30,30)
persp(x,y,z)
```

[4] For computational convenience, the maximum operator $\vee$ is approximated to have smooth edges, so that $w_1 \vee \ldots w_d = \bigvee_{i=1}^{d} w_i \approx \left(\sum_{i=1}^{d} w_i^{\lambda}\right)^{1/\lambda}$, where power λ defaults to 8 and can be changed by passing the argument `power`.

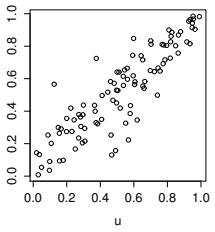


where instead of `persp` from package *graphics* a more impressive output is given by package *rgl* with function `persp3D`.

If definition lists do not contain explicit formulas for (constructing) density, the partial derivatives are approximated linearly. This is mostly the case with 3- and more-dimensional copulas.

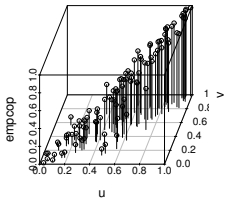
Sampling from the copula is, unsurprisingly, also provided.

```
sample <- rCopula(n=100,dim=2, generator=genGumbel(), gpars=3.5)
plot(sample)
```



Sometimes no assumption about parametric family of copula is made, instead an empirical distribution is of more interest. Then for a given data, say, the previous random sample, one may ask for value of empirical copula in specific point(s) and more easily in the points of its discontinuity.

```
> pCopulaEmpirical(c(0.2,0.3),base=sample)
[1] 0.14
> empcop <- pCopulaEmpirical(sample)
> scatterplot3d::scatterplot3d(cbind(sample,empcop),type="h",angle=70)
```



4 Estimation

Currently, there are two universal methods for parameters estimation implemented in the package (named `technique`): "ML", maximum (pseudo)likelihood method employing copula density, and "LS", least squares method minimizing distance to empirical copula. In short, given a random sample $\mathbf{U}_j$, $j = 1, \ldots, n$, from a continuous distribution with uniform (1-dimensional) margins, joint distribution function C_θ and density c_θ, define the maximum likelihood estimator as

$$\hat{\theta} = \arg\max_{\theta \in \mathcal{Q}} \sum_{j=1}^{n} \log[c_\theta(U_{j1}, \ldots, U_{jd})] \tag{4}$$

and least-squares estimator as

$$\hat{\theta} = \arg\min_{\theta \in \mathcal{Q}} \sum_{j=1}^{n} \left(C_n(U_{j1}, \ldots, U_{jd}) - C_\theta(U_{j1}, \ldots, U_{jd}) \right)^2, \tag{5}$$

where $\mathcal{Q}$ is parameter space of copula C_θ parameters set, $C_n(\mathbf{u}) = \frac{1}{n} \sum_{j=1}^{n} 1(U_{j1} \leq u_1, \ldots, U_{jd} \leq u_d)$, $\mathbf{u} \in [0,1]^d$, is the so-called empirical copula and $1(\cdot)$ is the indicator function which yields 1 whenever $\cdot$ is true and 0 otherwise. Both 'techniques' supplies function to perform optimization `procedure` over, thus finding those parameters that correspond to an optimum. The 'procedures' are three: "optim", "nlminb" and "grid". First two are system native, based on well-documented smart optimization methods, the third one uses brute force to get approximate global maximum/minimum and can be useful with multi-parameter copulas, at least to provide starting values for the other two 'procedures'.

For one-parameter bivariate copula families we also provide estimation method based on relation between copula parameter and rank-based measures of dependence (`technique="icorr"`), currently Kendall's tau (`corrtype="kendall"`),

$$\tau = 4 \iint_{[0,1]^2} C_\theta(u,v) dC_\theta(u,v) - 1, \tag{6}$$

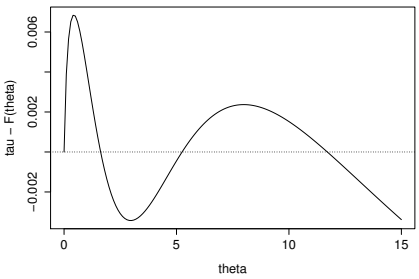
and Spearman's rho (`corrtype="spearman"`),

$$\rho = 12 \iint_{[0,1]^2} C_\theta(u,v) du v - 3. \tag{7}$$

Some of them have no closed form and need to be approximated, mostly by cumulative distribution function of the Pareto type IV distribution defined by

$$F_{\text{Pareto}}(x) = \begin{cases} 1 - \left(1 + \left(\frac{x - p_4}{p_1} \right)^{\frac{1}{p_2}} \right)^{-p_3} & x \geq \mu \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

with parameters $p_i > 0$, $i = 1, 2, 3$, and $p_4 \in \mathcal{R}$. Since F_{Pareto} is easily invertible, the estimation is fast and still acceptably precise keeping the approximation error in between -0.01 and +0.01 for dependence strength up to $\rho = 0.96$. For instance, relation between τ and Frank copula parameter (for positive dependence) is approximated by F_{Pareto} with parameters $p \approx (7.5, 1.3, 0.9, 0)$ and error $\tau(\theta) - \hat{F}_{\text{Pareto}}(\theta)$ plotted bellow.



The next few examples sketch various options one has got for copula fitting.

```
> eCopula(sample,gen=genClayton(),dep=depGumbel(),
+ technique="ML",procedure="optim",method="L-BFGS-B")
generator parameters:  0.09357958
   depfun parameters:  3.52958
   ML function value:  82.63223
    convergence code:  0
> eCopula(sample,gen=genClayton(),dep=depGumbel(),tech="ML",proc="nlminb")
```

```
generator parameters:  0.09183014
   depfun parameters:  3.533706
   ML function value:  82.63228
    convergence code:  0
> eCopula(sample,gen=genClayton(),dep=depGumbel(), tech="ML",proc="grid",
+ glimits=list(c(0),c(5)),dlimits=list(c(1),c(10)),pgrid=10)
generator parameters:  0.5555556
   depfun parameters:  3
   ML function value:  80.63322
    convergence code:
> eCopula(sample,gen=genGumbel(),technique="icorr",corrtype="kendall")
generator parameters:  3.442281
   depfun parameters:
   icorr function value:
    convergence code:
```

In addition, "optim" procedure has several `methods` to choose from: "L-BFGS-B", "Nelder-Mead", "BFGS", "CG", "SANN", "Brent".

So far, no precision for copula parameters is provided.

5 Testing

Having set of observations, it is often of great interest to test whether the estimated copula suffices to describe dependence structure in the data. For this purpose many goodness-of-fit tests were proposed, yet the principle remains to use different criterion than that employed with estimation of the copula parameters. Here we implement one of the 'blanket' tests described in [5] that is based on Kendall's transform of joint distribution function, $\mathcal{K}_\theta(t) = P(C_\theta(\mathbf{u}) \leq t)$, which reduces multivariate problem to one dimension. Its empirical version can be computed by $\mathcal{K}_n(t) = \frac{1}{n}\sum_{j=1}^{n} 1(C_n(\mathbf{U}_j) \leq t)$, $t \in [0,1]$. To test whether theoretical $\mathcal{K}$ matches the empirical one, the Cramér von-Mises test statistic $S_n = \int_0^1 \sqrt{n}(\mathcal{K}_\theta(t) - \mathcal{K}_n(t))^2 dt$ is available. As the asymptotic distributions of S_n depend on unknown copula C_θ and on θ in particular, approximate p-values must be found via simulation. The specific parametric bootstrap procedure is minutely described in [5].

In the example below normal copula is tested on the Gumbel copula sample data.

```
> gCopula(sample,cop=copNormal(),
+ etechnique="ML",eprocedure="optim",ncores=1,N=100)
Loading required package: mvtnorm
  |============================================================| 100%

 Blanket GOF test based on Kendall's transform

statistic       q95   p.value
0.1195500 0.1658125 0.1800000
-----------------------------
data:  sample
copula:  normal
estimates:
      pars     fvalue
 0.9155766 80.3420886
```

Although the p-value does not lead to rejection of the copula adequacy, its low value and small data length arouse suspicion. As for the other arguments, `N` sets number of bootstrap cycles and their parallel execution can be enabled by setting number of processor cores in `ncores` (not available under Windows OS). Package *mvtnorm* has been loaded to assist with simulation from normal copula, and when missing, internal but slower routine would be run instead.

The traditional parametric bootstrap-based procedure to approximate p-value, when theoretical probability distribution of the test statistic is unknown, is reliable yet computationally very exhaustive, therefore recently a method based on multiplier central limit theorem and proposed by [9] becomes popular with large-sample testing. Its implementation to testing goodness of parametric copula fit is scheduled

for future package updates. Nevertheless, the multiplier method takes part here in another test comparing two empirical copulas, i.e. dependence structure of two data sets, see [14] and package *TwoCop*. In the following example, random sample of the above Gumbel-Hougaard copula is tested for sharing common dependence structure with sample simulated from Clayton copula, parameter of which corresponds to the same Kendall's rank correlation ($\tau = 0.714$).

```
> sampleCl <- rCopula(n=100,dim=2,generator=genClayton(),gpars=5)
> gCopula(list(sample,sampleCl),ncores=1,N=100)
  |============================================================| 100%

 Test of equality between 2 empirical copulas

 statistic        q95    p.value
0.09791672 0.52893392 0.66000000
-----------------------------
data:  sample sampleCl
copula:
estimates:
NULL
```

Obviously, the test fails to distinguish copulas with differing tail dependence, at least having small and moderate number of observations, however it is sensitive enough to a difference in rank correlation.

The last procedure to mention checks the properties of a d-dimensional copula ($d \geq 2$), that is, being d-increasing as well as having 1 as neutral element and 0 as annihilator. The purpose is to assist approval of new copula constructs when theoretical proof is too complicated. The procedure examines every combination of discrete sets of copula parameters, in the very same fashion as within "grid" procedure of `eCopula`, by computing a) first differences recursively over all dimensions of an even grid of data points,i.e., C-volumes of subcopulas, b) values on the margin where one argument equals zero and c) where all arguments but one equals unity. Then whenever the result is a) negative, b) non-zero or c) other than the one particular argument, respectively, a record is made and first 5 are printed as shown below. In the example we examine validity of an assumed Archimedean copula generated by Gumbel-Hougaard generator family, only with a parameter being out of bounds.

```
> isCopula(generator=genGumbel(lower=0),dim=3,glimits=list(0.5,2),
+ dagrid=10,pgrid=4,tolerance=1e-15)

 Does the object appears to be a copula(?):  FALSE

 Showing 2 of 2 issues:

  dim property      value gpar
1   2    monot -0.1534827  0.5
2   3    monot -0.1402209  0.5
```

Three parameter values $(0.5, 1, 1.5, 2)$ were used, each supposed copula were evaluated in 10^3 grid nodes, and every violation of copula properties (the most extremal value per dimension and exceeding `tolerance`) were reported. Thus it is seen, that parameter value 0.5 does not result in copula because 3-monotonicity is not fulfilled (negative difference already in the second-dimension run). Note that without redefinition of lower bound the parameter value 0.5 would be excluded from the set of Gumbel-Hougaard copula parameters.

6 Utilities

For the *acopula* package to work many utility functions were created during development that were neither available in the basic R libraries nor they were found in contributed package under CRAN. Most of them are hidden within the procedures described above, however the two following are accessible on demand. The first to mention is a linear approximation of partial derivative of any-dimensional function and of any order with specification of increment (theoretically fading to zero) and area (to allow semi-differentiability)

```
> fun <- function(x,y,z) x^2*y*exp(z)
> nderive(fun,point=c(0.2,1.3,0),order=c(2,0,1),difference=1e-04,area=0)
[1] 2.600004
```

whereas the second utility function numerically approximates integration (by trapezoidal rule) such as demonstrated on example of joint standard normal density with zero correlation parameter

