Invariants of φ-transformations of uninorms and t-norms

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Abstract: *The paper deals with triangular norms and uninorms, and their constructions. Specifically, we study $\varphi-$transformations and their invariants. The work contains selected results of author's work in the student competition SVOČ.*

Keywords: *t-norm, uninorm, $\varphi-$transformation, invariant*

1 Preliminaries

The main topic of this article is a special type of constructions of triangular norms and uninorms. First we recall some important definitions and statements.

Definition 1.1. *[3] A triangular norm T (t-norm for short) is a commutative, associative, monotone binary operator on the unit interval $[0,1]$, fulfilling the boundary condition $T(x,1) = x$, for all $x \in [0,1]$.*

Uninorms were introduced by Yager and Rybalov in 1996 as a generalization of triangular norms and conorms [7].

Definition 1.2. *[7] An associative, commutative and increasing operation $U : [0,1]^2 \to [0,1]$ is called a uninorm, if there exists $e \in [0,1]$, called the neutral element of U, such that*

$$U(e,x) = U(x,e) = x \text{ for all } x \in [0,1].$$

There exist various constructions of t-norms, and we will deal with a method of constructing t-norms which gives the new t-norm from a previously known t-norm and a unary function φ.

Proposition 1.3. *[3] Let $\varphi : [0,1] \to [0,1]$ be a non-decreasing function and $T : [0,1]^2 \to [0,1]$ be a t-norm. Then the function defined by*

$$T_\varphi(x,y) = \begin{cases} \min\{x,y\}, & \text{if } \max\{x,y\} = 1, \\ \varphi^{(-1)}[T(\varphi(x),\varphi(y))], & \text{otherwise,} \end{cases}$$

is a t-norm. Note, that $\varphi^{(-1)}$ is a pseudo-inverse, which is a monotone extension of the ordinary inverse function and $\varphi^{(-1)}(x) = \sup\{z \in [0,1]; \varphi(z) < x\}$.

We can similarly construct uninorms:

Proposition 1.4. *[2] Let $\varphi : [0,1] \to [0,1]$ be a continuous, bijective function, and let there exist e' such that $e' = \varphi^{-1}(e)$, where e is the neutral element of a given uninorm U. Then the function*

$$U_\varphi(x,y) = \varphi^{-1}[U_e(\varphi(x),\varphi(y))]$$

is a uninorm with the neutral element e'.

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In this paper we will discuss the invariants of $\varphi-$transformation of t-norms and uninorms. It means, we will look for the uninorms and the bijective functions φ such that

$$\varphi(U(x,y)) = U(\varphi(x),\varphi(y)).$$

Finally, we include some necessary notions.

Definition 1.5. *[3] Let $T:[0,1]^2 \to [0,1]$ be a t-norm. Then a function $\delta_n:[0,1]\to[0,1]$ defined as*

$$\delta_1(x) = x, \quad \delta_{n+1}(x) = T(\delta_n(x), x), \quad \textit{for } x \in [0,1], n \in \mathbb{N},$$

is called the diagonal function of a t-norm T. The set of all diagonal functions of given t-norm T is denoted as $\Delta_T = \{\delta_n : n \in \mathbb{N}\}$.

Definition 1.6. *A t-norm T is called Archimedean if it has the Archimedean property, i.e., if for each x, y in the open interval $(0,1)$ there is a natural number n such that $\delta_n \leq y$.*

In this paper we deal with a specific class of uninorms, called simple uninorms.

Definition 1.7. *[2] A uninorm $U:[0,1]^2 \to [0,1]$ is called simple, if there exists left or right neighborhood of y for every $(x,y) \in [0,e) \times (e,1]$, where uninorm U has constant values, i.e.*

$$\forall (x,y) \in [0,e) \times (e,1], \forall y_1, y_2 \in U_\varepsilon^+(y) : U(x,y_1) = U(x,y_2) \quad (U_\varepsilon^-(y)).$$

2 Invariants of transformation on the set $[0,e) \times (e,1]$

In our investigation of invariants of uninorm transformations we start with the set $[0,e) \times (e,1]$.

Definition 2.1. *[2] Let $U : [0,1]^2 \to [0,1]$ be a uninorm with the neutral element e. Then we define $S(U) = \{(a_i,b_i) \times (c_i,d_i); i = 1,\cdots,n; n \in \mathbb{N}\}$ as a system of the sets, such that*

$$\forall J \in S(U) \textit{ and } \forall (x_1,y_1),(x_2,y_2) \in J : U(x_1,y_1) = U(x_2,y_2).$$

Moreover for every J must exists $\alpha_J \in H(J)$, such that

$$\forall p \in D(J) : U(p,\alpha_J) \neq U(x,\alpha_J), \textit{ where } x \in [0,e) \setminus D(J).$$

Definition 2.2. *[2] Let $U : [0,1]^2 \to [0,1]$ be a uninorm with the neutral element e. Then we define the set $M_x(U) = \{(a_1,b_1),\ldots,(a_n,b_n)\}$ as a set of x-coordinate discontinuities of uninorm U on $[0,e) \times (e,1]$. Similarly we define the set of y-coordinate discontinuities as $M_y(U)$.*

The following theorem deals with the properties of transformation function φ in the discontinuity points of given uninorm.