```
> nintegrate(function(x,y) mvtnorm::dmvnorm(c(x,y)),
+            lower=c(-5.,-5.),upper=c(0.5,1),subdivisions=30)
[1] 0.5807843
> pnorm(0.5)*pnorm(1)
[1] 0.5817583
```

fine-tuned by number of subdivisions. However, it must be admitted that numerical integration performs better with package *cubature*.

7 Conclusion

All the introduced and exemplified procedures are (a) extendible to arbitrary dimension, which is one of the significant contributions of the package. If explicit formulas are unavailable (through definition lists) then numerical approximation does the job. Another significant benefit is brought by (b) conditional probability and quantile function of the copula, as well as estimation methods based on least squares and grid complementing the usual maximum-likelihood method. Together with implementing (c) generalization of Archimedean and Extreme-Value by Archimax class with a (d) construction method of Pickands dependence function, (e) fast non-parametric estimation of one-parameter copulas , (f) numerical check of copula properties useful in new parametric families development, and (g) parallelized goodness-of-fit test based on Kendall's transform, these all (and under one roof) make the package competitive among both proprietary and open-source software tools for copula based analysis, to the date.

Yet because the routines are written solely in R language and rely on no non-standard packages (optionally), some tasks may take longer to perform. Nevertheless the source code is easy to access, understand and modify if necessary.

Future improvement is seen mainly in providing additional methods for parameters estimation (for multi-parameter copulas based on various dependence measures) and GoF tests, as well as connecting with other copula packages to simplify practical analysis.

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Efficient algorithm for computing certain graph-based monotone integrals: the l_p-indices

Marek Gagolewski [*†‡] Michał Dębski [§] Michał Nowakiewicz [†]

Abstract: *The Choquet, Sugeno and Shilkret integrals with respect to monotone measures are useful tools in decision support systems. In this paper we propose a new class of graph-based integrals that generalize these three operations. Then, an efficient linear-time algorithm for computing their special case, that is l_p-indices, $1 \leq p < \infty$, is presented. The algorithm is based on R.L. Graham's routine for determining the convex hull of a finite planar set.*

Keywords: *monotone measures, Choquet, Sugeno and Shilkret integral, l_p-index, convex hull, Graham's scan, scientific impact indices.*

1 Introduction

Many practical situations, especially in decision making, face us with the problem of aggregating numeric sequences not necessarily having equal lengths. For example, we observe a gradually increasing interest in developing objective and fair research performance evaluation methods of individual scientists by means of citations number of authored papers. Scientometricians try to find quantitative indicators that might complement, or even replace, expert judgment. Such tools could be used in deciding upon employment, grant allocation, etc.

Let us assume that we are given a set of vectors with elements in $\mathbb{I} = [0, \infty]$ and we would like to construct an aggregation operator F defined on the space $\mathbb{I}^{1,2,\cdots} := \bigcup_{i=1}^{\infty} \mathbb{I}^n$, and such that it is nondecreasing in each variable, arity-monotonic, and symmetric, cf. [6]. The most common approach is to assume that F is zero-insensitive, i.e. that it holds $\mathsf{F}(\mathbf{x}) = \mathsf{F}(\mathbf{x}, 0)$. It may be shown, cf. [7], that in such setting, a vector $\mathbf{x} \in \mathbb{I}^{1,2,\cdots}$ may be projected to the space $\mathcal{S}$ of infinite-length, nonincreasing vectors, $\widetilde{\mathbf{x}} = (x_{\{1\}}, x_{\{2\}}, \ldots, x_{\{n\}}, 0, 0, \ldots)$, where $x_{\{i\}}$ denotes the ith greatest value in $\mathbf{x}$, and then the construction of F is equivalent to considering $\mathsf{E} : \mathcal{S} \to \mathbb{I}$, $\mathsf{E}(0, 0, \ldots) = 0$, such that for all $\mathbf{x} \in \mathbb{I}^{1,2,\cdots}$ we have $\mathsf{F}(\mathbf{x}) = \mathsf{E}(\widetilde{\mathbf{x}})$.

In a very recent paper [7] we considered a uniform framework for the scientific impact assessment problem (and similar issues), where we have shown that most currently used bibliometric impact indices may be expressed by some universal integrals [10], see also [1, 2, 14] for other applications of monotone measures and integrals in scientometrics.

Recently, a very interesting class of so-called decomposition integrals [11] has been proposed. Some of these objects have a very nice graphical interpretation, which may be very important for the practitioners.

In this paper we propose another class of integrals that generalize the Sugeno, Choquet and Shilkret integrals, as well as some decomposition integrals. For example, they include the so-called "geometric" scientific impact indices proposed in [5]. Moreover, we introduce a linear-time algorithm for computing the l_p-indices and thus solve the open problem stated in [5]. The algorithm is an appealing modification of Graham's routine for the convex hull of a finite planar set [8].

[*]Corresponding author, *gagolews@ibspan.waw.pl*
[†]Systems Research Institute, Polish Academy of Sciences, Warsaw, Poland
[‡]Faculty of Mathematics and Information Science, Warsaw University of Technology, Warsaw, Poland
[§]Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Poland

2 Monotone measures and integrals

2.1 Monotone measures

Let $(Z, \mathcal{A})$ be a measurable space, i.e. a nonempty set Z equipped with a σ-algebra. We call $\mu : \mathcal{A} \to \mathbb{I}$ a *monotone measure* (a capacity), if (a) $\mu(\emptyset) = 0$, (b) $\mu(Z) > 0$, and (c) $\mu(U) \leq \mu(V)$ for $U \subseteq V$. Note that a monotone measure is not necessarily (σ-)additive. Moreover, let $\mathcal{M}^{(Z,\mathcal{A})}$ denote the set of all monotone measures.

A function $\mathsf{f} : Z \to \mathbb{I}$ is called *$\mathcal{A}$-measurable* if for each T in the σ-algebra of Borel subsets of $\mathbb{I}$, the inverse image $\mathsf{f}^{-1}(T) \in \mathcal{A}$. Let $\mathcal{F}^{(Z,\mathcal{A})}$ denote the set of all $\mathcal{A}$-measurable functions $\mathsf{f} : Z \to \mathbb{I}$.

2.2 Integrals

Let $\{u : \mathsf{f}(u) \geq t\} \in \mathcal{A}$ denote the so-called t-level set of $\mathsf{f} \in \mathcal{F}^{(Z,\mathcal{A})}$, $t \in \mathbb{I}$. It is easily seen that $\{u : \mathsf{f}(u) \geq t\}_{t\in\mathbb{I}}$ forms a left-continuous, nonincreasing chain (w.r.t. t). Thus, $\mathsf{h}^{(\mu,\mathsf{f})}(t) := \mu(\{u : \mathsf{f}(u) \geq t\})$ is a nonincreasing function of t, $\mathsf{h}^{(\mu,\mathsf{f})} : \mathbb{I} \to \mathbb{I}$.

Which function shall be called an integral of $\mathsf{f} \in \mathcal{F}^{(Z,\mathcal{A})}$ is still a disputable issue. Generally, it is agreed that an integral should map the space $\mathcal{M}^{(Z,\mathcal{A})} \times \mathcal{F}^{(Z,\mathcal{A})}$ into $\mathbb{I}$, should be at least nondecreasing with respect to each coordinate, and for $\mathsf{f} \equiv 0$ it should "output" the value 0.

Often integrals are defined as a function $\mathcal{I} : \mathcal{M}^{(Z,\mathcal{A})} \times \mathcal{F}^{(Z,\mathcal{A})} \to \mathbb{I}$ given by:

$$\mathcal{I}(\mu, \mathsf{f}) = \mathcal{J}(\mathsf{h}^{(\mu,\mathsf{f})}),$$

where $\mathcal{J} : \mathcal{F}^{(\mathbb{I},\mathcal{B}(\mathbb{I}))} \to \mathbb{I}$ is nondecreasing, $\mathcal{J}(0) = 0$.

For example, universal integrals, thoroughly discussed in [10], fulfill additional condition that for each $c, d \in \mathbb{I}$ we have $\mathcal{J}(d \cdot \mathbf{I}_{(0,c]}) = c \otimes d$, where $\otimes$ is some pseudo-multiplication.

2.3 Graph-based integrals

Let $\mathrm{Gr}(\mathsf{h}^{(\mu,\mathsf{f})}) = \{(x, y) \in \mathbb{I}^2 : y < \mathsf{h}^{(\mu,\mathsf{f})}(x)\}$. Recall that the Choquet integral is given by

$$\mathrm{Ch}(\mu, \mathsf{f}) = \int_{\mathbb{I}} \mu(\{u : \mathsf{f}(u) \geq t\})\, dt.$$

It may easily be shown that if $\mathrm{Gr}(\mathsf{h}^{(\mu,\mathsf{f})})$ is bounded and measurable, then $\mathrm{Ch}(\mu, \mathsf{f}) = \iint_{\mathbb{I}^2} d\mathrm{Gr}(\mathsf{h}^{(\mu,\mathsf{f})})$. Inspired with this fact and the notion of an decomposition integral [11], we introduce the so-called graph-based integrals.

Let $\mathcal{H} \subseteq 2^{2^{\mathbb{I}^2}}$, $\mathcal{H} \neq \emptyset$ be such that for all $\mathcal{P} \in \mathcal{H}$ it holds $p \cap p' = \emptyset$ for all $p, p' \in \mathcal{P}$, $p \neq p'$, i.e. it is a system of sets of disjoint Lebesgue-measurable subsets of $\mathbb{I}^2$. We define the **graph-based integral** corresponding to $\mathcal{H}$ as:

$$\mathrm{Gbi}_{\mathcal{H}}(\mu, \mathsf{f}) = \sup \left\{ \sum_{p\in\mathcal{P}} \lambda(p) : \mathcal{P} \in \mathcal{H}, \bigcup_{p\in\mathcal{P}} p \subseteq \mathrm{Gr}(\mathsf{h}^{(\mu,\mathsf{f})}) \right\}, \tag{1}$$

where λ is the Lebesgue measure, here in the space $(\mathbb{I}^2, \mathcal{B}(\mathbb{I}^2))$.

Intuitively, the calculation of a graph-based integral is done by finding the total area of the maximal "subcover" of $\mathrm{Gr}(\mathsf{h}^{(\mu,\mathsf{f})})$ by shapes from $\mathcal{H}$.

Here are some worth-noting instances of graph-based integrals.

Example 1. Let $\mathcal{H} = \{\{[x_1, x_2] \times [y_1, y_2]\}_{0\leq x_1\leq x_2, 0\leq y_1\leq y_2}\}$, i.e. each element of $\mathcal{H}$ is a set consisting of exactly one rectangle in $\mathbb{I}^2$. Then $\mathrm{Gbi}_{\mathcal{H}}(\mu, \mathsf{f})$ is equivalent to the Shilkret integral [12], $\mathrm{Sh}(\mu, \mathsf{f}) = \sup_{t\in\mathbb{I}}\{t \cdot \mu\{u : \mathsf{f}(u) \geq t\}\}$.

Example 2. Let $\mathcal{H} = \{\{[0,r] \times [0,r]\}_{0\leq r}\}$. Then $\sqrt{\mathrm{Gbi}_{\mathcal{H}}(\mu,\mathsf{f})}$ is equivalent to the Sugeno integral [13], $\mathrm{Su}(\mu,\mathsf{f}) = \sup_{t\in\mathbb{I}}\{t \wedge \mu\{u : \mathsf{f}(u) \geq t\}\}$. The same holds e.g. for $\mathcal{H} = \{\{[x,x+r] \times [y,y+r]\}_{0\leq r,0\leq x,0\leq y}\}$.

Example 3. Let $\mathcal{H}_k = \{\{[x_{1,i},x_{2,i}]\times[y_{1,i},y_{2,i}]\}_{i=1,\ldots,k,0\leq x_{1,i}\leq x_{2,i},0\leq y_{1,i}\leq y_{2,i}}\}$, such that for all $\mathcal{P} \in \mathcal{H}_k$, $p \cap p' = \emptyset$ for $p,p' \in \mathcal{P}$, $p \neq p'$, i.e. $\mathcal{P}$ is a set of k disjoint rectangles. Then each $\mathrm{Gbi}_{\mathcal{H}_k}(\mu,\mathsf{f})$ is an universal decomposition integral as defined in [11, Def. 4.4]. Moreover, $\lim_{k\to\infty}\mathrm{Gbi}_{\mathcal{H}_k}(\mu,\mathsf{f}) = \mathrm{Ch}(\mu,\mathsf{f})$, i.e. the Choquet integral [3].

2.4 The uniform model for bibliometric impact assessment

In [7] Gagolewski and Mesiar presented the following uniform model for bibliometric impact assessment problem. First of all, we need a transformation from the vector space $\mathcal{S}$ into the space $\mathcal{F}^{(\mathbb{I},\mathcal{B}(\mathbb{I}))}$. Given $\mathbf{x} \in \mathcal{S}$, let $\langle\mathbf{x}\rangle \in \mathcal{F}^{(\mathbb{I},\mathcal{B}(\mathbb{I}))}$ such that

$$\langle\mathbf{x}\rangle(t) = x_{\lfloor t+1\rfloor}, \quad t \in \mathbb{I}.$$

Let us consider the family Φ of aggregation operators $\mathsf{F} : \mathcal{S} \to \mathbb{I}$ given by the equation:

$$\mathsf{F}(\mathbf{x}) = \eta\Big(\mathcal{I}\big(\mu, \langle\varphi(\mathbf{x})\rangle\big)\Big) \tag{2}$$

where:

- $\varphi : \mathcal{S} \to \mathcal{S}$ – a function nondecreasing in each variable, $\varphi(0,0,\ldots) = (0,0,\ldots)$,
- $\mu : \mathcal{B}(\mathbb{I}) \to [0,\infty]$ – a monotone measure,
- $\mathcal{I}$ – an integral on $\mathcal{M}^{(\mathbb{I},\mathcal{B}(\mathbb{I}))} \times \mathcal{F}^{(\mathbb{I},\mathcal{B}(\mathbb{I}))}$,
- $\eta : \mathbb{I} \to \mathbb{I}$ – an increasing function, $\eta(0) = 0$.

It may be shown that an aggregation operator F may be expressed as (2) if and only if it is a zero-insensitive impact function, see [7]. Moreover, $\mathsf{h}^{(\mu,\langle\varphi(\mathbf{x})\rangle)}$ is a nonincreasing step function.

Example 4. For $p \geq 1$ let $\mathcal{H}_p = \{\{\mathrm{B}_p(r)|_{\mathbb{I}^2}\}_{r\geq 0}\}$, where $\mathrm{B}_p(r) = \{(x,y) : \|(x,y)\|_p \leq r\}$, i.e. it is a ball of radius r w.r.t. L_p distance, centered at $(0,0)$. Then for μ being a Lebesgue measure and $\varphi = \mathrm{id}$, $\lfloor\sqrt{\mathrm{Gbi}_{\mathcal{H}_\infty}(\mu,\langle\varphi(\mathbf{x})\rangle)}\rfloor$ is equivalent to the h-index [9], $\lfloor\sqrt{2\,\mathrm{Gbi}_{\mathcal{H}_1}(\mu,\langle\varphi(\mathbf{x})\rangle)}\rfloor$ is the w-index [15] and, more generally, $\lfloor\sqrt{p\,\mathrm{Gbi}_{\mathcal{H}_p}(\mu,\langle\varphi(\mathbf{x})\rangle)/B(1/p,1+1/p)}\rfloor$ gives the r_p-index [5], where B is the Euler beta function.

Example 5. For $p \geq 1$ let $\mathcal{H}_p = \{\{\mathrm{Bs}_p(a,b)|_{\mathbb{I}^2}\}_{a>0,b>0}\}$, where $\mathrm{Bs}_p(a,b)$ is a scaled L_p ball (L_p ellipse): $\mathrm{Bs}_p(a,b) = \{(x,y) : \|(x/a,y/b)\|_p \leq 1\}$, cf. Fig. 1. Then $p\,\mathrm{Gbi}_{\mathcal{H}_p}(\mu,\langle\varphi(\mathbf{x})\rangle)/B(1/p,1+1/p)$ is the (projected) l_p-index [5] if μ is the Lebesgue measure. Moreover, $\mathrm{Gbi}_{\mathcal{H}_\infty}(\mu,\mathsf{f})$ is equivalent to the Shilkret integral $\mathrm{Sh}(\mu,\mathsf{f})$ [12].

3 Determining the value of an l_p-index

As the definition of graph-based integrals for particular $\mathcal{H}$ sets may seem quite complicated, one should ask him- or herself a question whether there exist an algorithm that calculates the value of the integral efficiently. Of course, the Choquet, Sugeno, and Shilkret (and thus l_∞-index) integral for given $\mathsf{h}^{(\mu,\langle\varphi(\mathbf{x})\rangle)}$ may be calculated in linear time, i.e. $O(n)$, where $n = |\{x_i : x_i > 0\}|$.

Calculation of some graph-based integrals (like decomposition integrals from Example 3 for some k) seem to be an NP-Complete problem. Here we will derive an algorithm for calculating $\mathrm{Gbi}_{\mathcal{H}_p}(\mu,\langle\varphi(\mathbf{x})\rangle)$ for $1 \leq p < \infty$, where $\mathcal{H}_p$ is given in Example 5, i.e. we will get an l_p-index in particular. Such method is of practical interest, as a naïve implementation has computational complexity of $O(n^3)$, which even for moderate values of n (> 100) may require too much of computer processor time.

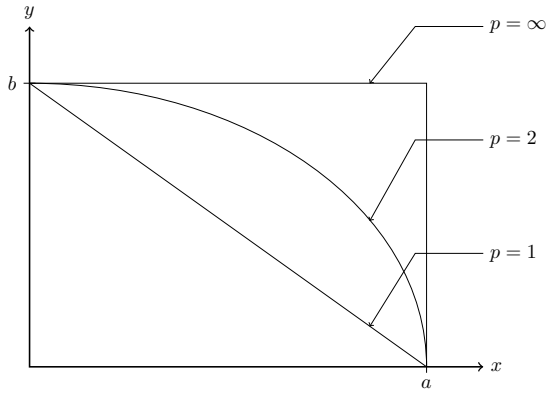

Figure 1: Boundaries of $\mathrm{Bs}_p(a,b)|_{\mathbb{I}^2}$ for different p.

Fix $\varphi, \mu, \mathbf{x}$ and $1 \leq p < \infty$, with $\mathsf{h}^{(\mu,\langle\varphi(\mathbf{x})\rangle)} \not\equiv 0$. As $\mathsf{h}^{(\mu,\langle\varphi(\mathbf{x})\rangle)}$ is a lower semicontinuous step function, let $\mathbf{Q} = (\mathbf{q}_0, \ldots, \mathbf{q}_n)$, $\mathbf{q}_i = (q_{ix}, q_{iy}) \in \mathbb{I}^2$, $q_{0x} = 0$, q_{ix} be (all) such that $\mathsf{h}^{(\mu,\langle\varphi(\mathbf{x})\rangle)}(q_{ix}^-) \neq \mathsf{h}^{(\mu,\langle\varphi(\mathbf{x})\rangle)}(q_{ix}^+)$ for $i > 1$, and $q_{iy} = \mathsf{h}^{(\mu,\langle\varphi(\mathbf{x})\rangle)}(q_{ix})$ for $i = 0, 1, \ldots, n$. Moreover, we assume that $q_{ix} < q_{j_x}$ for $i < j$.

Let $\mathbf{u} = (x_u, y_u)$ and $\mathbf{v} = (x_v, y_v)$ be arbitrary points in $\mathbb{I}^2$, for which $0 \leq x_u < x_v$ and $y_u > y_v \geq 0$. Let $\mathrm{Bs}_p(\mathbf{u}, \mathbf{v})$ denote the L_p ellipse interpolating these two points. It may be easily shown that $\mathrm{Bs}_p(\mathbf{u}, \mathbf{v}) = \mathrm{Bs}_p(a, b)$, where

$$a = \left(\frac{c}{y_v^p - y_u^p}\right)^{\frac{1}{p}}, \quad b = \left(\frac{-c}{x_v^p - x_u^p}\right)^{\frac{1}{p}},$$

and $c = x_u^p y_v^p - x_v^p y_u^p$.

The following lemma states that the graph-based integral of our interest may be determined by calculating the measure of an l_p-ellipse interpolating some two points from $\mathbf{Q}$.

Lemma 1. *There exist i, k, $i < k$, such that*

$$\lambda(\mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_k)) = \mathrm{Gbi}_{\mathcal{H}_p}(\mu, \langle\varphi(\mathbf{x})\rangle).$$

The proof is straightforward and therefore omitted.

The next lemma states that the boundaries of any two p-ellipses intersect in $\mathbb{I}^2$ at most in one point.

Lemma 2. *For any a, a', b, b' $|\partial\mathrm{Bs}_p(a,b)|_{\mathbb{I}^2} \cap \partial\mathrm{Bs}_p(a',b')|_{\mathbb{I}^2}| \leq 1$.*

The proof is left to the reader.

Lemma 3. *Let $0 \leq i < j < k \leq n$, such that $\mathbf{q}_i \notin \mathrm{Bs}_p(\mathbf{q}_j, \mathbf{q}_k)$. Then*

(i) $\mathbf{q}_k \notin \mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_j)$;

(ii) $\mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_j)|_{[0,q_{j_x}]\times\mathbb{I}} \supseteq \mathrm{Bs}_p(\mathbf{q}_j, \mathbf{q}_k)|_{[0,q_{j_x}]\times\mathbb{I}}$;

(iii) $\mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_j)|_{[q_{j_x},\infty)\times\mathbb{I}} \subseteq \mathrm{Bs}_p(\mathbf{q}_j, \mathbf{q}_k)|_{[q_{j_x},\infty)\times\mathbb{I}}$.

See Fig. 2 for an illustration of the lemma. The proof is omitted.

The proposed algorithm is given in Fig. 3. It is a modification of Graham's [8] routine for determining the convex hull of a finite planar set of points, also known as the Graham Scan.

The algorithm uses a stack, $\mathbf{S}$, i.e. a data structure on which the following operations may be performed: Push (adds an element to the top), Pop (removes the current top element) and $\#\mathbf{S}$ (returns the number of stored elements). Its elements may be accessed by an indexing operator $[\cdot]$, eg. $\mathbf{S}[\#\mathbf{S}]$ gets the element from the top of the stack.

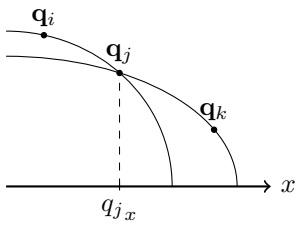

Figure 2: Illustration of Lemma 3.

Input: $p \in [1, \infty)$ and $\mathbf{Q} := (\mathbf{q}_0, \ldots, \mathbf{q}_n)$ determined by $\mathbf{x}, \mu, \varphi$ (see p. 3),
Result: $\mathrm{Gbi}_{\mathcal{H}_p}(\mu, \langle \varphi(\mathbf{x}) \rangle)$.

1 Create an empty stack $\mathbf{S} \subseteq \mathbf{Q}$;
2 Push $\mathbf{q}_0$ into $\mathbf{S}$;
3 Let $i := 1$;
4 **while** $(i < n)$ **and** $(q_{iy} = q_{0y})$ **do**
5 $\quad i := i + 1$;
6 Push $\mathbf{q}_i$ into $\mathbf{S}$;
7 **for** $j = i + 1, i + 2, \ldots, n$ **do**
8 $\quad$ **if** $(\mathbf{S}[\#\mathbf{S}]_y \neq q_{j\,y})$ **then** {
9 $\quad\quad$ **while** $(\#\mathbf{S} \geq 2)$ **and** $(\mathbf{S}[\#\mathbf{S} - 1] \in \operatorname{int} \mathrm{Bs}_p(\mathbf{S}[\#\mathbf{S}], \mathbf{q}_j))$ **do**
10 $\quad\quad\quad$ Pop from $\mathbf{S}$;
11 $\quad\quad$ Push q_j into $\mathbf{S}$;
12 $\quad$ }
13 **return** $B(1/p, 1 + 1/p) \cdot \max\{\mathbf{S}[i] \cdot \mathbf{S}[i + 1] : i = 1, 2, \ldots, \#\mathbf{S} - 1\}/p$,
i.e. $\max\{\lambda(\mathrm{Bs}_p(\mathbf{S}[i], \mathbf{S}[i + 1])) : i = 1, 2, \ldots, \#\mathbf{S} - 1\}$;

Figure 3: Algorithm for computing $\mathrm{Gbi}_{\mathcal{H}_p}(\mu, \langle \varphi(\mathbf{x}) \rangle)$, for $1 \leq p < \infty$.

The stack stores points in $\mathbb{I}^2$ which will be used to find the maximal p-ellipse. The algorithm scans through all the points from $\mathbf{Q}$, in the direction of increasing x (and nonincreasing y). At the j-th iteration, it repetitively pops elements from $\mathbf{S}$, until the p-ellipse interpolating $\mathbf{q}_j$ and the top-stack element does not contain the last-to-top element. After this process we consider only the p-ellipses obtained (by interpolation) from each two consecutive points on the stack. We state, that the p-ellipse of maximum area can be found among those.

Lemma 4. *Let* $\mathbf{S}^*$ *denote the contents of the stack after running the algorithm on arbitrary* $\mathbf{Q} = Q(\mathsf{h}^{(\mu,\mathsf{f})})$. *Then:*

(a) for all $i = 1, 2, \ldots, \#\mathbf{S}^* - 1$ *it holds that* $\mathrm{Bs}_p(\mathbf{S}^*[i], \mathbf{S}^*[i + 1]) \subseteq \mathrm{Gr}(\mathsf{h}^{(\mu,\mathsf{f})})$.

(b) for any $0 \leq i < k \leq n$ *such that* $\mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_k) \subseteq \mathrm{Gr}(\mathsf{h}^{(\mu,\mathsf{f})})$ *we have*

$$\max\{\lambda(\mathrm{Bs}_p(\mathbf{S}[i], \mathbf{S}[i + 1])) : i = 1, 2, \ldots, \#\mathbf{S} - 1\} \geq \lambda(\mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_k)).$$

Proof. (a) For brevity of notation, let $E_l := \mathrm{Bs}_p(\mathbf{S}^*[l], \mathbf{S}^*[l + 1])$ for $l = 1, 2, \ldots, \#\mathbf{S}^* - 1$. Let $\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k \in \mathbf{S}^*$ be any three consecutive points from the stack. Also, let l be an integer such that $\mathbf{S}^*[l] = \mathbf{q}_j$. We have $E_l \equiv \mathrm{Bs}_p(\mathbf{q}_j, \mathbf{q}_k)$. Then for every m such that $j < m < k$ it holds $\mathbf{q}_m \notin \operatorname{int} E_l$, due to Lemma 2 and the fact that $\mathbf{q}_j \in \mathrm{Bs}_p(\mathbf{q}_m, \mathbf{q}_k)$ (otherwise $\mathbf{q}_m$ would not be removed from the stack at some step, cf. line 10 of the algorithm).

By Lemma 3, for any $i < m < j$ we have $\mathbf{q}_m \notin \operatorname{int} E_l$. As $\mathbf{S}^*[l - 2] \notin \operatorname{int} E_{l-1}$ then also $\mathbf{S}^*[l - 2] \notin E_l$. By induction, for every $m < j$, $\mathbf{q}_m \notin \operatorname{int} E_l$.

Similarly we may show that for every $m > k$, $\mathbf{q}_m \notin \operatorname{int} E_l$. As $\mathbf{S}^*[1] = \mathbf{q}_0$ and $\mathbf{S}^*[\#\mathbf{S}^*] = \mathbf{q}_n$,

(b) Let $i < k$ with $\mathbf{q}_i, \mathbf{q}_k$ not being two consecutive elements from $\mathbf{S}^*$. Consider two cases:

(i) Assume that $\mathbf{q}_i \in \mathbf{S}^*$ and $\mathbf{q}_k \in \mathbf{S}^*$. Let $\mathbf{q}_j$ be the element from the stack that directly precedes $\mathbf{q}_k$. By (a), $\mathbf{q}_j \neq \mathbf{q}_i$. We have $\mathbf{q}_i \notin \operatorname{int} \mathrm{Bs}_p(\mathbf{q}_j, \mathbf{q}_k)$. Lemma 2 states that p-ellipses $\mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_j)$ and $\mathrm{Bs}_p(\mathbf{q}_j, \mathbf{q}_k)$ intersect only in $\mathbf{q}_j$. That implies $\mathbf{q}_j \in \operatorname{int} \mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_k)$, thus $\mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_k)$ cannot generate a proper solution to our task.

(ii) Assume that $\mathbf{q}_i \notin \mathbf{S}^*$. Let $j = \min\{m > i : \mathbf{q}_m \in \mathbf{S}^*\}$, i.e. $\mathbf{q}_j$ be the element from the stack the nearest to $\mathbf{q}_i$ on the right. Also, let l be an integer such that $\mathbf{S}^*[l] = \mathbf{q}_j$. By Lemma 2, we either have $\mathbf{q}_j \in \operatorname{int} \mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_k)$ or $\mathbf{q}_j \notin \operatorname{int} \mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_k)$ but $\mathbf{S}^*[l-1] \in \operatorname{int} \mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_k)$, because $\mathbf{S}^*[l-1] \in \operatorname{int} \mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_j)$ (line 10 of the algorithm). Thus,

$$\lambda(\mathrm{Bs}_p(\mathbf{q}_i, \mathbf{q}_k)) < \max\left\{\lambda(\mathrm{Bs}_p(\mathbf{S}[i], \mathbf{S}[i+1])) : i = 1, 2, \ldots, \#\mathbf{S} - 1\right\}.$$

(iii) Case $\mathbf{q}_k \notin \mathbf{S}^*$ is similar to the previous one, therefore the proof is complete.

□

Now we may approach to the final conclusion.

Theorem 5. *For any $1 \leq p < \infty$, nonincreasingly sorted $\mathbf{x}$, μ, and φ, $\mathrm{Gr}(\mathsf{h}^{(\mu,\mathsf{f})})$ may be determined in linear time using the algorithm in Fig. 3.*

Proof. By Lemma 1, the maximal p-ellipse interpolates some two points from $\mathbf{Q}$. The algorithm finds in $O(n)$ time the only (Lemma 4) p-ellipses which may generate the desired solution. In the last step of the algorithm, the largest p-ellipse is determined in $O(n)$, thus the proof is complete. □

4 Conclusions

In this paper we introduced the notion of a graph-based integral, which generalizes the Choquet, Shilkret, and Sugeno integrals, as well as some decomposition integrals, and which have a very appealing graphical interpretation. Moreover, for a particular class of those integrals, the l_p-indices, we developed an efficient, linear-time algorithm. The routine being, on its own, an interesting modification of Graham's Scan, was shown to be potentially useful in practical applications. Its implementation for $\mu = \lambda$ and $\varphi = \mathrm{id}$ has been included in the agop package for R, see [4].

Future work should definitely explore formally the properties of graph-based integrals and their relation with a universal integral.

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Construction of copulas by means of measure-preserving transformations

L'ubomíra Horanská [*,†], Anna Kolesárová [*,‡]

Abstract: *The aim of the paper is the presentation of a construction of n-copulas which is based on an arbitrary n-copula and some special measure-preserving transformations. We also show an equivalent alternative approach for obtaining such copulas. In the last section, properties of resulting 2-dimensional copulas are investigated.*

Keywords: *Copula, Measure-preserving transformation.*

1 Introduction

Copulas are functions describing the dependence structure of random vectors. By the Sklar theorem [13], copulas join multivariate distribution functions of random vectors to their one-dimensional marginal distribution functions. More precisely, for each random vector $(X_1, X_2, \ldots, X_n)$, $n \in \mathbb{N}$, $n \geq 2$, there exists a copula C such that the joint distribution function H of a random vector $(X_1, X_2, \ldots, X_n)$ and marginal distribution functions $F_1, F_2, \ldots, F_n$ of the random variables $X_1, X_2, \ldots, X_n$, respectively, are related by

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)), \text{for all } x_1, \ldots, x_n \in \overline{\mathbb{R}}.$$

Copulas can be seen as the restrictions to the unit n-box of joint distribution functions with marginals uniformly distributed over the unit interval. From practical point of view, the importance of copulas follows from the fact that they enable to separate the modeling of a complex n-dimensional random process into two parts, namely, into looking for appropriate one-dimensional (marginal) distribution functions and for an appropriate copula. There are plenty of applications of copulas, for instance, in quantitative finance, engineering, medicine, weather and climate research, etc. Mathematically, copulas can be introduced as follows.

Definition 1. *Let $\mathbf{I} = [0, 1]$. A function $C : \mathbf{I}^n \to \mathbf{I}$ is an n-copula if it satisfies the following conditions:*

(C1) $C(x_1, \ldots, x_n) = 0$ *if* $x_i = 0$ *for some* $i \in \{1, \ldots, n\}$, *i.e.,* 0 *is the annihilator of* C,

(C2) $C(x_1, \ldots, x_n) = x_i$ *if* $x_j = 1$ *for all* $j \neq i$, $i, j \in \{1, \ldots, n\}$, *i.e.,* 1 *is a neutral element of* C,

(C3) C *is n-increasing, i.e., the C-volume V_C of each n-box* $\prod_{i=1}^{n} [x_i, y_i] \subseteq \mathbf{I}^n$ *is non-negative:*

$$V_C(\prod_{i=1}^{n}[x_i, y_i]) := \sum_{\mathbf{v} \in \prod_{i=1}^{n} \{x_i, y_i\}} (-1)^{N(\mathbf{v})} C(\mathbf{v}) \geq 0,$$

where $N(\mathbf{v}) = \mathrm{card}(\{j : v_j = x_j\})$.

[*] Institute of Information Engineering, Automation and Mathematics, Faculty of Chemical and Food Technology, Slovak University of Technology in Bratislava

[†] *lubomira.horanska@stuba.sk*

[‡] *anna.kolesarova@stuba.sk*

Note that for each n-copula C and all $(x_1, \ldots, x_n) \in \mathbf{I^n}$ it holds

$$W(x_1, \ldots, x_n) \leq C(x_1, \ldots, x_n) \leq M(x_1, \ldots, x_n),$$

where $W(x_1, \ldots, x_n) = \max\{x_1 + \cdots + x_n - n + 1, 0\}$ and $M(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}$. The upper bound M is a copula for each number n of arguments, while the lower bound W is a copula only for $n = 2$. An important n-copula (for each $n \geq 2$) is the product copula Π, modeling the independence of random variables, given by $\Pi(x_1, \ldots, x_n) = x_1 \cdots x_n$. For more details on copulas we refer to [11].

In the last period copulas have been studied very intensively. A large number of recent papers have been devoted to constructions of copulas, e.g., [1, 2, 3, 10, 12, 8, 5, 9], among others. Basic methods of constructing copulas are also studied in the monograph [11]. The aim of this paper is to present a construction method for n-copulas by means of measure-preserving transformations. We also show an alternative way for obtaining these copulas. In the last section we prove several results for binary copulas.

2 Copulas and measure-preserving transformations

Let us briefly describe a correspondence between copulas and measure-preserving transformations on the unit interval. More details can be found, e.g., in [6], also see the references therein.

Let us denote by $\mathcal{B}(\mathbf{I})$ the system of all Borel subsets of the unit interval $\mathbf{I}$. We say that a mapping $f : \mathbf{I} \to \mathbf{I}$ is a measure-preserving transformation on the unit interval, if for every $B \in \mathcal{B}(\mathbf{I})$, the pre-image $f^{-1}(B) \in \mathcal{B}(\mathbf{I})$ and $\lambda(f^{-1}(B)) = \lambda(B)$, where λ is the standard Lebesgue measure on $\mathcal{B}(\mathbf{I})$.

Let us assign to each number $a \in \mathbf{I}$ a function $f_a \colon \mathbf{I} \to \mathbf{I}$ in the following way: for $a = 0$ let f_a be the identity function, $f_0(t) = t$, for $a = 1$ let $f_1(t) = 1 - t$, and for any $a \in]0, 1[$ let f_a be a piecewise linear function, defined by

$$f_a(t) = \max\left\{1 - \frac{t}{a}, \frac{t-a}{1-a}\right\}, \text{ i.e., } f_a(t) = \begin{cases} 1 - \frac{t}{a} & \text{if } t \in [0, a], \\ \frac{t-a}{1-a} & \text{if } t \in]a, 1]. \end{cases} \tag{1}$$

It is easy to see that $f_a^{-1}([0, x]) = [a(1 - x), x + a(1 - x)]$ and $\lambda(f_a^{-1}([0, x])) = \lambda([0, x]) = x$, see Fig.1. Clearly, functions f_a, $a \in \mathbf{I}$, are measure-preserving transformations on the unit interval. Note that to simplify the notation, in what follows, instead of the notation $f_a^{-1}([0, x])$ we will write $f_a^{-1}[0, x]$ only.

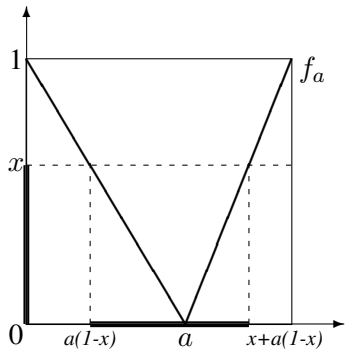

Figure 1: The graph of $f_a, a \in]0, 1[$. It holds $f_a^{-1}[0, x] = [a(1 - x), x + a(1 - x)]$.

The following theorem describes the above mentioned correspondence between copulas and measure-preserving transformations on the unit interval, see, e.g., [6].

Theorem 1. *If $\varphi_1, \varphi_2, \ldots, \varphi_n$ are measure-preserving transformations on the unit interval, then the function $C_{\varphi_1,\varphi_2,\ldots,\varphi_n} : \mathbf{I^n} \to \mathbf{I}$ defined by*

$$C_{\varphi_1,\varphi_2,\ldots,\varphi_n}(x_1, x_2, \ldots, x_n) := \lambda(\varphi_1^{-1}[0, x_1] \cap \varphi_2^{-1}[0, x_2] \cap \cdots \cap \varphi_n^{-1}[0, x_n]) \tag{2}$$

is an n-copula. Conversely, for every n-copula C, there exist n measure-preserving transformations $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that

$$C = C_{\varphi_1,\varphi_2,\ldots,\varphi_n}. \tag{3}$$

Note that the representation of an n-copula C in the form (3) is not unique. If C is determined by measure-preserving transformations $\varphi_1, \varphi_2, \ldots, \varphi_n$, and $\varphi\colon \mathbf{I} \to \mathbf{I}$ is any measure-preserving transformation, then it also holds $C = C_{\varphi\circ\varphi_1,\varphi\circ\varphi_2,\ldots,\varphi\circ\varphi_n}$.

Relation (2) can be understood as a method for constructing new n-copulas.

Example 1. *The mappings* $\varphi_1,\ \varphi_2 : [0,1] \to [0,1]$

$$\varphi_1(t) = \begin{cases} 2t & \text{if } t \in [0, 1/2[, \\ 2t-1 & \text{if } t \in [1/2, 1] \end{cases}, \qquad \varphi_2(t) = t,$$

are measure-preserving transformations. The copula C_{φ_1,φ_2} is given in Fig.2.

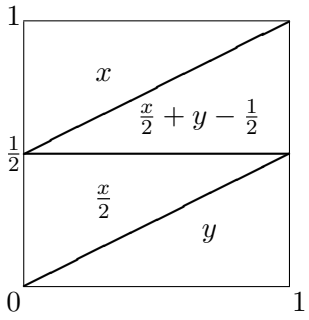

Figure 2: Copula C_{φ_1,φ_2} from Example 1

Given an n-copula C generated by measure-preserving transformations $\varphi_1, \varphi_2, \ldots, \varphi_n$, and measure-preserving transformations f_{a_i}, $a_i \in \mathbf{I}$, $i = 1, \ldots, n$, introduced above, we can construct a new n-copula in the following way.

Definition 2. *Let $C\colon \mathbf{I^n} \to \mathbf{I}$ be an n-copula generated by measure preserving transformations $\varphi_1, \varphi_2, \ldots, \varphi_n\colon \mathbf{I} \to \mathbf{I}$, i.e., $C = C_{\varphi_1,\varphi_2,\ldots,\varphi_n}$. For each $i = 1, \ldots, n$, let a_i be a number in $\mathbf{I}$ and $f_{a_i}\colon \mathbf{I} \to \mathbf{I}$ the corresponding measure-preserving transformation defined by (1). We define the function $Ca_1, \ldots, a_n\colon \mathbf{I^n} \to \mathbf{I}$ as follows*

$$Ca_1, \ldots, a_n(x_1, \ldots, x_n) = C_{f_{a_1}\circ\varphi_1,\ldots,f_{a_n}\circ\varphi_n}(x_1, \ldots, x_n). \tag{4}$$

Clearly, the function $Ca_1, \ldots, a_n$ defined by (4) is an n-copula. We can write

$$C_{a_1,\ldots,a_n}(x_1, \ldots, x_n) = \lambda\left(\bigcap_{i=1}^{n} \varphi_i^{-1} \circ f_{a_i}^{-1}[0, x_i]\right), \tag{5}$$

too.

Copulas $Ca_1, \ldots, a_n$ can also be obtained consecutively. As f_0 is an identity mapping on $\mathbf{I}$, it is easy to show that for each $a_1 \in \mathbf{I}$, it holds

$$C_{a_1,0,\ldots,0} = C_{f_{a_1}\circ\varphi_1,\varphi_2,\ldots,\varphi_n}.$$

Similarly, for all $a_1, a_2 \in \mathbf{I}$,

$$(C_{a_1,0,\ldots,0})_{0,a_2,0,\ldots,0} = C_{f_{a_1}\circ\varphi_1,f_{a_2}\circ\varphi_2,\varphi_3,\ldots,\varphi_n},$$

etc., and finally, for all $a_1, \ldots, a_n \in \mathbf{I}$, it holds

$$(\cdots((C_{a_1,0,\ldots,0})_{0,a_2,0,\ldots,0})\cdots)_{0,\ldots,0,a_n} = C_{f_{a_1}\circ\varphi_1, f_{a_2}\circ\varphi_2,\ldots,f_{a_n}\circ\varphi_n} \overset{def}{=} C_{a_1,\ldots,a_n}.$$

For most copulas it is not easy to determine measure-preserving transformations generating them. An alternative formula for copulas $C_{a_1,\ldots,a_n}$ is given in Theorem 2 whose proof is based on the previous property and the following lemma which is–for simplicity–formulated for the first step of the previous approach only.

Lemma 1. *For each $a_1 \in \mathbf{I}$ and all $(x_1, \ldots, x_n) \in \mathbf{I}^n$ it holds*

$$C_{a_1,0,\ldots,0}(x_1, \ldots, x_n) = V_C(f_{a_1}^{-1}[0, x_1] \times [0, x_2] \times \ldots \times [0, x_n]). \tag{6}$$

Proof. On the one hand, as $f_{a_1}^{-1}[0, x_1] = [a_1(1 - x_1), x_1 + a_1(1 - x_1)]$, due to the properties of the transformation φ_1 and the Lebesgue measure, it holds

$$\begin{aligned}
&C_{a_1,0,\ldots,0}(x_1, \ldots, x_n)\\
&= \lambda\left(\varphi_1^{-1} \circ f_{a_1}^{-1}[0, x_1] \cap \varphi_2^{-1}[0, x_2] \cap \ldots \cap \varphi_n^{-1}[0, x_n]\right)\\
&= \lambda\left(\varphi_1^{-1}([0, x_1 + a_1(1 - x_1)] \setminus [0, a_1(1 - x_1)]) \cap \varphi_2^{-1}[0, x_2] \cap \ldots \cap \varphi_n^{-1}[0, x_n]\right)\\
&= \lambda\left(\varphi_1^{-1}[0, x_1 + a_1(1 - x_1)] \cap \varphi_2^{-1}[0, x_2] \cap \ldots \cap \varphi_n^{-1}[0, x_n]\right)\\
&\quad -\lambda\left(\varphi_1^{-1}[0, a_1(1 - x_1)] \cap \varphi_2^{-1}[0, x_2] \cap \ldots \cap \varphi_n^{-1}[0, x_n]\right)\\
&= C(x_1 + a_1(1 - x_1), x_2, \ldots, x_n) - C(a_1(1 - x_1), x_2, \ldots, x_n).
\end{aligned}$$

On the other hand, by definition of V_C and the fact that zero is the annihilator of C, we get

$$\begin{aligned}
&V_C(f_{a_1}^{-1}[0, x_1] \times [0, x_2] \times \ldots \times [0, x_n])\\
&= V_C([a_1(1 - x_1), x_1 + a_1(1 - x_1)] \times [0, x_2] \times \ldots \times [0, x_n])\\
&= C(x_1 + a_1(1 - x_1), x_2, \ldots, x_n) - C(a_1(1 - x_1), x_2, \ldots, x_n),
\end{aligned}$$

and the claim follows.

□

Theorem 2. *Let $C_{a_1,\ldots,a_n}$ be an n-copula introduced by (4). Then it holds*

$$C_{a_1,\ldots,a_n}(x_1, \ldots, x_n) = V_C\left(\prod_{i=1}^{n} f_{a_i}^{-1}[0, x_i]\right). \tag{7}$$

Equation (7) can also be written as

$$C_{a_1,\ldots,a_n}(x_1, \ldots, x_n) = V_C\left(\prod_{i=1}^{n} [a_i(1 - x_i), x_i + a_i(1 - x_i)]\right). \tag{8}$$

Note that for $n = 2$, copulas defined by formula (8) have already been mentioned in [11]. Special cases which are often of interest are, e.g., copulas

$$\begin{aligned}
C_{0,1}(x, y) &= x - C(x, 1 - y),\\
C_{1,0}(x, y) &= y - C(1 - x, y),\\
C_{1,1}(x, y) &= x + y - 1 + C(1 - x, 1 - y).
\end{aligned}$$

Note that $C_{0,0} = C$. The copula $C_{1,1} = \hat{C}$ is the so-called survival copula of a copula C and the copulas $C_{1,0}$ and $C_{0,1}$ are flipped copulas. Particularly, for the basic copulas M, $M(x, y) = \min\{x, y\}$, and W, $W(x, y) = \max\{x + y - 1, 0\}$, it holds:

$$M_{0,1} = M_{1,0} = W \text{ and } M_{1,1} = M,$$

$$W_{0,1} = W_{1,0} = M \text{ and } W_{1,1} = W.$$

3 Several results for binary copulas

A deeper study of binary copulas (copulas, for short) obtained from a copula $C = C_{\varphi_1,\varphi_2}$ by (5) or equivalently by (7), i.e., copulas given by

$$C_{a_1,a_2}(x,y) = \lambda\left(\varphi_1^{-1} \circ f_{a_1}^{-1}[0,x] \cap \varphi_2^{-1} \circ f_{a_2}^{-1}[0,y]\right),$$

or

$$C_{a_1,a_2}(x,y) = V_C\left([a_1(1-x), x + a_1(1-x)] \times [a_2(1-y), y + a_2(1-y)]\right),$$

can be found in [4]. Note that for binary copulas the 2-increasing property means that

$$V_C([x_1,x_2] \times [y_1,y_2]) = C(x_2,y_2) - C(x_1,y_2) + C(x_1,y_1) - C(x_2,y_1) \geq 0$$

for each rectangle $[x_1,x_2] \times [y_1,y_2] \subseteq [0,1]^2$.

In [4] we have also proved that by repeating the construction in binary case we obtain a copula

$$(C_{a_1,a_2})_{b_1,b_2}(x,y) = \lambda\left(\varphi_1^{-1} \circ f_{a_1,b_1}^{-1}[0,x] \cap \varphi_2^{-1} \circ f_{a_2,b_2}^{-1}[0,y]\right),$$

where $f_{a_i,b_i} = f_{b_i} \circ f_{a_i}$, $i = 1,2$, or equivalently,

$$(C_{a_1,a_2})_{b_1,b_2}(x,y) = V_C\left(f_{a_1}^{-1} \circ f_{b_1}^{-1}[0,x] \times f_{a_2}^{-1} \circ f_{b_2}^{-1}[0,y]\right),$$

and moreover, a geometrical interpretation of this result has also been shown.

Let us show several properties of copulas M_{a_1,a_2}, W_{a_1,a_2} obtained from the basic copulas M and W. First of all, let us mention that as the simplest measure-preserving transformations generating the minimum copula M we can take identity functions on $\mathbf{I}$, i.e., $\varphi_1(t) = \varphi_2(t) = t$, as it can be seen from

$$\lambda\left(\varphi_1^{-1}[0,x] \cap \varphi_2^{-1}[0,y]\right) = \lambda\left([0,x] \cap [0,y]\right) = \min\{x,y\} = M(x,y).$$

Similarly, functions φ_1, φ_2, where $\varphi_1(t) = 1-t$ and $\varphi_2(t) = t$, are measure-preserving transformations generating the copula W because

$$\begin{aligned} \lambda\left(\varphi_1^{-1}[0,x] \cap \varphi_2^{-1}[0,y]\right) &= \lambda\left([1-x,1] \cap [0,y]\right) \\ &= \left\{ \begin{array}{ll} 0 & \text{if } x+y \leq 1 \\ x+y-1 & \text{if } x+y \geq 1 \end{array} \right\} = W(x,y). \end{aligned}$$

Now, consider measure-preserving transformations $f_a \colon \mathbf{I} \to \mathbf{I}$, given for any $a \in]0,1[$ by (1) and $f_0(t) = t$, $f_1(t) = 1-t$. As for each $x \in \mathbf{I}$,

$$\begin{aligned} f_1^{-1} \circ f_a^{-1}[0,x] &= f_1^{-1}[a(1-x), x + a(1-x)] \\ &= [1 - x - a(1-x), 1 - a(1-x)] \\ &= [(1-a)(1-x), 1 - a(1-x)] \end{aligned}$$

and

$$f_{1-a}^{-1}[0,x] = [(1-a)(1-x), x + (1-a)(1-x)] = [(1-a)(1-x), 1 - a(1-x)],$$

we get $f_1^{-1} \circ f_a^{-1} = f_{1-a}^{-1}$. Similarly, $f_0^{-1} \circ f_a^{-1} = f_a^{-1}$.

Proposition 1. *Let $a_1, a_2 \in \mathbf{I}$. Then*

(i) $M_{a_1,a_2} = M_{1-a_1,1-a_2}$,

(ii) $W_{a_1,a_2} = M_{1-a_1,a_2}$, $W_{a_1,a_2} = M_{a_1,1-a_2}$.

Proof.
(i) As M is generated by identity transformations, $M_{1,1} = M$ and for each $a \in \mathbf{I}$, $f_1^{-1} \circ f_a^{-1} = f_{1-a}^{-1}$; for each $(x, y) \in \mathbf{I^2}$, we can write

$$\begin{aligned} M_{a_1,a_2}(x,y) &= (M_{1,1})_{a_1,a_2}(x,y) = \lambda\left(f_1^{-1} \circ f_{a_1}[0,x] \cap f_1^{-1} \circ f_{a_2}[0,y]\right) \\ &= \lambda\left(f_{1-a_1}^{-1}[0,x] \cap f_{1-a_2}^{-1}[0,y]\right) = M_{1-a_1,1-a_2}(x,y). \end{aligned}$$

(ii) Clearly, $M_{1,0} = W$. Thus

$$\begin{aligned} W_{a_1,a_2}(x,y) &= (M_{1,0})_{a_1,a_2}(x,y) = \lambda\left(f_1^{-1} \circ f_{a_1}[0,x] \cap f_0^{-1} \circ f_{a_2}[0,y]\right) \\ &= \lambda\left(f_{1-a_1}^{-1}[0,x] \cap f_{a_2}^{-1}[0,y]\right) = M_{1-a_1,a_2}(x,y). \end{aligned}$$

Moreover, by (i), $M_{1-a_1,a_2} = M_{a_1,1-a_2}$.

□

Note that the product copula Π is invariant with respect to our construction, $\Pi_{a_1,a_2} = \Pi$ for all $a_1, a_2 \in \mathbf{I}$, as can easily be shown by (8).

Example 2. *Consider the minimum copula M and any $a_1, a_2 \in \mathbf{I}$. If $a_1 = a_2$ then $M_{a_1,a_2} = M$. Suppose that $a_1 < a_2$. Then*

$$M_{a_1,a_2}(x,y) = \min\{x, y, \max\{0, (1-a_1)x + a_2 y + a_1 - a_2\}\},$$

see Fig.3(left). M_{a_1,a_2} is a singular copula with support uniformly distributed over the segments connecting the vertices $(0, (a_2-a_1)/a_2)$ and $((a_2-a_1)/(1-a_1), 0)$, next $(0, (a_2-a_1)/a_2)$ and $(1,1)$, and finally, $((a_2-a_1)/(1-a_1), 0)$ and $(1,1)$. For $a_1 > a_2$ we can use property (i) in Proposition 1 and the previous formula, see Fig.3(right). Note that by using (ii) in Proposition 1, the formulas for W_{a_1,a_2} can be obtained.

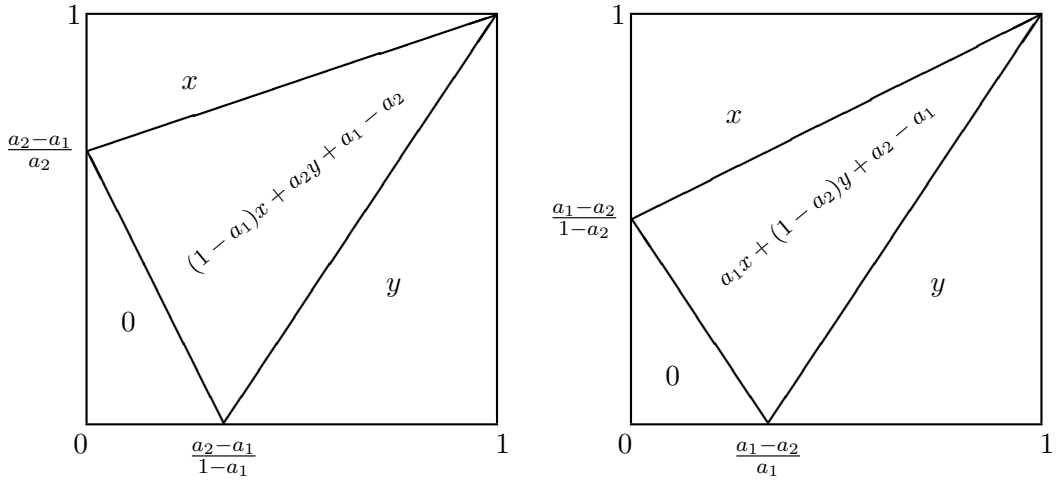

Figure 3: Copulas M_{a_1,a_2} for $a_1 < a_2$ (left) and for $a_1 > a_2$ (right)

Now, consider the family of copulas $\{M_{a_1,a_2}\}_{a_1,a_2 \in [0,1]}$ and observe the tail dependence coefficients of its members. Recall that if (X, Y) is a random vector with continuous marginal distribution functions F_X, F_Y and a copula C, then the upper tail dependence coefficient is a number $\lambda_U \in [0,1]$ given by

$$\lambda_U := \lim_{u \to 1^-} P\left(Y > F_Y^{-1}(u) | X > F_X^{-1}(u)\right) = \lim_{u \to 1^-} \frac{1 - 2u + C(u,u)}{1-u}$$

(if the limit exits). Similarly, the lower tail dependence coefficient is a number $\lambda_L \in [0,1]$ given by

$$\lambda_L := \lim_{u \to 0^+} P\left(Y \leq F_Y^{-1}(u) | X \leq F_X^{-1}(u)\right) = \lim_{u \to 0^+} \frac{C(u,u)}{u}.$$

As tail dependence is a copula property, we will write $\lambda_U(C)$ and $\lambda_L(C)$.

While for the minimum copula M it holds $\lambda_U(M) = 1$, $\lambda_U(M_{a_1,a_2})$ attains the value in $[0,1]$, which depends on a_1, a_2 as follows.

Proposition 2. *Let $a_1, a_2 \in \mathbf{I}$. Then $\lambda_U(M_{a_1,a_2}) = 1 - |a_1 - a_2|$.*

Proof. Let $a_1 \leq a_2$. Then

$$\begin{aligned}\lambda_U(M_{a_1,a_2}) &= \lim_{u\to 1^-} \frac{1-2u+(1-a_1)u+a_2u+a_1-a_2}{1-u} \\ &= \lim_{u\to 1^-} \frac{(1+a_1-a_2)(1-u)}{1-u} = 1+a_1-a_2.\end{aligned}$$

Similarly, if $a_1 > a_2$, then $\lambda_U(M_{a_1,a_2}) = 1 - a_1 + a_2$.

□

On the other hand, note that $\lambda_L(M) = 1$, but for each $a_1 \neq a_2$, $\lambda_L(M_{a_1,a_2}) = 0$.

Proposition 3. *Let $C\colon \mathbf{I^2} \to \mathbf{I}$ be a copula, $a_1, a_2 \in \mathbf{I}$. Then*

$$(\hat{C})_{a_1,a_2} = C_{1-a_1,1-a_2}.$$

Proof. Let $C = C_{\varphi_1,\varphi_2}$. As $\hat{C} = C_{1,1}$, for all $(x,y) \in [0,1]^2$ it holds

$$\begin{aligned}(\hat{C})_{a_1,a_2}(x,y) &= (C_{1,1})_{a_1,a_2}(x,y) = \lambda\left(\varphi_1^{-1} \circ f_{1,a_1}^{-1}[0,x] \cap \varphi_2^{-1} \circ f_{1,a_2}^{-1}[0,y]\right) \\ &= \lambda\left(\varphi_1^{-1} \circ f_1^{-1} \circ f_{a_1}^{-1}[0,x] \cap \varphi_2^{-1} \circ f_1^{-1} \circ f_{a_2}^{-1}[0,y]\right) \\ &= \lambda\left(\varphi_1^{-1} \circ f_{1-a_1}^{-1}[0,x] \cap \varphi_2^{-1} \circ f_{1-a_2}^{-1}[0,y]\right) = C_{1-a_1,1-a_2}(x,y).\end{aligned}$$

□

Corollary 1. *Let $C\colon \mathbf{I^2} \to \mathbf{I}$ be a radially symmetric copula, i.e., a copula satisfying the property $C = \hat{C}$. Then $C_{a_1,a_2} = C_{1-a_1,1-a_2}$.*

Note that property (i) in Proposition 1 is covered by this claim because $M = \hat{M}$.

The following property concerns copulas constructed from absolutely continuous copulas. Recall that a copula C is *absolutely continuous*, if for all $(x,y) \in \mathbf{I^2}$,

$$C(x,y) = \int_0^x \int_0^y \frac{\partial^2 C(x,y)}{\partial x \partial y}\,\mathrm{d}x\,\mathrm{d}y,$$

where $\frac{\partial^2 C(x,y)}{\partial x \partial y}$ is a joint density of C considered as a joint distribution function (restricted to $\mathbf{I^2}$).

Proposition 4. *Let $C\colon \mathbf{I^2} \to \mathbf{I}$ be an absolutely continuous copula with joint density φ and let $a_1, a_2 \in \mathbf{I}$. Then the copula C_{a_1,a_2} is absolutely continuous with joint density φ_{a_1,a_2}, whose value at each point (x,y) is equal to a convex combination of the values of φ at vertices of the rectangle $f_{a_1}^{-1}[0,x] \times f_{a_2}^{-1}[0,y]$.*

Proof. Applying formula (7) for C_{a_1,a_2} and a formula for a partial derivative of a function composition we get

$$\begin{aligned}&\varphi_{a_1,a_2}(x,y) = \frac{\partial^2 C_{a_1,a_2}(x,y)}{\partial x \partial y} \\ =\ & \frac{\partial^2}{\partial x \partial y}\left(C(x+a_1(1-x), y+a_2(1-x)) - C(x+a_1(1-x), a_2(1-x))\right. \\ -\ & \left. C(a_1(1-x), y+a_2(1-x)) + C(a_1(1-x), a_2(1-x))\right) \\ =\ & \varphi(x+a_1(1-x), y+a_2(1-x))(1-a_1)(1-a_2) \\ +\ & \varphi(x+a_1(1-x), a_2(1-x))(1-a_1)a_2 \\ +\ & \varphi(a_1(1-x), y+a_2(1-x))a_1(1-a_2) \\ +\ & \varphi(a_1(1-x), a_2(1-x))a_1a_2\end{aligned}$$

Since $(1-a_1)(1-a_2)+(1-a_1)a_2+a_1(1-a_2)+a_1a_2=1$, the above combination is convex, and the claim follows.

□

4 Concluding remarks

We have shown that starting from any n-copula C and any numbers $a_1,\ldots,a_n \in [0,1]$, we can construct another n-copula $C_{a_1,\ldots,a_n}$ by using measure-preserving transformations corresponding to the considered copula C and to the numbers $a_1,\ldots,a_n$. However, because practically it is often not easy to determine measure-preserving transformations generating the copula C, it is important that the same n-copula can be constructed by means of C-volumes V_C of special n-boxes depending on numbers $a_1,\ldots,a_n$. The fact, which of these two equivalent approaches is used, depends on the problem to be solved. In our future work we intend to study, e.g., the relationship between the studied construction and some other constructions of copulas, e.g., ordinal sums, but also the properties of resulting n-copulas for $n>2$.

Acknowledgement

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Issues in construction of linguistic summaries

Miroslav Hudec[1]

Abstract. *Linguistic summaries are convenient approach for revealing intensity of relational knowledge in the data. Two main parts of summaries are summarizers defined as predicates and quantifiers. The validity of a rule critically depends on constructed fuzzy sets for predicates and quantifiers. This paper deals with the construction of membership function for predicates from the current content of a data set and the construction of membership function for quantifiers in the [0, 1] interval. The second aim is building complex summaries. Moreover, linguistic summaries can be used as flexible queries for ranking entities on higher hierarchical level using data on lower hierarchical level.*

Keywords**:** *linguistic quantifiers, linguistic summarizers, construction of membership functions, fuzzy queries.*

1 Introduction

Linguistic summaries are able to express relational knowledge and its intensity about collected data. For people linguistic summarisation is a desirable way how to communicate in natural language and obtain validity of uncertain rules from a data set. Linguistic summaries are of well-known structure *Q entities in database are (have) S* where *S* is a summarizer defined as linguistic term on the domain of examined attribute and *Q* is a fuzzy quantifier in sense of Zadeh (1983). An example of simple linguistic summary is: *most customers are middle aged*. Linguistic summaries could be more complex e.g. *most highly situated (altitude above the sea level) and small municipalities have high unemployment and small migration*.

A linguistic summary is a short sentence that describes relational knowledge in large data sets. The concept of linguistic summaries has been initially introduced in (Yager, 1982) and further developed in (Rasmussen and Yager, 1997; Kacprzyk and Yager, 2001; Kacprzyk and Zadrozny, 2009). Truth value of summaries is usually called validity and gets values from the [0, 1] interval by agreement. Data summarization is one of basic capabilities needed to any "intelligent" system (Kacprzyk and Zadrozny, 2009). In order to use advantages of the Structured Query Language (SQL) and linguistic summaries Rasmussen and Yager (1997) have created the SummarySQL language. FQUERY for Access (Kacprzyk and Zadrozny, 2009) makes possible to use fuzzy terms in usual fuzzy queries and for summarisation.

Galindo (2008) concluded that when the system uses badly defined membership functions, it will not work properly. So, these functions have to be carefully defined. In the same way this holds for linguistic summaries, because it is required to calculate the proportion of entities that satisfies (fully or partially) the summarizer *S* and validity of a rule.

This paper is focused on developing linguistic summaries by dynamically constructing fuzzy sets for summarizers *S* from the current database content applying results of Tudorie (2008) and Hudec and Sudzina (2012) and defining quantifiers. Section 2 describes the concept of linguistic summaries. Section 3 is devoted to construction of membership functions for predicates and quantifiers. Short illustrative examples are provided in Section 4. Section 5 examines further development of summarizers by preferences. Finally, conclusions are drawn in Section 6.

[1] Faculty of Economic Informatics, University of Economics in Bratislava, Slovakia

2 Linguistic summaries by fuzzy queries

According to Zadrożny and Kacprzyk (2009) an imprecise (fuzzy) query is a query containing natural language expressions, referred to as linguistic terms, to specify:

a) imprecise values e.g. *low salary*;
b) imprecise comparison operators e.g. *salary much greater than 2 000*;
c) non-standard aggregation scheme of the fulfilment degrees to partial conditions e.g. *most of municipalities have small migration*.

In this paper we are focused on the third issue. Nevertheless, evaluation of imprecise values in query conditions is the basis for the linguistic summaries.

Because for the humans the usual means of communication is natural language, an uncertain proposition (linguistic summary) would be desirable way to express relational knowledge about the data (Kacprzyk and Zadrożny, 2009; Rasmussen and Yager, 1997).

2.1 Linguistic summaries for extracting relational knowledge

Examples of linguistic summaries are as follows:

(a) Few municipalities have high altitude;
(b) Most municipalities have high unemployment and small migration;
(c) Most low polluted municipalities have high altitude and small number of inhabitants.

Linguistically quantified propositions are written in a general form:

$$Qx(Px) \tag{1}$$

where Q is a linguistic quantifier, $X = \{x\}$ is a universe of disclosure (e.g. the set of all municipalities) and $P(x)$ is a predicate depicting summariser S e.g. *small migration*. Predicate P is a fuzzy set $P \in \mathcal{F}(X)$. where $\mathcal{F}(X)$ is a family of fuzzy sets defined on the domain of an examined variable.

The truth value of a statement (rule) is computed by the following equation (Zadrożny and Kacprzyk, 2009):

$$T(Qx(Px)) = \mu_Q\left(\frac{1}{\mathrm{n}}\sum_{i=1}^{n}\mu_{\mathrm{P}}(x_i)\right) \tag{2}$$

where n is the cardinality of a data set (number of entities), $\frac{1}{\mathrm{n}}\sum_{i=1}^{n}\mu_{\mathrm{P}}(x_i)$ is the proportion of objects in a data set that satisfy $P(x)$ and μ_Q is the membership function of a quantifier.

Measure of validity can be calculated by quantifiers defined in Zadeh (1983) or using the OWA (Ordered Weighted Averaging) operator (Yager and Kacprzyk, 1997; Zadrożny and Kacprzyk, 2009). In this paper the former approach is used.

First type of summary is of the form:

Q entities in database are (have) S

Examples are rules (a) and (b) where summary (b) consist of two elementary conditions connected with the *and* aggregation operator. If summarizer consists of several atomic predicates $\mu_{\mathrm{P}}(x_i)$ is calculated in the following way:

$$\mu_{\mathrm{P}}(x_i) = f(\mu_{Pj}(x_i)) \tag{3}$$

where P_j is the *j*-th atomic predicate and *f* is either t-norm or t-conorm. The truth value of a statement (rule) is computed by the eq. (2).

Second type of summary is of the form:

Q R entities in database are (have) S

The example is the rule (c). The procedure for calculating truth value has the following form (Rassmusen and Yager, 1997):

$$T(Qx(Px)) = \mu_Q\left(\frac{\sum_{i=1}^{n} t(\mu_{\mathrm{P}}(x_i), \mu_{\mathrm{R}}(x_i))}{\sum_{i=1}^{n} \mu_{\mathrm{R}}(x_i)}\right) \tag{4}$$

where $\frac{\sum_{i=1}^{n} t(\mu_{\mathrm{P}}(x_i), \mu_{\mathrm{R}}(x_i))}{\sum_{i=1}^{n} \mu_{\mathrm{R}}(x_i)}$ is the proportion of the *R* objects in a database that satisfy *S*, *t* is a t-norm, μ_Q is the membership function of a quantifier. The same discussion as for $\mu_{\mathrm{P}}(x_i)$ (eq. 3) applies for $\mu_{\mathrm{R}}(x_i)$.

2.2 Linguistic summaries for flexible queries

An example of query is the following *select regions where most of municipalities have small unemployment and low migration*. In the first step, validity of summaries is calculated for each region. In the second step regions are ranked downwards starting with region having the highest value of the rule validity.

The procedure for calculating validity of summary *Q entities in database are (have) S* for each data cluster (group) is created as the extension of (2):

$$T_i(Qx(Px)) = \mu_Q\left(\frac{1}{\mathrm{N_i}} \sum_{j=1}^{N_i} \mu_{\mathrm{P}}(x_{ji})\right), \quad i = 1, \ldots, R, \quad \sum_{i=1}^{R} \mathrm{N_i} = n \tag{5}$$

where *n* is the number of entities in whole database, N_i is the number of entities in cluster *i* (municipalities in region *i*), *R* is the number of clusters in a database (e.g. regions), T_i is validity of rule for *i*-th cluster, and $\mu_p(x_{ji})$ is the proportion of objects in *i*-th cluster that satisfy summarizer *S*.

The procedure for calculating validity of summary *Q R entities in database are (have) S* for each cluster is created using the extension of (4):

$$T_i(Qx(Px)) = \mu_Q\left(\frac{\sum_{j=1}^{N_i} t(\mu_{\mathrm{S}}(x_{ji}), \mu_{\mathrm{R}}(x_{ji}))}{\sum_{j=1}^{N_i} \mu_{\mathrm{R}}(x_{ji})}\right) \quad , i = 1 \ldots R, \quad \sum_{i=1}^{R} \mathrm{N_i} = n \ . \tag{6}$$

The meaning of variables is the same as in (4, 5).

3 Construction of membership functions for predicates and quantifiers

The matching degree of each database record to query condition critically depends on constructed membership functions of predicates. Therefore, these functions have to be carefully constructed. The same holds for quantifiers. In the paper we are focused on relative quantifiers *most*, *about half* and *few*.

3.1 Construction of membership functions for predicates

Let D_{min} and D_{max} be the lowest and the highest domain values of attribute A i.e. Dom(A) = [D_{min}, D_{max}] and L and H be the lowest and the highest values in the current content of a database respectively (Hudec and Sudzina, 2012). Usually attribute's domain is defined in a way that all theoretically possible values could be stored. In practice, collected data are often far from the values of D_{min} and D_{max}; that is, [L, H] $\subset$ [D_{min}, D_{max}] (either [D_{min}, L] or [H, D_{max}] are empty or even both of them are empty). This fact should be considered in linguistic summaries.

The uniform domain covering method (Tudorie, 2008) is an appropriate method for construction of membership functions for these tasks. At the beginning, values of L and H are obtained from the current database content. The length of fuzzy set core β and the slope α (Figure 1) are calculated using the following equations (Tudorie, 2008):

$$\alpha = \frac{1}{8}(H - L), \tag{7}$$

$$\beta = \frac{1}{4}(H - L). \tag{8}$$

Required parameters A, B C and D (Figure 1) are calculated using (7, 8):
$A = L + \beta$; $B = L + \beta + \alpha$; $C = H - \beta - \alpha$; $D = H - \beta$.

The uniform domain covering method is adequate because the main goal is to reveal relational dependencies among data where distribution of stored data should be reflected in the membership functions.

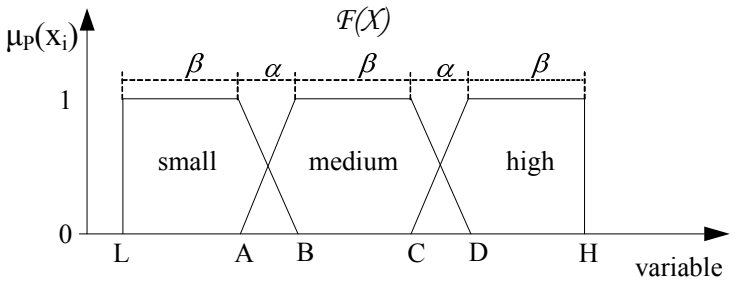

Figure 1: Linguistic and crisp domain of an attribute

3.2 Construction of membership functions for quantifiers

The validity of summaries examined in the paper is computed by relative quantifiers. A quantifier is constructed by a fuzzy set on the [0, 1] interval (Zadrożny and Kacprzyk, 2009). For compatibility with the construction of predicates, explained in Section 3.1, the [0, 1] interval is marked as the domain of a family of quantifiers.

For a regular non-decreasing quantifier (e.g. *most*) its membership function should meet the following property:

$$x \le y \Rightarrow \mu_Q(x) \le \mu_Q(y)\,;\, \mu_Q(0) = 0;\;\; \mu_Q(1) = 1\;. \tag{9}$$

The quantifier might be given as (Kacprzyk and Zadrożny, 2009):

$$\mu_Q(y) = \begin{cases} 1, & for\;\; y > 0.8 \\ 2y - 0.6 & for\;\; 0.3 \le y \le 0.8 \\ 0, & for\;\; y < 0.3 \end{cases}\;. \tag{10}$$

The second way for modelling a linguistic quantifier is realised by the OWA operator. If quantifier is a regular non-decreasing (9) then the weight vector of an OWA operator is defined in the following way (Yager, 1988):

$$w_i = \mu_Q\left(\frac{i}{m}\right) - \mu_Q\left(\frac{i-1}{m}\right),\;\; i = 1,...,m\;. \tag{11}$$

The first approach is appropriate for ordinal summaries e.g. *most of municipalities have small gas consumption*. Number of municipalities that meet the predicate to some extent could be high (value of m in eq. 11) and it is time consuming to calculate all values of w_i for such a long vector. In this case (10) is a rational option.

Having an "aggregated" linguistic quantifier e.g.: *most of the predicates {P_i} are satisfied (i=1… n)* then the quantifier could be represented by the OWA operator using (11). Number of predicates is significantly smaller than number of entities in a database.

Equivalently, non-increasing quantifier e.g. *few* could be created as a "mirror picture" of (10) in the following way:

$$\mu_Q(y) = \begin{cases} 1, & for\;\; y < 0.2 \\ 1.4 - 2y & for\;\; 0.2 \le y \le 0.7 \\ 0, & for\;\; y > 0.7 \end{cases}\;. \tag{12}$$

Having in mind the uniform domain method for construction of family of membership functions $\mathcal{F}(X)$ on domain of attribute for summarizers (Section 3.1) we can create the family of membership functions $\mathcal{Q}$ for quantifiers in the same way depicted in Figure 2.

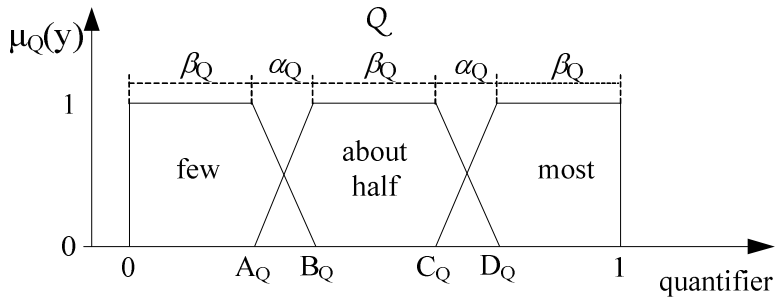

Figure 2: The domain for quantifiers

The length of fuzzy set core β and the slope α are calculated using (7) and (8). In this case the values are following:

$$\alpha_Q = \frac{1}{8},\; \beta_Q = \frac{1}{4},\; A_Q = 0.25,\; B_Q = 0.375,\; C_Q = 0.625,\; D_Q = 0.75\;. \tag{13}$$

Applying (13), parameters of the quantifier *most* are calculated in the following way:

$$\mu_Q(y) = \begin{cases} 1, & for \;\; y > 0.75 \\ \dfrac{y}{0.125} - 5 & for \;\; 0.625 \le y \le 0.75 \\ 0, & for \;\; y < 0.625 \end{cases} . \tag{14}$$

In this way quantifier is more restrictive than quantifier defined in (10). On the other hand, quantifiers are uniformly distributed in their domain (Figure 2).

Comparison of both approaches is depicted in Figure 3. The increasing part of the quantifier *most* in (10) starts earlier and inclines slower in comparison with (14). The core of (10) is shorter than for (14). In addition, intersection of fuzzy sets most and few defined by (13) is empty because these quantifiers are opposite and there is also the space for the quantifier *about half* which has overlapping boundaries with quantifiers *most* and *few*.

Presumably, the following question might appear: which approach for the quantifiers' construction is more appropriate? The discussion is provided in Section 4.

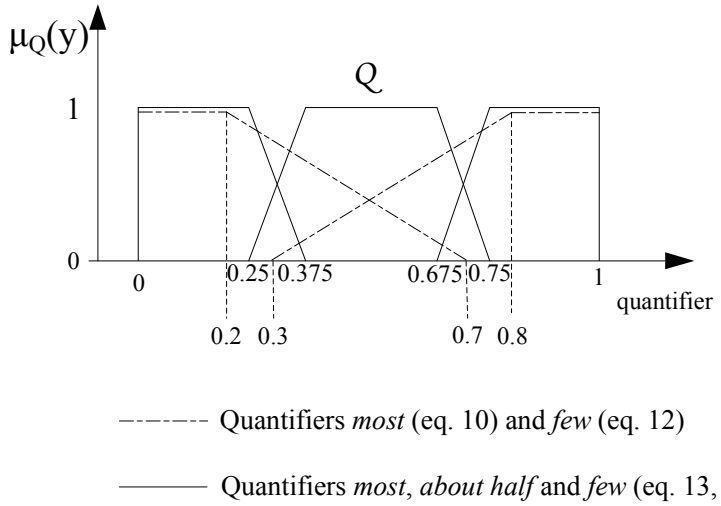

Figure 3: Comparison of definitions of quantifiers.

Moreover, if we want to extend family of fuzzy sets $\mathcal{A}(X)$ to five: *very small*, *small*, *medium*, *high* and *very high* we need only to divide the domain to five linguistic terms uniformly distributed (Tudorie, 2009). The same could hold for the family of quantifiers $\mathcal{Q}$

4 Illustrative examples

This section is devoted to small examples of both kind of summaries examined in the Section 3.

4.1 Summaries for extraction relational knowledge in the data

In the following three short examples quantifiers defined in (10) and (14) were evaluated.

Example 1. Let's have 10 entities of which 6 fully meet the summarizer (value of 1), 3 partially do with values of 0.9, 0.7 and 0.2 respectively and one record does not meet the condition (value of 0) then the proportion of objects in a data set that satisfy *P(x)* obtains the value of 0.78. The validity of rule calculated by (10) is 0.96 and by (14) is 1.

Example 2. Let's have 10 entities with the following membership degrees to the summarizer 0; 0; 0; 0.4; 0.4; 0.4; 0.4; 0.5; 0.5; 1, then the proportion of objects in a data set that satisfy *P(x)* obtains the value of 0.36. The validity of rule calculated by (10) is 0.12 and by (14) is 0.

Example 3. Let's have again 10 entities with the following membership degrees to the summarizer 0; 0.3; 0.4; 0.6; 0.8; 0.9; 1; 1; 1; 1, then the proportion of objects in a data set that satisfy *P(x)* obtains the value of 0.7. The validity of rule calculated by (10) is 0.8 and by (14) is 0.6.

Results are more or less similar. However, partially belonging of value 0.36 to the quantifier *most* (10) even with small value is disputable.

Additional analysis of both approaches is required. Therefore, it is a topic for the further research. Anyway, user opinion of the strength of the quantifiers should be incorporated into the construction of quantifiers' membership functions.

4.2 Summaries as flexible queries

For example we want to know to which extent is the following rule (query) satisfied *most of municipalities has small attitude above sea level*. The result for all eight regions of the Slovak Republic is presented in Table 1 (Hudec, 2013). Table 1 shows that regions Bratislava, Trnava and Nitra are flat whereas regions Žilina and Prešov are hilly. Region Trenčín is more flat than hilly. The same holds for region Košice but it is a slightly hillier than region Trenčín. Data about municipalities were used for summaries but result is visible for regions only ranked according to value of rule validity.

Table 1: Linguistic summary for each region

Region	**Validity of the summary**
Bratislava	1
Trnava	1
Nitra	1
Trenčín	0.7719
Košice	0.6314
Bánska Bystrica	0.2116
Žilina	0
Prešov	0

5 Further improvements of linguistic summaries

In summarizers not all elementary predicates always have the same importance. The aim of preferences is to distinguish elementary conditions according to their importance inside the overall summarizer.

Applying preferences linguistic summaries become more sophisticated covering additional class of problems e.g. *most of municipalities have high altitude above sea level and low pollution where the second condition is more important than the first one*. In order to calculate validity of the rule weights should be associated with each elementary condition.

This issue could be solved by appropriate fuzzy implications (Zadrożny et al, 2008). The idea how to calculate the matching degree of an elementary predicate *Pi* according to an importance weight w_j and fuzzy implications has the following form (Zadrożny et al, 2008):

$$\mu(P_j^*, x_i) = (w_j \Rightarrow \mu(P_j, x_i)) \tag{15}$$

where $\Rightarrow$ is a fuzzy implication, P_j is *j*-th elementary predicate and x_i, is *i*-th entity in database which meet the summarizer. In order to be meaningful, weights should satisfy several requirements (Dubois and Prade, 1997). One of them is the following:

if w_i=0 then result should be such as if P_i does not exist.

Applying this requirement shows us that Mamdani implication is not adequate whereas Kleene-Dienes, Godel and Gougen implications match this requirement. Examples of the first two implications are briefly outlined below.

The Kleene-Dienes implication has the following structure:

$$\mu(P_j^*, x_i) = \max(\mu(P_j, x_i), 1 - w_j) \ . \tag{16}$$

Apparently, for small importance of P_j (w_j is close or equal to 0), the satisfaction of atomic predicate P_j has a very small influence moving to no influence on the query satisfaction ($w_j \to 0 \Rightarrow \mu(P_j^*, x_i) \to 1$). In another case when w_j is close to 1, the satisfaction of P_j is essential for the satisfaction of the overall condition ($w_j \to 1 \Rightarrow \mu(P_j^*, x_i) \to \mu(P_j, x_i)$).

Contrary, the Mamdani implication is not suitable for this approach. It can be shown on the following example:

$$\mu(P_j^*, x_i) = \min(\mu(P_j, x_i), w_j) \ . \tag{17}$$

Because of the small importance of w_i the overall matching degree is close to 0. In case when w_i=0, the overall matching is 0 regardless of other elementary conditions. It implies that the requirement *if w_i=0 then result should be such as if Pi does not exist* is not satisfied for the implication (17).

The proportion of objects in a database that satisfy *P(x)* applying the Kleene-Dienes implication (16) is calculated in the following way:

$$\frac{1}{\mathrm{n}} \sum_{i=1}^{n} t(\max_{j=1,\ldots,N}(\mu(P_j, x_i), 1 - w_j)) \ . \tag{18}$$

Finally, validity of rule is expressed by the equation:

$$T(Qx(Px)) = \mu_Q(\frac{1}{\mathrm{n}} \sum_{i=1}^{n} t(\max_{j=1,\ldots,N}(\mu(P_j, x_i), 1 - w_j))) \quad . \tag{19}$$

6 Conclusion

The paper demonstrates how we can start with a simple linguistic summary and build more complex summaries. Although fuzzy set theory has been already established as an adequate framework to deal with linguistic summaries, there is still space for improvements. The critical parts are construction of membership functions for linguistic terms (summarizers) *small*, *medium*, *high* and construction of relative quantifiers *few*, *about half*, *most*. The former can be satisfactorily solved if we calculate parameters of membership functions directly from the current database content using the uniform domain covering method. The later can be satisfactorily solved if we calculate parameters of relative quantifiers in the [0, 1] interval by the same method as for summarizers. Finally, summarizers were extended by preferences described as fuzzy implications.

Relational knowledge about the data is valuable either for decision making or for broad audience. Both of them usually are not interested in data itself but in the relational knowledge that could support decision making or can satisfy their curiosity.

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Ordering Based on Implications

M. Nesibe Kesicioğlu * Radko Mesiar †‡

Abstract: *Implication function I on a bounded lattice L is defined by means of some boundary conditions and monotonicity constraints. On the other hand, each implication function I on L defines a special relation which, in some cases, can be a (partial) order on L. We study the properties of I resulting into such (partial) orders. A special attention is given do situations yielding new bounded lattices.*

Keywords: *Implication, bounded lattice, partial order.*

1 Introduction

Fuzzy implications are one of the most important operations in fuzzy logic having a significant role in many applications, viz., approximate reasoning, fuzzy control, fuzzy image processing, etc. (see [1, 5, 12, 13, 16]). They generalize the classical implication, which takes values in $\{0, 1\}$, to fuzzy logic, where the truth values belong to the unit interval $[0, 1]$. In general situation, since $[0, 1]$ is a bounded lattice, like in the case of other logical operators, the problem of introducing implications on a bounded lattice laid bare and Ma and Wu [11] have introduced them at first. Several authors have investigated the implications on a bounded lattice and their relations to the other logical operators [9, 14, 15, 18, 19, 20].

In this paper, we introduce an order by means of an implication possessing some special properties on a lattice and discuss some of its properties. The paper is organized as follows. We shortly recall some basic notions in Section 2. In Section 3, we determine the relationship between the order induced by an implication and the order on the lattice. Giving example, we show that a bounded lattice needs not be a lattice with respect to the order induced by an implication. Also, we give an example for an implication making the unit interval $[0, 1]$ a lattice with respect to the order induced by it. Moreover, we obtain that such a generating method of an order is independent from the order induced by an adjoint t-norm (T-partial order) [8]. We prove that under the conditions required to define implication based order, the considered implication must be an S-implication, and so we obtain that the order induced by an implication coincides with the order which is generated in a similar way from a t-conorm. Consequently, we obtain that an implication on the unit interval $[0, 1]$ is continuous if and only if the implication based order and the dual of the natural order on $[0, 1]$ coincide.

2 Notations, definitions and a review of previous results

Definition 1. [2] Let $(L, \leq, 0, 1)$ be a bounded lattice. A binary operation T (S) on L is called a t-norm (t-conorm) if it satisfies the following conditions:

(1) $T(T(a, b), c) = T(a, T(b, c))$ (associative law),

(2) $T(a, b) = T(b, a)$ (commutative law),

(3) $b \leq c \Rightarrow T(a, b) \leq T(a, c)$ (monotonicity),

(4) $T(a, 1) = a$ ($S(a, 0) = a$) (boundary condition),

where a, b and c are any elements of L.

*Corresponding author, Department of Mathematics, Recep Tayyip Erdoğan University 53100 Rize, Turkey, *m.nesibe@gmail.com*, Tel:(+90)(464-223 61 26), Fax:(+90)(464-223 40 19)

†Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Slovakia, *mesiar@math.sk*

‡Institute for Research and Application of Fuzzy Modeling, University of Ostrava, Czech Republic

Definition 2. [11] Let $(L, \leq, 0, 1)$ be a bounded lattice. A decreasing function $N : L \to L$ is called a negation if $N(0) = 1$ and $N(1) = 0$. An implication N on L is called strong if it is an involution, i.e., $N(N(x)) = x$, for all $x \in L$.

On each bounded lattice L we have two extremal negations $N^+, N^- : L \to L$ given by
$N^-(x) = \begin{cases} 1 & if \quad x = 0, \\ 0 & \text{otherwise} \end{cases}$ and $N^+(x) = \begin{cases} 0 & if \quad x = 1, \\ 1 & \text{otherwise.} \end{cases}$
Obviously, for any negation $N : L \to L$, it holds $N^- \leq N \leq N^+$.

Definition 3. [1, 11] Let $(L, \leq, 0, 1)$ be a bounded lattice. A binary operator $I : L^2 \to L$ is said to be an implication function, shortly an implication, if it satisfies
(I1) For every elements a, b with $a \leq b$, $I(a, y) \geq I(b, y)$ for all $y \in L$.
(I2) For every elements a, b with $a \leq b$, $I(x, a) \leq I(x, b)$ for all $x \in L$.
(I3) $I(1, 1) = 1$, $I(0, 0) = 1$ and $I(1, 0) = 0$.
Note that from the definition, it follows that
$I(0, x) = 1$ and $I(x, 1) = 1$, for all $x \in L$. Special interesting properties for implications are:

- The exchange principle (EP)
  $I(x, I(y, z)) = I(y, I(x, z))$ for all $x, y, z \in L$
- The left neutrality principle (NP)
  $I(1, y) = y$, for every $y \in L$
- The contrapositive symmetry to a negation N (CP-N)
  $I(x, y) = I(N(y), N(x))$, for every $x, y \in L$
- The left contrapositive symmetry to a negation N (L-CP(N))
  $I(N(x), y) = I(N(y), x)$, for every $x, y \in L$

Obviously, for a strong negation N, the left contrapositive symmetry and the contrapositive symmetry coincide.

Definition 4. [1] Let $(L, \leq, 0, 1)$ be a lattice and I be an implication on L. The function $N_I : L \to L$ given by
$N_I = I(x, 0)$ for all $x \in L$
is a negation and it is called the natural negation of I.

Definition 5. [9] Let $(L, \leq, 0, 1)$ be a lattice. An implication $I : L^2 \to L$ is called an S- implication if there exists a t-conorm S and a strong negation N such that for every $x, y \in L$

$$I(x, y) = S(N(x), y).$$

Definition 6. [8] Let L be a bounded lattice, T be a t-norm on L. The order defined as following is called a $T-$ *partial order* (triangular order) for t-norm T:

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.$$

From the definition, it follows that $a \preceq_T b$ implies that $a \leq b$ for any elements $a, b \in L$.

Definition 7. [10] Let $T : [0, 1]^2 \to [0, 1]$ be a left-continuous t-norm. The function $I_T : [0, 1]^2 \to [0, 1]$ given by

$$I_T(x, y) = sup\{z \in [0, 1] | T(x, z) \leq y\}. \tag{1}$$

is an implication and it is called as a residual implication.

Observe that the definition (1) can be applied to any t-norm $T : L^2 \to L$ acting on a bounded lattice L, and the resulting function $I_T : L^2 \to L$ is an implication on L.

3 An *I*-based ordering

Definition 8. Let $(L, \leq, 0, 1)$ be a bounded lattice and $I : L^2 \to L$ be an implication. Define the relation $\preceq_I$ on L as follows:
For every $x, y \in L$

$$y \preceq_I x :\Leftrightarrow \exists \ell \in L \quad \text{such that} \quad I(\ell, x) = y. \tag{2}$$

Proposition 1. *The relation $\preceq_I$ is a partial order on L, whenever $I : L^2 \to L$ is an implication satisfying the exchange property (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation N_I.*

We will call such an order defined in (2) as an I- *based ordering*.

Proposition 2. *Let $(L, \leq, 0, 1)$ be a bounded lattice and $I : L^2 \to L$ be an implication satisfying (EP) and (CP) with respect to the strong natural negation N_I. If $(x, y) \in \preceq_I$, then $(y, x) \in \leq$.*

Remark 1. Let $(L, \leq, 0, 1)$ be a bounded lattice and I be an implication satisfying (EP) and (CP-N_I).
(i) It is clear that 0 and 1 are the greatest and the least element with respect to $\preceq_I$, respectively.
(ii) The converse of Proposition 2 may not be satisfied. For example:
Consider the lattice $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$ whose lattice diagram is displayed in Figure 1:

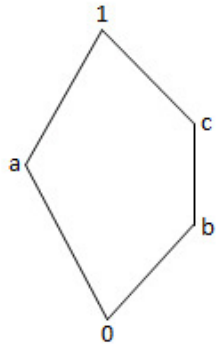

Figure 1: $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$

Define the function $I : L^2 \to L$ as follows:

I	0	a	b	c	1
0	1	1	1	1	1
a	a	1	1	1	1
b	c	1	1	1	1
c	b	1	1	1	1
1	0	a	b	c	1

Obviously, I is an implication on L satisfying the exchange principle (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation N_I defined as

$$N_I(x) = \begin{cases} a & if \quad x = a, \\ c & if \quad x = b, \\ b & if \quad x = c, \\ 1 & if \quad x = 0, \\ 0 & if \quad x = 1. \end{cases}$$

Although $b \leq c$, $c \not\preceq_I b$ since there does not exist an element $k \in L$ such that $I(k, b) = c$. The order $\preceq_I$ on L has its Hasse diagram as follows:

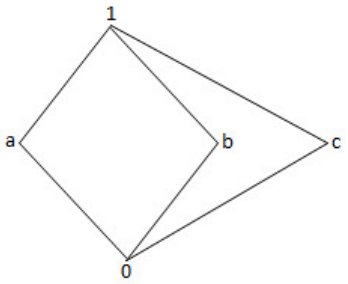

Figure 2: $(L = \{0, a, b, c, 1\}, \leq, 0, 1)$

(iii) Even if $(L, \leq, 0, 1)$ is a chain, the partially ordered set $(L, \preceq_I)$ may not be a chain. For example: consider $L = [0, 1]$ and take the Fodor implication $I = I_{FD}$ defined as

$$I_{FD}(x, y) = \begin{cases} 1 & if \quad x \leq y, \\ max(1 - x, y) & if \quad x > y. \end{cases} \tag{3}$$

It is clear that I_{FD} satisfies the exchange principle (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation $N_{I_{FD}} = N_C$, $N_C(x) = 1 - x$. Since $1/2$ and $3/4$ are not comparable with respect to $\preceq_{I_{FD}}$, $([0, 1], \preceq_{I_{FD}})$ is not a chain.

Remark 2. Let T be a left continuous t-norm on $[0, 1]$ and I_T be the corresponding residual implication. Then, the implication based ordering and the T-partial order are independent.

L needs not be a lattice w.r.t. $\preceq_I$. The following example illustrates this case.

Example 1. Let $L = [0, 1]$ and take the implication I_{FD} given by (3). $([0, 1], \preceq_{I_{FD}})$ is not a lattice.

Proposition 3. *For every implication I satisfying (EP) and (CP-N_I), there exists a t-conorm S such that*

$$I(x, y) = S(N_I(x), y),$$

that is, I is an S- implication.

Corollary 4. *Let the implication $I : L^2 \to L$ satisfy (EP) and CP-N_I. Then, for any $a, b \in L$*
$a \preceq_I b$ if and only if $N_I(a) \preceq_T N_I(b)$,
where $T : L^2 \to L$ is a t-norm given by $T(x, y) = N_I(I(x, N_I(y)))$.

Theorem 5. *Let $I : [0, 1]^2 \to [0, 1]$ be a fuzzy implication satisfying (EP) and the contrapositive symmetry (CP) with respect to the natural strong negation N_I and $\preceq_I$ be the order linked to the implication I. Then, I is continuous if and only if $\preceq_I = \geq$.*

One can wonder whether L is a bounded lattice w.r.t. an order obtained from an implication (under which conditions). In the next Proposition, we give some sufficient conditions.

Proposition 6. *Let $(L, \leq, 0, 1)$ be a bounded lattice and $I : L^2 \to L$ an implication on L defined as $I(x, y) = 1$ when $x \neq 1$ and $y \neq 0$, satisfying the exchange principle (EP) and the contrapositive symmetry (CP) with respect to the strong natural negation N_I. Then, $(L, \preceq_I)$ is a lattice.*

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A note on a copula construction method

Vadoud Najjari [*†‡] Hasan Bal [†] Salih Çelebioğlu [†]

Abstract: *The main endeavor in this work is to comment on the study by Kim et al. which generalizes Rodríguez-Lallena and Úbeda-Flores' result to any given copula family. In this study we concentrate on the proposed interval of parameter in their work and then we comment on the proposed interval.*

Keywords: *Absolutely continuous functions; Copulas*

1 Introduction

A copula is a function $C : [0,1]^2 \to [0,1]$ which satisfies:

(a) for every $u,\ v$ in $[0,1]$, $C(u,0) = 0 = C(0,v)$ and $C(u,1) = u$ and $C(1,v) = v$;

(b) for every u_1, u_2, v_1, v_2 in [0, 1] such that $u_1 \leq u_2$ and $v_1 \leq v_2$, $V_C(R) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$ (in other word, for all rectangles $R = [u_1,u_2] \times [v_1,v_2]$ whose vertices lie in $[0,1]^2$, C-volume is non-negative).

Copulas are multivariate distributions in modeling the dependence structure between variables, irrespective of their marginal distributions. Obviously with a wide range of copulas we are able to capture more miscellaneous dependence structures. Hence there is a wide effort on constructions of copulas in the literature (see for instance, [1, 2, 3, 4, 5, 6]). Also Nelsen [7] summarizes different methods of constructing copulas.

Rodríguez-Lallena and Úbeda-Flores [6] introduced a class of bivariate copulas of the form:

$$C_\lambda(u,v) = uv + \lambda\, f(u)g(v), \qquad (u,v)\ \in\ [0,1]^2 \tag{1.1}$$

where f and g are two non-zero absolutely continuous functions such that $f(0) = f(1) = g(0) = g(1) = 0$ and the admissible range of the parameter λ is

$$\frac{-1}{\max(\alpha\gamma, \beta\delta)} \leq \lambda \leq \frac{-1}{\min(\alpha\delta, \beta\gamma)} \tag{1.2}$$

where

$$\begin{aligned} &\alpha = \inf\{f^{'}(u) :\ u \in A\} < 0 \quad , \quad \beta = \sup\{f^{'}(u) :\ u \in A\} > 0 \\ &\gamma = \inf\{g^{'}(v) :\ v \in B\} < 0 \quad , \quad \delta = \sup\{g^{'}(u) :\ u \in B\} > 0 \\ &A = \{u \in [0,1] : f^{'}(u)\ exists\} \quad , \quad B = \{v \in [0,1] : g^{'}(u)\ exists\}. \end{aligned} \tag{1.3}$$

[*]Corresponding author, *vnajjari@gazi.edu.tr*, Tel: +90 312 2021465, Fax: +90 312 212 2279

[†]Department of Mathematics, Islamic Azad University, Maragheh Branch, Maragheh, Iran

[‡]Gazi University, Faculty of Science, Statistics Department, Ankara, Turkey

This class of copulas provides a method for constructing bivariate distributions with a variety of dependence structures and generalizes several known families such as the Farlie-Gumble-Morgenstern (FGM) distributions. Dolati and Úbeda-Flores [2] provided procedures to construct parametric families of multivariate distributions which generalize (1.1).

Kim et al. [8] generalized Rodríguez-Lallena and Úbeda-Flores' study to any given copula family. They presented an extension for any given copula family C as below

$$C^*_\lambda(u,v) = C(u,v) + \lambda\, f(u)g(v), \qquad (u,v)\ \in\ [0,1]^2 \tag{1.4}$$

where for any non-trivial rectangle R, the parameter λ satisfies on the following inequalities

$$\frac{-V_C(R)}{\triangle \times \max(\alpha\gamma, \beta\delta)} \leq \lambda \leq \frac{-V_C(R)}{\triangle \times \min(\alpha\gamma, \beta\delta)} \tag{1.5}$$

in these inequalities $\alpha,\ \beta,\ \gamma,\ \delta$ are same as (1.3), $\triangle = (u_2 - u_1)(v_2 - v_1)$, u_1, u_2, v_1, v_2 are in $[0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$ and f, g are two non-zero absolutely continuous functions defined on $[0,1]$ such that $f(0) = f(1) = g(0) = g(1) = 0$. Kim et al.'s results had been discussed by Mesiar et al. in [9] and also by Bekrizadeh et al. in [10, 11].

In this study we concentrate on interval of the λ parameter given by (1.5) and we investigate inaccuracy of this interval.

2 Comments on inaccuracy of the interval

In this section we concentrate on the bounds in (1.5) to specify some comments about this relation. We impose on C^* in (1.4) the property of being 2-increasing. For any u_1, u_2, v_1, v_2 in [0, 1] such that $u_1 < u_2$ and $v_1 < v_2$ and $R = [u_1, u_2] \times [v_1, v_2]$, with simple calculation we get

$$V_{C^*}(R) = V_C(R) + \lambda(f(u_2) - f(u_1))(g(v_2) - g(v_1)) \geq 0 \tag{2.1}$$

then

$$\lambda(f(u_2) - f(u_1))(g(v_2) - g(v_1)) \geq -V_C(R). \tag{2.2}$$

If f and g are absolutely continuous functions, as in Rodríguez-Lallena and Úbeda-Flores' study we get

$$\frac{-V_C(R)}{\max(\alpha\gamma, \beta\delta)(u_2 - u_1)(v_2 - v_1)} \leq \lambda \leq \frac{-V_C(R)}{\min(\alpha\delta, \beta\gamma)(u_2 - u_1)(v_2 - v_1)} \tag{2.3}$$

and to get the optimal interval we have

$$\sup\left(\frac{-V_C(R)}{\triangle \times \max(\alpha\gamma, \beta\delta)}\right) \leq \lambda \leq \inf\left(\frac{-V_C(R)}{\triangle \times \min(\alpha\delta, \beta\gamma)}\right) \tag{2.4}$$

where $\triangle,\ \alpha,\ \beta,\ \gamma,\ \delta$ are same as (1.3). From the comparison of (2.4) and (1.5) the inaccuracy of the interval by (1.5) is clear. Moreover we show inaccuracy of the mentioned interval in (1.5) by several counter-examples:

Counterexample 2.1. Let f, g are of the form $f(u) = u(1-u),\ g(v) = v(1-v)$ then $\alpha = -1,\ \beta = 1,\ \gamma = -1\ ,\ \delta = 1$ and $\max(\alpha\gamma, \beta\delta) = \min(\alpha\gamma, \beta\delta) = 1$. For any rectangle R

$$\lambda = \frac{-V_C(R)}{(u_2 - u_1)(v_2 - v_1)} \tag{2.5}$$

and then there does not exist any λ satisfying the above equalities whenever C is different from the product copula. Moreover, for the product copula, there is the unique solution $\lambda = -1$, i.e., this fact also contradicts the result of Rodríguez-Lallena and Úbeda-Flores' study if we let $C(u, v) = uv$.

Counterexample 2.2. Let $C(u, v) = min(u, v)$ and f, g for all u, v in $[0, 1]$ are non-zero absolutely continuous functions as below

$$f(u) = u^2(1-u) \quad , \quad g(v) = v(1-v) \tag{2.6}$$

then $\alpha = -0.3333,\ \beta = 1,\ \gamma = -1\ ,\ \delta = 1$ and hence

$$\max(\alpha\gamma, \beta\delta) = 1 \quad , \quad \min(\alpha\gamma, \beta\delta) = 0.3333. \tag{2.7}$$

In the Kim's approach we obtain empty set for lambda (see R=$[0.2, 0.3] \times [0.1, 0.2]$ and R=$[0, 1]^2$). Note that by (2.4) we get $\lambda = 0$.

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Some remarks on level dependent capacities based Sugeno integral

Peter Smrek[1]

Abstract. *The standard Sugeno integral has several equivalent ways to be introduced. This equivalence fails when generalizing the standard capacities into level dependent capacities. We discuss several possible types of Sugeno integral based on level dependent capacities. Some illustrative examples are added.*

Keywords: *capacity; level dependent capacity; Sugeno integral*

1 Introduction

For a measurable space $(X, \mathcal{A})$, a monotone set function m: $\mathcal{A} \rightarrow [0,1]$ is called a capacity whenever m(∅) = 0 and m(X) = 1. Observe that capacities are sometimes called also fuzzy measures [6]. Sugeno has introduced his integral in 1974 [6], considering fuzzy events, i.e. $\mathcal{A}$-measurable functions $f: X \rightarrow [0,1]$, as a functional $S_m(f)$: $\mathcal{F} \rightarrow [0,1]$, where $\mathcal{F}$ is the class of all fuzzy events on $(X, \mathcal{A})$. S_m was given by

$$S_m(f) = sup\ \{\ min\ (a, m(A)) \mid a \cdot 1_A \leq f\ \}\ . \tag{1}$$

Equivalently, S_m can be expressed as

$$S_m(f) = sup\ \{\ min\ (a, m(f \geq a)) \mid a \in [0,1]\}, \tag{2}$$

or

$$S_m(f) = sup\ \{\ min\ (m(A),\ min(f(x) \mid x \in A) \mid A \in \mathcal{A})\ \}. \tag{3}$$

In [2], another equivalent definition of Sugeno integral was introduced, namely

$$S_m(f) = inf\ \{\ max\ (a, m(f \geq a)) \mid a \in [0,1]\}. \tag{4}$$

Recently, the concept of capacities was extended to level dependent capacities [1], see also [3, 4].
A mapping $M: \mathcal{A} \times [0,1] \rightarrow [0,1]$ such that for each $t \in [0,1]$, $M(\cdot, t) = m_t$ is a capacity, is called a level dependent capacity.
The aim of this paper is to discuss Sugeno integral with respect to level dependent capacities, discussing its different forms based on extension of formulae (1) – (4). The paper is organized as follows. In the next section, we introduce extremal forms of level dependent capacities based Sugeno integral following the approach from [3], and versions of this integral deduced from formulae (1) – (4). In section 3, some examples are given. Finally, some concluding remarks are added.

2 Sugeno integral and level dependent capacities

Sugeno integral, as introduced in [6], is a special instance of universal integrals proposed by Klement et al. in [4]. In the framework of universal integrals, all information contained in a capacity m and a fuzzy event f is summarized into one special function $h_{m,f}: [0,1] \rightarrow [0,1]$ given by $h_{m,f}(t) = m(f \geq t)$. This function can be seen as generalized survival function (i.e., a complement to distribution function). In the case of universal integrals extended for level dependent capacities, Klement at al. have proposed in [3] to consider the function $h_{M,f}: [0,1] \rightarrow [0,1]$ given by $h_{M,f}(t) = M(\{f \geq t\}, t) = m_t(f \geq t)$. Observe that while $h_{m,f}$ is a decreasing function (and thus Borel measurable), these properties need not

[1] Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia

be satisfied for $h_{M,f}$. Then, again following [3], two decreasing boundaries $(h_{M,f})_*, (h_{M,f})^*$: $[0,1] \to [0,1]$ of $h_{M,f}$ can be considered,

$$(h_{M,f})_* = sup\ \{ h : [0,1] \to [0,1] \mid h\ is\ decreasing, h \leq h_{M,f} \} \quad \text{and}$$

$$(h_{M,f})^* = inf\ \{ h : [0,1] \to [0,1] \mid h\ is\ decreasing, h \geq h_{M,f} \}.$$

It is not difficult to check that for each $t \in [0,1]$ it holds

$$(h_{M,f})_*(t) = inf\ \{h_{M,f}(u) \mid u \in [0,t]\} \quad \text{and}$$

$$(h_{M,f})^*(t) = sup\ \{h_{M,f}(v) \mid v \in [t,1]\}.$$

Obviously $(h_{M,f})_* = h_{M,f} = (h_{M,f})^*$ if and only if $h_{M,f}$ is decreasing. Following [3], the smallest Sugeno integral based on level dependent capacities $(Su_M)_*: \mathcal{F} \to [0,1]$ is given by

$$(Su_M)_*(f) = sup\ \{ min\ (t,\ (h_{M,f})_*(t)) \mid t \in [0,1] \}. \tag{5}$$

Similarly, the greatest Sugeno integral based on level dependent capacities $(Su_M)^*: \mathcal{F} \to [0,1]$ is given by

$$(Su_M)^*(f) = sup\ \{ min\ (t,\ (h_{M,f})^*(t)) \mid t \in [0,1] \}. \tag{6}$$

Evidently, it holds

$$(Su_M)^*(f) = sup\ \{ min\ (t, h_{M,f}(v)) \mid 0 \leq t \leq v \leq 1\} \tag{7}$$

and

$$(Su_M)_*(f) = inf\ \{ max\ (t, h_{M,f}(u)) \mid 0 \leq u \leq t \leq 1 \}. \tag{8}$$

Rewriting formulae (1) – (4) for level dependent capacities we get the next possible forms of level dependent capacities based Sugeno integral:

$$Su_M^{(1)}(f) = sup\ \{ min\ (a, m_a(A)) \mid a \cdot 1_A \leq f \}, \tag{9}$$

$$Su_M^{(2)}(f) = sup\ \{ min\ (a, m_a(f \geq a)) \mid a \in [0,1] \}, \tag{10}$$

$$Su_M^{(3)}(f) = sup\ \{ min\ (t, m_t(A)) \mid A \in \mathcal{A}, t = min(f(x) \mid x \in A)\}, \tag{11}$$

$$Su_M^{(4)}(f) = inf\ \{ max\ (a, m_a(f \geq a)) \mid a \in [0,1]\}. \tag{12}$$

It is not difficult to check that $Su_M^{(2)} = Su_M^{(3)}$ due to the monotonicity of $\mathrm{m_a}$ for each fixed $a \in [0,1]$. Moreover, $Su_M^{(1)} \geq Su_M^{(2)}$ because of the fact that $a \cdot 1_{\{f \geq a\}} \leq f$ for each $a \in [0,1]$. We have the following Hasse diagram of all introduced versions of Sugeno integral based on level dependent capacities, see Figure 1.

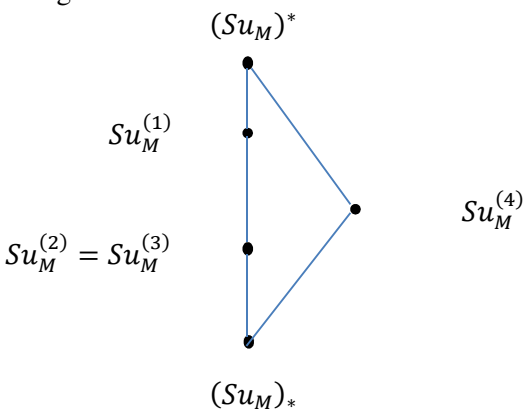

Fig. 1 Hasse diagram of different Sugeno integrals based on level dependent capacities

3 Examples

Obviously, if $M = (m_t)_{t \in [0,1]}$ is a constant level dependent capacity, $m_t = m$ for all $t \in [0,1]$, then all introduced integrals coincide.

Example 1.
For an arbitrary but fixed measurable space $(X, \mathcal{A})$,

i) consider $M = (m_t)_{t \in [0,1]}$, $m_t = m^*$ if $t \in \left[0, \frac{1}{2}\right]$ and $m_t = m_*$ if $t \in]\frac{1}{2}, 1]$,

where m^*, $m_* : \mathcal{A} \to [0, 1]$ are given by

$$m^*(A) = \begin{cases} 0 \; if \; A = \emptyset, \\ 1 \quad else, \end{cases}$$

and

$$m_*(A) = \begin{cases} 1 \; if \; A = X, \\ 0 \quad else. \end{cases}$$

Then for all integrals introduced in Section 2 we have the same result, namely

$$med\left(sup\, f, \frac{1}{2}, inf\, f\right).$$

Observe that the coincidence of all integrals from Section 2 appears, whenever the system $M = (m_t)_{t \in [0,1]}$ is decreasing in t, as then the function $h_{M,f}$ is decreasing, independently of f.

ii) Consider now $M = (m_t)_{t \in [0,1]}$ given by $m_t = m_*$ if $t \in \left[0, \frac{1}{2}\right]$, and $m_t = m^*$ if $t \in]\frac{1}{2}, 1]$.

Then:

- if $sup\, f \leq \frac{1}{2}$, all integrals from Section 2 have value $inf\, f$,
- if $inf\, f \geq \frac{1}{2}$, all integrals from Section 2 have value $sup\ f$,
- in the remaining case, $(Su_M)_*(f) = inf\, f = Su_M^{(4)}(f)$, $(Su_M)^*(f) = sup\ f = Su_M^{(i)}(f), i = 1,2,3.$

Example 2.
Fix $X = [0,1]$ and $\mathcal{A} = \mathcal{B}([0,1])$ (Borel subsets of $[0,1]$). Consider now $M = (m_t)_{t \in [0,1]}$ given by $m_t = m^*$ if $t \in \left[0, \frac{1}{4}\right]$, $m_t = m_*$ if $t \in]\frac{1}{4}, \frac{1}{2}[$, $m_t(A) = \sqrt{\lambda(A)}$ for $A \in \mathcal{A}$, where λ is the standard Lebesgue measure on $\mathcal{B}([0,1])$ if $t \in \left[\frac{1}{2}, 1\right]$, and $f_1(x) = x$. Then we have

$$h_{M,f_1}(t) = \begin{cases} 1 \; if \; t \in \left[0, \frac{1}{4}\right], \\ 0 \; if \; t \in]\frac{1}{4}, \frac{1}{2}[, \\ \sqrt{1-t} \; if \; t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

$$\left(h_{M,f_1}\right)^*(t) = \begin{cases} 1 \; if \; t \in \left[0, \frac{1}{4}\right], \\ \sqrt{0.5} \; if \; t \in]\frac{1}{4}, \frac{1}{2}[, \\ \sqrt{1-t} \; if \; t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

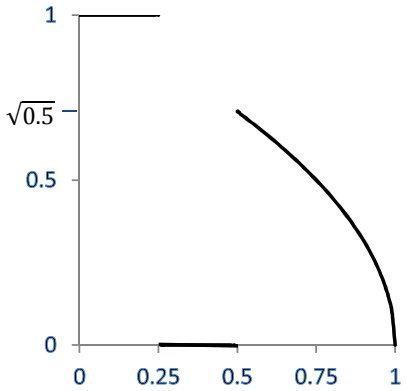

Fig. 2 Function h_{M,f_1}

$$\left(h_{M,f_1}\right)_*(t) = \begin{cases} 1 \; if \; t \in \left[0, \frac{1}{4}\right], \\ 0 \; if \; t \in]\frac{1}{4}, 1]. \end{cases}$$

Then introduced integrals are :

$$(\mathrm{Su_M})_*(f_1) = \frac{1}{4} = Su_M^{(4)}(f_1)\ , (\mathrm{Su_M})^*(f_1) = \frac{\sqrt{5}}{2} - \frac{1}{2} = Su_M^{(i)}(f_1), i = 1,2,3.$$

Example 3.
Fix $= [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$, f(x) = x. For measurable space $(X, \mathcal{A})$ consider

i) $M = (m_t)_{t \in [0,1]}$ given by

$$m_{\frac{1}{3}} = m_*, \qquad m_{\frac{2}{3}} = m^*, \qquad else\ m_t(A) = \lambda(A)$$

for $A \in \mathcal{A}$, where λ is the standard Lebesgue measure on $\mathcal{B}([0,1])$.
Then introduced integrals are :

$$(\mathrm{Su_M})_*(f) = \frac{1}{3} = Su_M^{(4)}(f), (\mathrm{Su_M})^*(f) = \frac{2}{3} = Su_M^{(i)}(f), i = 1,2,3.$$

ii) Consider now $M = (m_t)_{t \in [0,1]}$ given by

$$m_0 = m_*,\ m_{\frac{1}{3}} = m^*,\ m_{\frac{2}{3}} = m_*\ , m_1 = m^*\ , else\ m_t(A) = t \; if \; A \notin \{\emptyset, X\}.$$

Then introduced integrals are :

$$(\mathrm{Su_M})_*(f) = 0 = Su_M^{(4)}(f), (\mathrm{Su_M})^*(f) = 1 = Su_M^{(i)}(f), i = 1,2,3.$$

4 Concluding remarks

We have introduce several versions of Sugeno integral for level dependent capacities, following the ideas from [3] . Our examples suggest $(Su_M)_* = Su_M^{(4)}$ and $(Su_M)^* = Su_M^{(i)}, i = 1,2,3.$ This problem is an open problem for our further investigation. In our next study, we will also focus on copula-based Sugeno integral for level dependent capacities, and study the properties of all introduced integrals. A special focus will be put on the (comonotone) maxitivity of introduced functionals, where we aim to relate our results to those presented in [5] .

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Probabilistic summation of fuzzy numbers

Andrea Stupňanová *

Abstract: *We introduced an alternative look on fuzzy numbers based on two random variables and to their summation. This approach covers triangular norm - based approach, however, it is much more general. We illustrate it on some examples.*

Keywords: *copula, fuzzy arithmetics, fuzzy number, triangle function, triangular norm.*

1 Introduction

Fuzzy numbers and the related arithmetics were deeply studied since 1975 [3]. For overview of most important definitions and results we recommend [8, 1].

Definition 1. Consider a fuzzy subset A of $\mathbb{R} =]-\infty, \infty[$. Let μ_A be its membership function. *A fuzzy number* is the fuzzy subset A whenever

- it is normal, i.e., $\mu_A(x_0) = 1$ for some $x_0 \in \mathbb{R}$,
- function μ_A is upper semi-continuous and convex, i.e.,

  $$\bigcup_{\alpha>\beta} \{x \in \mathbb{R} |\ \mu_A \geq \alpha\} = \{x \in \mathbb{R} |\ \mu_A \geq \beta\}$$

  for each $\alpha, \beta \in [0, 1]$, and

  $$\mu_A(\lambda x + (1-\lambda)y) \geq \min(\mu_A(x), \mu_A(y))$$

  for each $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,
- it has bounded support, i.e., there are $x_1, x_2 \in \mathbb{R}, x_1 < x_2$ such that

  $$\mu_A(x) = 0 \text{ whenever } x < x_1 \text{ or } x > x_2.$$

In any type of fuzzy arithmetics, multiplication by a real constant $c \in \mathbb{R}$ gives either 0 if $c = 0$, or, if $c \neq 0$ then $\mu_{cA}(x) = \mu_A(\frac{x}{c})$. Similarly, concerning $c + A$, we have $\mu_{c+A}(x) = \mu_A(x - c)$.

However, for the processing of proper fuzzy numbers, several approaches have been proposed, so far. They are based on a given (left-continuous) triangular norm (t-norm) $T : [0, 1]^2 \to [0, 1]$ (i.e., an associative, commutative and monotone binary operation on $[0, 1]$ with neutral element $e = 1$; for more details see [5]), modifying the Zadeh extension principle from [14]. We will focus on the summation only, and for a given t-norm T, it is defined as follows:

Definition 2. Let A, B be fuzzy numbers and T a triangular norm. Then the T-sum $C = A \boxplus_T B$ has the membership function

$$\mu_C(z) = \sup \{T(\mu_A(x), \mu_B(z - x)) |\ x \in \mathbb{R}\}. \tag{1}$$

*Slovak University of Technology, Faculty of Civil Engineering, Bratislava, Slovakia, *andrea.stupnanova@stuba.sk*

Note that there are also alternative definition of fuzzy numbers, using the distribution functions, or survival functions, see [4, 7, 10], but still exploiting formula (1).

Inspired by the above mentioned alternative definitions, we offer in this contribution an alternative look on fuzzy numbers and their summation.

The contribution is organized as follows. In the next section, we use a probabilistic look on fuzzy numbers as a pair of random variables $A = (X, Y)$, and propose a probabilistic approach to summation of fuzzy numbers. In Section 3, several examples are introduced. Finally, concluding remarks are added, especially towards the probabilistic approach to multiplication of fuzzy numbers.

2 Probabilistic approach to the summation of fuzzy numbers

Consider a fuzzy number A and a real value x_0 such that $\mu_A(x_0) = 1$. It is evident that the functions $F_A, S_A : \mathbb{R} \to [0, 1]$ given by

$$F_A(x) = \begin{cases} \mu_A(x) & \text{if } x \leq x_0, \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad S_A(x) = \begin{cases} 1 & \text{if } x \leq x_0, \\ \mu_A(x) & \text{otherwise,} \end{cases}$$

are a distribution function and a survival function, respectively. Moreover, considering random variables X and Y defined on some probabilistic space $(\Omega, \mathcal{A}, P)$, and related to F_A and S_A, respectively, it is evident that $X \leq Y$ in strict sense, i.e., $X(\omega_1) \leq Y(\omega_2)$ for any $\omega_1, \omega_2 \in \Omega$. Also the range of X is contained in $[x_1, x_0]$, and the range of Y is contained in $[x_0, x_2]$, compare Definition 1. Moreover, observe that $\mu_A = \min(F_A, S_A)$. We summarize the above facts in the next proposition.

Proposition 1. *A fuzzy subset A of $\mathbb{R}$ is a fuzzy number (with membership function μ_A) if and only if there is a pair of random variables (X, Y) with non-overlapping ranges, $X \leq Y$, related to a distribution function F_X and survival function S_Y, respectively, so that $\mu_A = \min(F_X, S_Y)$.*

Consider fuzzy numbers $A \sim (X_A, Y_A)$ and $B \sim (X_B, Y_B)$. Our aim is to introduce a sum C of A and B. If we consider only functions F_A, S_A, F_B, S_B, one can apply any triangle function τ [11, 12]. Observe that triangle functions are defined on distance distribution functions, i.e., those with support in $[0, \infty]$, and thus one should consider F_{A,x_1} given by $F_{A,x_1} = F_A(x + x_1)$, F_{S_A,x_0} given by $F_{S_A,x_0}(x) = 1 - S_A(x + x_0)$, etc.

If one considers triangle function τ_T defined by means of a t-norm T,

$$\tau_T(F_{X_1}, F_{X_2})(z) = \sup \{T(F_{X_1}(x), F_{X_2}(y))|\ x + y = z\},$$

we recover the formula (1).

However, considering random variables X_A, Y_A, X_B, Y_B, it is natural to look for the sum $X_C = X_A + X_B$ (and then to the related distribution function F_C), and to the sum $Y_C = Y_A + Y_B$ (and then to the related survival function S_C). It is well-known that the distribution function F_{X_C} of a random variable $X_C = X_A + X_B$ depends on the joint distribution function $F_{X_A,X_B} : \mathbb{R}^2 \to [0, 1]$,

$$F_{X_C}(z) = P(\{(x, y) \in \mathbb{R}^2|\ x + y \leq z\}), \tag{2}$$

where the probability P is introduced by the joint distribution function F_{X_A,X_B}. Since Sklar [13] we know that for each couple of random variables X_A and X_B there is a copula $K : [0, 1]^2 \to [0, 1]$ so that

$$F_{X_A,X_B}(x, y) = K(F_{X_A}(x), F_{X_B}(y)).$$

For more details on copulas we recommend lecture notes [9]. Now, we are ready to introduce probabilistic approach to summation of fuzzy quantities.

Definition 3. Let $K : [0, 1]^2 \to [0, 1]$ be a fixed copula and $(X_A, Y_A), (X_B, Y_B)$ be couples of random variables linked to fuzzy numbers A and B, respectively. Then the K-sum of A and B, $C = A \oplus_K B$, is linked to a couple (X_C, Y_C) of random variables, $X_C = X_A + X_B, Y_C = Y_A + Y_B$, with distribution functions F_{X_C} and F_{Y_C}, respectively, given by (2) and considering respectively join distribution functions $F_{X_A,X_B} = K(F_{X_A}, F_{X_B})$ and $F_{Y_A,Y_B} = K(F_{Y_A}, F_{Y_B})$.

Observe that formula (1) cannot decrease the uncertainty characterized be the spreads of incoming fuzzy numbers (it is always between the maximal incoming spread, and the sum of incoming spreads), see [1], what is not the case of our approach. Note that this phenomenon was till now obtained only when the constraint fuzzy arithmetics was considered [6, 2].

3 Examples

For the sake of simplicity, we will consider only random variables uniformly distributed over a given interval. We denote by $X_{[a,b]}$ ($Y_{[a,b]}$) a random variable uniformly distributed over $[a, b]$.

Example 1. A pair $(X_{[a,b]}, Y_{[c,d]})$ is linked to a fuzzy number A if and only if $b \leq c$, and then

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } x \in [a, b], \\ 1 & \text{if } x \in [b, c], \\ \frac{d-x}{d-c} & \text{if } x \in [c, d], \\ 0 & \text{otherwise,} \end{cases}$$

i.e., A is a trapezoidal fuzzy number, $A = TPFN(a, b, c, d)$. If $b = c$, then A is triangular fuzzy number, $A = TFN(a, b, d)$, see [1].

Example 2. Consider three basic copulas $W, \Pi, M : [0, 1]^2 \to [0, 1]$

$$\begin{aligned} W(x, y) &= \max(0, x + y - 1), \\ \Pi(x, y) &= xy, \\ M(x, y) &= \min(x, y). \end{aligned}$$

Then:

i) The copula M models the total positive dependence. Thus, if we consider $X_{[a_1,b_1]}$ and $X_{[a_2,b_2]}$, necessarily $X_{[a_2,b_2]} = a_2 + (b_2 - a_2) \cdot \frac{X_{[a_1,b_1]} - a_1}{b_1 - a_1}$, and thus

$$X_{[a_1,b_1]} + X_{[a_2,b_2]} = X_{[a_1+a_2,b_1+b_2]}.$$

Then, if $A = TPFN(a_1, b_1, c_1, d_1)$ and $B = TPFN(a_2, b_2, c_2, d_2)$, it holds

$$C = A \oplus_M B = TPFN(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2),$$

recovering the traditional min-based sum of trapezoidal fuzzy numbers given by formula (1) (in general, $\oplus_M = \boxplus_M$).

ii) The copula W models the total negative dependence, and then

$$X_{[a_2,b_2]} = b_2 - (b_2 - a_2) \cdot \frac{X_{[a_1,b_1]} - a_1}{b_1 - a_1}.$$

Consequently,

$$X_{[a_1,b_1]} + X_{[a_2,b_2]} = X_{[\min(a_1+b_2,a_2+b_1),\max(a_1+b_2,a_2+b_1)]},$$

which in the case $a_1 + b_2 = a_2 + b_1$ (i.e., the intervals $[a_1, b_1]$ and $[a_2, b_2]$ have the same length), yields $X_{[a_1,b_1]} + X_{[a_2,b_2]} = a_1 + b_2$ (i.e., the Dirac distribution in point $a_1 + b_2$ is obtained). Thus

$$\begin{aligned} A \oplus_W B = TPFN(&\min(a_1 + b_2, a_2 + b_1), \max(a_1 + b_2, a_2 + b_1), \\ &\min(c_1 + d_2, c_2 + d_1), \max(c_1 + d_2, c_2 + d_1)). \end{aligned}$$

Consider $A = B = TFN(0, 1, 2)$. Then $A \oplus_W B = 1$ is a crisp real number.

iii) The copula Π models the independence, and thus the random vector $(X_{[a_1,b_1]}, X_{[a_2,b_2]})$ is uniformly distributed over the rectangle $[a_1, b_1] \times [a_2, b_2]$ (and its density is constant $\frac{1}{(b_1-a_1)(b_2-a_2)}$).

For the simplicity consider $A = B = TFN(0, 1, 2)$, i.e., $b_1 - a_1 = b_2 - a_2 = 1$. Then, for $X_C = X_A + X_B$ its distribution function $F_{X_C} = F_{X_A} * F_{X_B}$ is the convolution of distribution function F_{X_A} and F_{X_B}, i.e.,

$$F_{X_C}(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{x^2}{2} & \text{if } x \in [0, 1], \\ \frac{4x-2-x^2}{2} & \text{if } x \in [1, 2], \\ 1 & \text{otherwise.} \end{cases}$$

Using a similar reasoning for random variables Y_A, Y_B, Y_C, we see that $F_{Y_C}(x) = F_{X_C}(x - 2)$, and thus

$$S_{Y_C}(x) = \begin{cases} 1 & \text{if } x \leq 2, \\ \frac{4x-2-x^2}{2} & \text{if } x \in [2, 3], \\ \frac{(4-x)^2}{2} & \text{if } x \in [3, 4], \\ 0 & \text{otherwise,} \end{cases}$$

and thus the fuzzy number $C = A \oplus_\Pi B$ has a membership function μ_C given by

$$\mu_C(x) = \begin{cases} \frac{x^2}{2} & \text{if } x \in [0, 1], \\ \frac{4x-2-x^2}{2} & \text{if } x \in [1, 3], \\ \frac{(4-x)^2}{2} & \text{if } x \in [3, 4], \\ 0 & \text{otherwise.} \end{cases}$$

Observe that, considering $A = B = TFN(0, 1, 2)$, it holds:

- $A \oplus_M B = TFN(0, 2, 4) = A \boxplus_M B$;
- $A \oplus_W B = 1$ but $A \boxplus_W B = TFN(1, 2, 3)$, i.e., $A \oplus_W B$ is a fuzzy subset of $A \boxplus_W B$;
- for $D = A \boxplus_\Pi B$,

$$\mu_D(x) = \begin{cases} (\frac{x}{2})^2 & \text{if } 0 \leq x \leq 2, \\ (\frac{4-x}{2})^2 & \text{if } 2 \leq x \leq 4, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $A \boxplus_\Pi B$ is a fuzzy subset of $A \oplus_\Pi B$.

4 Concluding remarks

We have introduced a new look on fuzzy numbers by means of random variables. This fact has opened the door to a new type of probabilistic fuzzy arithmetics. In this paper, only the probabilistic summation of fuzzy numbers was considered and exemplified. The other parts of probabilistic fuzzy arithmetics will be the topic of our further investigations.

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Properties of aggregation operators extended via extension principle

Zdenko Takáč *

Abstract: *Recently an extension of aggregation operators via extension principle was proposed in the literature. This is a tool for aggregation of fuzzy truth values (fuzzy sets in $[0,1]$). We study some properties of these extended aggregation operators with respect to the properties of original aggregation operators. We show that basic properties are preserved by the extension: symmetry, idempotency, neutral element and annihilator.*

Keywords: *Aggregation operator, Fuzzy truth values, Extension principle, Type-2 fuzzy sets, Type-2 aggregation operator.*

1 Introduction

The theory of aggregation of real numbers is well established (see e.g. [11], [3], [18]). It is useful in fuzzy logic systems based on fuzzy sets (we will refer to as type-1 fuzzy sets). The concept of type-2 fuzzy sets was introduced by Zadeh [19] as an extension of classical fuzzy sets. The membership grades of type-2 fuzzy sets are classical fuzzy sets in $[0,1]$, we will refer to as fuzzy truth values. The type-2 fuzzy sets are very useful in circumstances where it is difficult to determine an exact membership function for a fuzzy set [7]. This makes them to be an attractive tool in many real problems. However, there is no sufficiently developed theory allowing us to aggregate fuzzy truth values. This is one of several obstacles for applicability of the systems based on type-2 fuzzy sets.

Some particular generalized aggregation operators were studied: type-2 t-norms and type-2 t-conorms in [4], [6], [13], [14]; type-2 implications in [5]; α-level approach to type-1 OWA operator is developed in [20], and it is applied in [2]; an overview of linguistic aggregation operators is given in [17] summarizing results from [1], [8], [9], [12]. Theoretical aspects of aggregation operators for fuzzy truth values are presented in [10] and [16]. The authors of [10] focus on multi-dimensional aggregation of fuzzy numbers, especially with trapezoidal shape. They applied the extension principle to multi-dimensional functions (with certain conditions) and obtained multi-dimensional aggregation functions on the lattice of fuzzy numbers. In [15] is proposed an extension of aggregation operators via convolution. The resulting operator aggregates fuzzy truth values. We study some basic properties of this extended aggregation operators: symmetry, idempotency, neutral element and annihilator based on the properties of original classical aggregation operator.

The paper is organized as follows. Section 2 contains basic definitions and notations that are used in the remaining parts of the paper. Section 3 presents the extension of aggregation operator via convolution and some properties of the extended aggregation operators are studied. Conclusions are drawn in Section 4.

2 Definitions and notations

In this section we present some basic concepts and terminology that will be used throughout the paper.

*Institute of Information Engineering, Automation and Mathematics, Faculty of Chemical and Food Technology, Slovak University of Technology in Bratislava, Radlinského 9, 812 37 Bratislava, Slovak Republic

A mapping $f : X \to [0,1]$ is called a fuzzy set in a set X, the value $f(x)$ is called a membership grade of x. A fuzzy set f in X is normal if there exists $x \in X$ such that $f(x) = 1$. The crisp set $Ker(f) = \{x \in X \mid f(x) = 1\}$ is called a kernel of f. Let X be a linear space, a fuzzy set f in X is convex if it is satisfied $f(\lambda x_1 + (1-\lambda)x_2) \geq \min(f(x_1), f(x_2))$ for all $\lambda \in [0,1]$, where x_1, x_2 are arbitrary elements of X.

Let $\mathcal{F}$ denotes a class of all fuzzy sets in $[0,1]$. Elements of $\mathcal{F}$ are called fuzzy truth values.[1] Moreover, we denote by $\mathcal{F}_N$, $\mathcal{F}_C$ a class of all normal, convex fuzzy truth values, respectively.

Definition 1. *A function $A : [0,1]^n \to [0,1]$ is called an n-ary aggregation operator on $[0,1]$ if and only if it satisfies the conditions:*

$(A1)$ $A(0,\ldots,0) = 0$;

$(A2)$ $A(1,\ldots,1) = 1$;

$(A3)$ $x_1 \leq y_1, \ldots, x_n \leq y_n$ *implies* $A(x_1,\ldots,x_n) \leq A(y_1,\ldots,y_n)$.

for all $x_1, y_1, \ldots x_n, y_n \in [0,1]$.

An n-ary aggregation operator A is called: symmetric if for each permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ and each $x_1,\ldots,x_n \in [0,1]$ it holds $A(x_1,\ldots,x_n) = A(x_{\sigma(1)},\ldots,x_{\sigma(n)})$; idempotent if for each $x \in [0,1]$ it holds $A(x,\ldots,x) = x$. An element $a \in [0,1]$ is called an annihilator of n-ary aggregation operator A if for each $x_1,\ldots,x_n \in [0,1]$ when $x_k = a$ for some $k = 1,\ldots,n$, then $A(x_1,\ldots,x_n) = a$. An element $e \in [0,1]$ is called a neutral element of n-ary aggregation operator A if for each $k = 1,\ldots,n$ and each $x \in [0,1]$ it holds

$$A(\underbrace{e,\ldots,e}_{(k-1)-times}, x, \underbrace{e,\ldots,e}_{(n-k)-times}) = x.$$

3 Extension of aggregation operators and its properties

3.1 Extension of aggregation operators

According to Zadeh's extension principle [19] n-ary aggregation operator $A : [0,1]^n \to [0,1]$ can be extended by the convolution with respect to minimum $\wedge$ and maximum $\vee$ to n-ary operator $\widetilde{A} : \mathcal{F}^n \to \mathcal{F}$ as follows:

$$\widetilde{A}(f_1,\ldots,f_n)(y) = \sup_{A(x_1,\ldots,x_n)=y} (f_1(x_1) \wedge \ldots \wedge f_n(x_n)), \tag{1}$$

where $y, x_1,\ldots,x_n \in [0,1]$ and $f_1,\ldots,f_n \in \mathcal{F}$.

Clearly, if $f_1,\ldots,f_n$ are crisp values from $[0,1]$ considered as fuzzy subsets of $[0,1]$, the obtained result $\widetilde{A}(f_1,\ldots,f_n)$ is fuzzy truth value corresponding to the crisp aggregation of considered values, i.e., the original classical aggregation is embedded into fuzzy extension (1).

3.2 Symmetry

An extended aggregation operator $\widetilde{A}$ is called symmetric if for each permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ and each $f_1,\ldots,f_n \in \mathcal{F}$ the following holds:

$$\widetilde{A}(f_1,\ldots,f_n) = \widetilde{A}(f_{\sigma(1)},\ldots,f_{\sigma(n)}). \tag{2}$$

Theorem 1. *Let A be an n-ary aggregation operator and $\widetilde{A} : \mathcal{F}^n \to \mathcal{F}$ be an extended aggregation operator on fuzzy truth values given by (1). Then A is symmetric if and only if $\widetilde{A}$ is symmetric.*

[1] The reason is that the elements of $\mathcal{F}$ are grades of type-2 fuzzy sets. Recall that type-2 fuzzy sets are fuzzy sets whose membership grades are fuzzy sets in $[0,1]$, i.e. type-2 fuzzy set is a mapping $\widetilde{f} : X \to \mathcal{F}$.

Proof. Sufficiency: Straightforward from (1).

Necessity: Let A be asymmetric. Then there exist $z_1, \ldots, z_n \in [0,1]$ and a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $A(z_1, \ldots, z_n) \neq A(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$. Let f_i, for all $i = 1, 2, \ldots, n$, be defined by: $f_i(z_i) = 1$ and $f_i(x) = 0$, for $x \in [0,1] - \{z_i\}$. Let $y = A(z_1, \ldots, z_n)$. Then

$$\widetilde{A}(f_1, \ldots, f_n)(y) = \sup_{A(x_1,\ldots,x_n)=y} (f_1(x_1) \wedge \ldots \wedge f_n(x_n)) = f_1(z_1) \wedge \ldots \wedge f_n(z_n) = 1$$

and from $y \neq A(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ it follows

$$\widetilde{A}(f_{\sigma(1)}, \ldots, f_{\sigma(n)})(y) = \sup_{A(x_{\sigma(1)},\ldots,x_{\sigma(n)})=y} (f_{\sigma(1)}(x_{\sigma(1)}) \wedge \ldots \wedge f_{\sigma(n)}(x_{\sigma(n)})) = 0.$$

So also $\widetilde{A}$ is asymmetric. □

3.3 Idempotency

An extended aggregation operator $\widetilde{A}$ is called idempotent if for each $f \in \mathcal{F}$ the following holds:

$$\widetilde{A}(f, \ldots, f) = f. \tag{3}$$

Theorem 2. *Let A be an idempotent n-ary aggregation operator. Then an extended aggregation operator on convex fuzzy truth values $\widetilde{A} : \mathcal{F}_C^n \to \mathcal{F}_C$ given by (1) is idempotent.*

Proof. Let $f \in \mathcal{F}_C$. Then for each $y \in [0,1]$:

$$\widetilde{A}(f, \ldots, f)(y) = \sup_{A(x_1,\ldots,x_n)=y} (f(x_1) \wedge \ldots \wedge f(x_n)) \geq f(y), \tag{4}$$

due to the idempotency of A. Now we need to prove the opposite inequality. Suppose that there exist $x_1, \ldots, x_n$ with $A(x_1, \ldots, x_n) = y$ such that $f(x_i) > f(y)$ for all $i \in \{1, \ldots, n\}$. From the convexity of f it follows $x_i > y$ for all $i \in \{1, \ldots, n\}$, or $x_i < y$ for all $i \in \{1, \ldots, n\}$. If the former is true (the proof of the latter one is similar): let $x_0 = \min\{x_1, \ldots, x_n\}$, then $A(x_1, \ldots, x_n) \geq A(x_0, \ldots, x_0) = x_0 > y$ and we have a contradiction with $A(x_1, \ldots, x_n) = y$. □

3.4 Neutral element

A function $g : [0,1] \to [0,1]$ is called a neutral element of extended aggregation operator $\widetilde{A}$ if for each $k = 1, \ldots, n$ and each $f \in \mathcal{F}$ it holds that:

$$\widetilde{A}(\underbrace{g, \ldots, g}_{(k-1)-times}, f, \underbrace{g, \ldots, g}_{(n-k)-times}) = f.$$

Theorem 3. *Let A be an n-ary aggregation operator with neutral element e. Then a function $g : [0,1] \to [0,1]$ is a neutral element of extended aggregation operator on fuzzy truth values $\widetilde{A} : \mathcal{F}^n \to \mathcal{F}$ given by (1) if and only if*

$$g(x) = \begin{cases} 1 & , \textit{if } x = e, \\ 0 & , \textit{otherwise}. \end{cases} \tag{5}$$

Proof. Let g be given by (5). Then for all $f \in \mathcal{F}$, $k \in \{1, \ldots, n\}$ it holds:

$$\widetilde{A}(\underbrace{g, \ldots, g}_{(k-1)-times}, f, \underbrace{g, \ldots, g}_{(n-k)-times})(y) =$$

$$= \sup_{A(x_1,\ldots,x_n)=y} (g(x_1) \wedge \ldots \wedge g(x_{k-1}) \wedge f(x_k) \wedge g(x_{k+1}) \wedge \ldots \wedge g(x_n)) = f(y).$$

The uniqueness follows from the uniqueness of neutral element of aggregation operators in general. □

3.5 Annihilator

A function $g : [0,1] \to [0,1]$ is called an annihilator of extended aggregation operator $\widetilde{A}$ if for each $f_1, \ldots, f_n \in \mathcal{F}$ when $f_k = g$ for some $k = 1, \ldots, n$, then

$$\widetilde{A}(f_1, \ldots, f_n) = g.$$

Theorem 4. *Let A be an n-ary aggregation operator with annihilator a. Then a function g is an annihilator of extended aggregation operator on normal fuzzy truth values $\widetilde{A} : \mathcal{F}_N^n \to \mathcal{F}_N$ given by (1) if and only if*

$$g(x) = \begin{cases} 1 & , \textit{if } x = a, \\ 0 & , \textit{otherwise}. \end{cases} \tag{6}$$

Proof. Let g be given by (6). Then for all $f_1, \ldots, f_n$, where $f_k = g$ for some $k \in \{1, \ldots, n\}$, it holds:

$$\widetilde{A}(f_1, \ldots, f_n)(a) =$$

$$= \sup_{A(x_1,\ldots,x_n)=a} (f_1(x_1) \wedge \ldots \wedge f_{k-1}(x_{k-1}) \wedge g(x_k) \wedge f_{k+1}(x_{k+1}) \wedge \ldots \wedge f_n(x_n)) = 1,$$

because for $x_k = a$ and $x_i \in Ker(f_i)$, for $i \neq k$, it holds $f_1(x_1) = \ldots = f_n(x_n) = 1$. Moreover, for $y \neq a$ we have:

$$\widetilde{A}(f_1, \ldots, f_n)(y) = 0,$$

because $A(x_1, \ldots, x_n) \neq a$ implies $x_i \neq a$ for all $i = 1, \ldots, n$ and consequently $g(x_k \neq a) = 0$.

The uniqueness follows from the uniqueness of annihilator of aggregation operators in general. □

4 Conclusions

Recently an extension of aggregation operators via extension principle was proposed in the literature. This approach leads to the following constructing method: we consider a type-1 aggregation operator with well known properties and extend it via convolution (extension principle) to operator that aggregate functions. In this paper we showed that the extended operator have similar properties as original aggregation operator. More precisely, we proved that the extension preserves symmetry, idempotency, neutral element and annihilator.

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A note on the construction of large graphs and digraphs of given degree and diameter

Mária Ždímalová *

Abstract: *The undirected (directed) version of the degree - diameter problem is determination of the largest order of a graph (digraph) of a given maximum (out -) degree and diameter. In this note we present some of known constructions of the large graphs and digraphs and discuss using of the voltage assignment approach as well as the property to be a Cayley graph (digraph) in some known constructions of large graphs and digraphs.*

1 Introduction

The undirected (directed) degree-diameter problem is to determine the largest order n of a graph (digraph) of a given maximum (out -) degree Δ and diameter D. It also involves the problem of determination of the corresponding extremal graphs and digraphs. For more details we refer to the survey [6] for a summary of the history and the current state-of-the-art.

A spanning tree argument shows that for the number of vertices n

$$n(\Delta, D) \leq M(\Delta, D),$$

where

$$M(\Delta, D) = 1 + \Delta + \Delta(\Delta - 1) + ... + \Delta(\Delta - 1)^{D-1} \tag{1}$$

is the Moore bound. The graphs of maximum degree Δ and diameter at most D will be referred to as (Δ, D) **- graphs**. A Moore graph is a (Δ, D)-graph of order equal to the Moore bound (Δ, D). Another way to study large graphs close to the Moore bound is constructing large graphs of a given degree and diameter in order to improve the lower bound on the maximum possible order of graphs for given Δ and D.

In the case of directed graphs, there is also an upper bound $\overrightarrow{n}(\Delta, D)$ on the order of directed graphs, for given maximum out-degree Δ and diameter D. Let $\overrightarrow{n}_i$, for $0 \leq i \leq D$, be the number of vertices at distance i from v. Then $\overrightarrow{n_i} \leq \Delta^i$, for $0 \leq i \leq D$, and therefore

$$\begin{aligned} \overrightarrow{n}(\Delta, D) = \sum_{i=1}^{D} n_i &\leq 1 + \Delta + \Delta^2 + \ldots + \Delta^D \\ &= \begin{cases} \frac{\Delta^{D+1}-1}{\Delta-1} & \text{if } \Delta > 1 \\ D + 1 & \text{if } \Delta = 1 \end{cases} \end{aligned} \tag{2}$$

The right-hand side of (2), denoted $\overrightarrow{M}_{\Delta,D}$ is called *Moore bound* for digraphs. If the order of the digraph is equal to the Moore bound, such digraph is called a *Moore digraph*.

*Slovak University of Technology in Bratislava, *zdimalova@math.sk*

Various restricted versions of the problem have been studied and we shall be interested in an analysis of certain large *vertex-transitive* graphs and digraphs of a given degree and diameter. Interest in vertex-transitive graphs and digraphs within the undirected as well as in the directed degree-diameter problem is also motivated by computer generation of graphs (digraphs) with an extremely large number of vertices, where fast diameter checking is essential. Chcecking diameter in the process of computer generation of graphs of order "more than a millon", would not be possible in real time if the graphs were not vertex - transitive. This gave rise to investigation of the vertex - transitive and Cayley version of the (Δ, D) - problem.

The aim of this note is to show some known constructions of (Δ, D) - graphs and digraphs and to show how it is posible to describe them in the language of voltage assignment approach and discuss which of the selected construction have the property to be a Cayley graph (digraph).

The paper is organized as follows. In Section 2 are given Preliminaries, in Section 3 will be described the voltage assignment technique [4]. In Section 4 we discusse the construction of Kautz digraph [3, 5].

2 Preliminaries

Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. The order of the graph is the number of vertices. The degree of a vertex is the number of darts adjacent to the vertex. A graph is Δ - regular if the degree of all vertices is equal to Δ. The distance $d(u, v)$ from a vertex u to a vertex v is the number of darts of a shortest directed path from u to v, and its maximum value over all pairs of vertices, $D = max_{u,v \in V} d(u, v)$ is the diameter of the digraph. A graph is vertex symmetric if its automorphism group acts transitively on its set of vertices.

Let Γ be a finite group and let X be unit-free generating set for X such that $X = X^{-1}$, that is, we assume that X is closed under taking inverse elements. The Cayley graph $Cay(\Gamma, X)$ has vertex set Γ, and two vertices $g, h \in \Gamma$ are joined by an edge if $g^{-1}h \in X$. Since this condition is equivalent to $h^{-1}g \in X$ because of $X = X^{-1}$, the Cayley graph $Cay(\Gamma, X)$ is undirected. The degree of $Cay(\Gamma, X)$ is $|X|$.

Let $G = (V, D)$ be a digraph with the vertex set V and the dart set D. The order of the digraph is the number of vertices. The out - degree (in - degree) of a vertex is the number of darts leaving (entering) the vertex. A digraph is Δ - regular if the in - degree and out - degree of all vertices is equal to Δ. All digraphs considered in this article are strongly connected. The distance $d(u, v)$ from a vertex u to a vertex v is the number of darts of a shortest directed path from u to v, and its maximum value over all pairs of vertices, $D = max_{u,v \in V} d(u, v)$ is the diameter of the digraph. A digraph is vertex symmetric if its automorphism group acts transitively on its set of vertices.

Let Γ be a finite group and let X be a subset of Γ which generates Γ and does not contain the identity, the Cayley digraph of Γ with respect to X is the directed graph with vertex set Γ and dart set $\{(u, v); v = ux$ for some $x \in X\}$.

3 Covering techniques

Graph coverings are a special case of coverings of topological spaces, for more details see [4]. A graph can be regarded as one-dimensional simplicial complex. Therefore, known results from algebraic topology can be transferred to graphs. This method enables to "blow up" a given "base graph" to a larger graph (called lift) which is a regular covering space of the base graph.

The lift is best described in terms of the base graph and a mapping, called a voltage assignment. As shown in [1], many of the currently known largest examples of graphs of given degree and diameter can indeed be obtained by the covering graph construction. The covering graph construction has a very good

potential for producing examples of large graph of given degree and diameter. For more details see the monograph of Gross and Tucker [4].

3.1 Ordinary voltage assignments-undirected case

Let G be an undirected graph, possibly with loops and / or parallel edges and semiedges, that is, dangling edges with just one end incident to a vertex. Although the graph G itself is undirected, it will be advantage to assign (for auxiliary purposes) directions to its edges. Each edge (inclusive loops) which is not a semiedge can be assigned one of the two possible directions; an edge with a direction is called a dart. A semiedge has only one possible direction (toward the incident vertex). In this way, every edge which is not a semiedge underlies a pair of mutually reverse darts. The reverse of a dart e is denoted by e^{-1}. For convenience, if e is a semiedge we may still used the symbol e^{-1} but we understand that $e = e^{-1}$ in such a case.

We say that e is a dart at v if the orientation of e points towards the (incident) vertex v. If e is a dart at v than e is also said to terminate at v; at the same time e^{-1} is said to emanate from v. If a dart e emanates from a vertex u and terminates at the vertex v we often say that e is a dart from u to v. The set of all darts of a graph G will be denoted by $D(G)$. Note that $|D(G)| = 2|E(G)| + |S(G)|$, where $S(G)$ stands for the set of all semiedges of G.

By a $u - v$ walk of length k we understand a sequence $W = e_1e_2...e_k$ where e_i are darts of G, such that e_1 emanates from u, e_k terminates at v, and terminal vertex of e_{i-1} coincides with the initial vertex of $e_i, 2 \leq i \leq k$. (If, say, e_i is a dart arising from a semiedge, then its initial and terminal vertices are identical.) We also admit a trivial walk based as u; it consists just of the vertex u and has no darts. By W^{-1} we denote the reverse of W; formally $W^{-1} = e_k^{-1}...e_2^{-1}e_1^{-1}$.

Let Γ be an arbitrary finite group. A mapping $\alpha : D(G) \to \Gamma$ is a **voltage assignment** if, for each dart $e \in D(G)$,

$$\alpha(e^{-1}) = (\alpha(e))^{-1}. \tag{3}$$

The values of α are called voltages an the group Γ is the voltage group. Observe that for each semiedge e we have $(\alpha(e))^2 = 1_\Gamma$.

The pair $\langle G, \Gamma \rangle$ enables us to define a new graph G^α, called an ordinary lift of G. The vertex set of the lift is $V(G^\alpha) = V(G) \times \Gamma$ and the dart set of the lift is $D(G) \times \Gamma$. Incidence in G^α is defined as follows: A dart $(e, i) \in D(G^\alpha)$ emanates from the vertex (u, i) and terminates at the vertex (v, j) if and only if e is a dart from u to v and $j = i\alpha(e)$. We will sometimes use u_i and e_i in place of (u, i) and (e, i).

The ordinary lift G^α is an undirected graph, since the darts (e, i) and $(e^{-1}, i\alpha(e))$ are mutually reverse and form an undirected edge on G^α. Note that a semiedge e in G lifts in G^α either to $|\Gamma|$ loops (if $\alpha(e) = 1_\Gamma$) or to $|\Gamma|/2$ edges which are not loops (if $\alpha(e)$ is a nontrivial element of order 2 in Γ).

The ordinary voltage-graph construction was first suggested by Gross (1974) and immediately improved by Gross and Tucker (1974). Its advantage over various formalistic "covering graph" constructions, all essentially equivalent, is largely its visual suggestiveness. Voltage graphs are usually given by pictures, rather than combinatorial descriptions.

3.1.1 Fibers and Natural Projection

Let us have two graphs H and G. A homomorphism from H to G is a mapping $f : D(H) \to D(G)$, from H into G if f maps any two darts at a common vertex of H onto a pair of darts at a common vertex of G, and if $f(e^{-1}) = (f(e))^{-1}$ for any dart $e \in D(H)$.

A bijective homomorphism $G \to G$ is called an automorphism of G. The collection of all automorphism forms the automorphism group $Aut(G)$ of the graph G.

A covering is a homomorphism $f : H \to G$ of two graphs if for each vertex $v \in V(H)$ the set of darts at v is mapped bijectively by f onto the sets of darts at $f(v)$.

The sets $f^{-1}(u)$ and $f^{-1}(e)$ are called fibers above a vertex u and a dart e, respectivelly. If e is a dart running from a vertex u to a vertex v in the graph G and if e is assigned voltage b, the each edge e_a in the fiber over e runs from the vertex u_a in the fiber over the initial point u to the vertex v_{ab} in the fiber over the terminal point v. We can say that the edge fiber over e matches the vertices in the vertex fiber over u one-to-one onto the vertices of the vertex fiber over v. Thus, the fiber over a proper edge is isomorphic to the disjoint union of $|\Gamma|$ copies of K_2 and the fiber over a loop forms a set of cycles (if the voltage group is fined). If the voltage b on a v-based loop e has order n in the group Γ then each cycle in the edge fiber over e must have length n, and there must be $|\Gamma|/n$ such cycles.

3.1.2 Walk Lifting

Many properties of the lift can be identified by examing walks in the base graph. Let $W = e_1e_2...e_k$ be a walk in the graph G. Then the voltage $\alpha(W)$ of the walk W is defined by $\alpha(W) = \alpha(e_1)\alpha(e_2)...\alpha(e_k)$. Observe that the voltage of W and the voltage of its reverse W^{-1} are related by $\alpha(W^{-1}) = (\alpha(W))^{-1}$. By default the voltage of a trivial walk is defined to be 1_Γ, the identity element of the group Γ.

A 'lift' of a walk W in the base graph G is a walk $\widetilde{W} = \widetilde{e_1}\widetilde{e_2}...\widetilde{e_k}$ in the derived graph G^α such that for $i = 1, ..., n$ the edge $\widetilde{e_i}$ is in the fiber over the edge e_i.

Let W be a walk in the ordinary voltage graph such that the initial vertex W is u. For each vertex u_a in the fiber over u, there is unique lift of W that starts at u_a. It makes sense to designate the lift of a walk W starting at the vertex u_a by W_a. Observe that if W is a walk from u to v in G and b is a net voltage on W, then the lift W_a starting at u_a terminates at the vertex v_{ab}. We can say that for each $u \to v$ walk W in G and each $g \in \Gamma$ there exists a unique walk W_g^α in the lift G^α emanating from u_g and such that $f(W_g^\alpha) = W$; it has the same length as W and terminates at the vertex $v_{g\alpha(W)}$.

3.1.3 Ordinary voltage assignments-directed case

The covering and lifting method described in previously subsections can be use for generating large constructing large digraphs. Most of facts can be applied to digraphs without minor changes. Let G be a *base digraph*, let $A(G)$ be its dart set and let Γ be a finite group. We define a *voltage assignment* on G in Γ as *any* mapping $\alpha : D(G) \to \Gamma$. We need no extra voltages, because edge directions are a part of the description of the digraphs G. The description of the lift is the same as it was by undirected case. The lift is automatically a digraph. For more details see [6].

3.2 Permutation voltage assignments

We now introduce permutation voltage assignment that allow for an alternative description of graph coverings. The permutation on voltage-graph construction is also due to Gross and Tucker(1977).

Let $\sum_n$ be the symmetric group on the set $[n] = \{1, 2, ..., n\}$. A mapping $\sigma : D(G) \to \sum_n$ is said to be a permutation voltage assignment on the graph G if $\sigma(e^{-1}) = (\sigma(e))^{-1}$, for any dart $e \in D(G)$. The permutation lift G^σ is defined by setting $V(G^\sigma) = V(G) \times [n]$ and $D(G^\sigma) = D(G) \times [n]$. As for ordinary derived graphs, one uses the pair (v, i) and (e, i), or the subscripted notations, v_i and e_i. If the dart e of the base graph G runs from the vertex u to vertex v, and if the voltage on e is the permutation π, then for $i = 1, ..., n$, the edge e_i of the derived graph G^σ runs from the vertex u_i to the vertex $v_{\pi(i)}$.

The natural projection $p : G^\alpha \to G$ for permutation voltage graph $\langle G, \alpha\rangle_n$ is the graph map that takes any vertex v_i or edge e_i of the derived graph to the vertex v or the edge e of the base graph. The set of vertices $\{v_i | i = 1, .., n\}$ is called the "fiber" over v and the set of edges $\{e_i | i = 1, .., n\}$ is called the "fiber" over e. The natural projection $p : G^\alpha \to G$ associated with any permutation voltage graph $\langle G, \alpha\rangle_n$ is a covering projection on each component of its domain.

We can define the net permutation voltage on a walk as the product of the voltages encountered in traversal of that walk, exactly as for ordinary voltages. A lift of a walk W in the base graph G is a walk $\widetilde{W}$ in the permutation derived graph such that the natural projection Π maps the edges $\widetilde{W}$ onto the edges of W precisely in the order of traversal. If W is a walk in the base space of a permutation voltage graph $\langle G, \alpha \rangle_n$ such that the initial vertex of W is u, then for each vertex u_i in the fiber over u, there is a unique lift of W that starts at u_i.

4 Kautz digraphs

Kautz digraphs [5] $K(\Delta, D)$ gives the general lower bound on the largest order for the degree diameter problem. These digraphs can be obtained by $(D-1)$ - fold iteration of the line digraph construction applied to the complete digraph of order $\Delta + 1$. As regards symmetry properties of Kautz digraphs, $K(\Delta, D)$ are vertex-transitive if and only if $D \leq 2$. Because of growing interest in vertex-transitive and Cayley digraphs in the degree-diameter problem our aim was in [7] to determine all the values of n for which the line digraph of the complete digraph of order n is a Cayley digraph.

Line digraphs of complete digraphs are a special case of the so - called Kautz digraphs $K(\Delta, D)$ obtained by applying $(D-1)$ - times the line digraphs construction to the complete digraph on $\Delta+1$ vertices [5]. (We recall that in a complete digraph, for any ordered pair of distinct vertices u, v there is an arc from u to v). The digraph $K(\Delta, D)$ is Δ-regular, has $\Delta^D + \Delta^{D-1}$ vertices and diameter D.

Kautz digraphs of diameter two are line digraphs of complete digraphs (in which for any ordered pair of vertices u, v there is an arc from u to v). We have determined [7] all the values of n for which the line digraph of a complete digraph of order n is a Cayley digraph.

Let K_n be the complete digraph of order n (that is with n vertices) and let $L(K_n)$ be its line digraph. As we know, $L(K_n)$ is the Kautz digraph $K(n-1, 2)$. We are now ready to present our characterization.

Theorem 1 *[7] The Kautz digraph $L(K_n)$ is a Cayley digraph if and only if n is a prime power.*

The Kautz digraphs $K(\Delta, D), \Delta \geq 2$ can alternatively be described as follows [2]:

Vertices are labeled with words $x_1 x_2 \dots x_{n-1}$, where x_i belongs to an alphabet of $\Delta+1$ letters and $x_i \neq x_{i+1}$ for $1 \leq i \leq D-1$. A vertex $x_1 x_2 \dots x_{n-1}$ is adjacent to the Δ-vertices $x_2 x_3 \dots x_D x_{D+1}$, where x_{D+1} can be any letter different from x_D. Hence, the digraph $K(\Delta, D)$ is Δ-regular, has $\Delta^D + \Delta^{D-1}$ vertices and diameter D. For $D = 2$ the Kautz digraphs are vertex symmetric.

We remark that assertion of Thm. 1 can be interpreted as follows.

Theorem 2 *If $D = 2$ and $\Delta = q$, where q is a prime power, Kautz digraphs are Cayley digraphs.*

Proof. We recall that for $D = 2$ Kautz digraphs are in fact the digraphs of Faber-Moore-Chen construction [3] and hence the proof of this result in Thm.1. □

Observation 1

$K(\Delta, 2)$ can be described as ordinary lift of bouquet of circles in the following way:
From 2 follows, that for $\Delta = q - 1$, where q is a prime power, and $D = 2$, Kautz digraphs are Cayley digraphs.

The base digraph is a bouquet of circles, that means single vertex digraph with $\Delta = q - 1$ loops. Then we define an ordinary voltage assignment α in a group $A(1, q) = \{x \mapsto ax + b,\ a \neq 0, a, b \in GF(q)\}$ and on every loop we give the voltage $f_{a,1} = ax + 1$, where $a \neq 0, a \in GF(q)$. Then we will obtain a lift, which is isomorphic to the Kautz digraph of diameter 2 and $\Delta = q - 1$.

Observation 2

$K(\Delta, 2)$ can be described as lifts.

As the base graph H we consider complete graphs K_Δ, with one loop at each vertex and $V(H) = Z_{d+1} \setminus \{O\}$. The voltage assignment α is defined: $\alpha(e) = i$, in the voltage group $Z_{\Delta+1}$. Then we will obtained the lift H^α, where $V(H^\alpha) = \Delta.(\Delta + 1)$ which is isomorphic to the Kautz digraph of degree Δ and diameter 2. For more details see [1].

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