Theorem 2.3. *[2] Let $U:[0,1]^2 \to [0,1]$ be a simple uninorm and $M_x(U)$ be a finite set of x-coordinate discontinuities of uninorm U. Further we consider nondecreasing bijection $\varphi : [0,1] \to [0,1]$. Then if the original uninorm is formed by the φ-transformation, then $\forall (a_i,b_i) \in M_x(U) : \varphi(a_i) = a_i$.*

The proof is based on an examination of the cases $\varphi(a_i) > a_i$ and $\varphi(a_i) < a_i$. Note that in a very similar way we can prove this statement for the set $M_y(U)$, i.e, that $\forall (x,y) \in M_y(U) : \varphi(y) = y$. The following example shows the importance of finiteness of the set $M_x(U)$ from the previous theorem.

Example 2.4. *Let us consider continuous bijective function $f : [0,1] \to [0,1]$ given by following formula*

$$f(x) = \begin{cases} \sqrt[3]{\frac{x}{4}} & \textit{if } x \leq \frac{1}{2}, \\ x & \textit{otherwise}. \end{cases}$$

Further more consider a uninorm $U^* : [0,1]^2 \to [0,1]$ with neutral element $e = \frac{1}{2}$ given as:

$$U^*(x,y) = \begin{cases} 1 & \text{if } \min\{x,y\} > \frac{1}{2}, \\ \min\{x,y\} & \text{if } \max\{x,y\} = \frac{1}{2}, \\ \max\{x,y\} & \text{if } \min\{x,y\} = \frac{1}{2}, \\ f^{i+1}\left(\frac{1}{4}\right) & \text{if } \max\{x,y\} > \frac{1}{2} \text{ and} \\ & \quad \min\{x,y\} \in \left(f^i\left(\frac{1}{4}\right), f^{i+1}\left(\frac{1}{4}\right)\right] \text{for } i \in \mathbb{Z}, \\ 0 & \text{otherwise}. \end{cases}$$

We study transformation given by the function $\varphi = f$. Here we show only the most interesting case of proving the invariance. Therefore we assume $x \in \left(\varphi^i(\frac{1}{4}), \varphi^{i+1}(\frac{1}{4})\right], y \in \left(\frac{1}{2}, 1\right]$. Then

$$U^*(\varphi(x), \varphi(y)) = \varphi^{i+2}\left(\frac{1}{4}\right) = \varphi \circ \varphi^{i+1}\left(\frac{1}{4}\right) = \varphi(U^*(x,y)).$$

Other cases could be proved similarly. The uninorm U^* with the function φ give us an example of a φ-transformation, in which the fixed points of the function φ in discontinuities of U^* are not necessary for invariant.

Theorem 2.5. *[2] Let $U : [0,1]^2 \to [0,1]$ be a simple uninorm and $\varphi : [0,1] \to [0,1]$ be a continuous bijective function. If the original uninorm is formed by the φ-transformation, then*

$$\forall J \in S(U) : \varphi(\sup J_x) = \sup J_x \text{ and } \varphi(\inf J_x) = \inf J_x.$$

Proof. The proof is based on generating the set $M(U)$ using an iteration of the function φ. Since the set $S(U)$ is finite, the set $M(U)$ is finite as well and hence there exists a fixed point of the function φ at the points $\inf J_x$ and $\sup J_x$ for $J \in S(U)$. □

Corollary 2.6. *[2] Let $U : [0,1]^2 \to [0,1]$ be a simple uninorm, $\varphi : [0,1] \to [0,1]$ be a continuous bijective function and $M_y(U)$ be a finite set. If the original uninorm is formed by the φ-transformation, then the interval $(e,1]$ can be divided into subintervals $I_i = (y_i, y_{i+1}]$ for which $\varphi(y_i) = y_i$ holds.*

Theorem 2.7. *[2] Let $U : [0,1]^2 \to [0,1]$ be a simple uninorm, $I_i = (y_i, y_{i+1}]$ be sub-intervals from Corollary 2.6 and a function $\varphi : [0,1] \to [0,1]$ be a continuous bijection for which $\varphi(y_i) = y_i$ holds. Further we assume a function $\psi_i(x) = U(x, y_i)$ for $x \in [0,e)$ and $y \in I_i$. Then the original uninorm on the set $[0,e) \times (e,1]$ is formed by the φ-transformation iff*

$$\varphi \circ \psi_i(x) = \psi_i \circ \varphi(x), \quad \forall x \in [0,e), i \leq n, \tag{1}$$

where n is the number of intervals.

Proof. We use the definition of a φ-tranformation and the previous corollary. In short we get

$$\varphi \circ \psi_i(x) = \psi_i \circ \varphi(x) \Leftrightarrow \varphi(U(x,y)) = U(\varphi(x), y) \Leftrightarrow \varphi(U(x,y)) = U(\varphi(x), \varphi(y))$$

for $x < e, y \in I_i$. □

If we denote a set of all functions φ, satisfying equation (1) as $\mathcal{F}_i$, then a set of all functions $\mathcal{F}_\varphi$ forming the original uninorm by the φ-transformation on the set $[0,e) \times (e,1]$ is given as follows

$$\varphi \in \mathcal{F}_i \Leftrightarrow \varphi \circ \psi_i(x) = \psi_i \circ \varphi(x) \quad \text{and} \quad \mathcal{F}_\varphi = \bigcap_{i=0}^{n} \mathcal{F}_i.$$

In the following text we deal with solving the functional equation (1). Functions satisfying this equation are called as permutable functions.

2.1 Chebyshev polynomials

The first partial solution of equation (1) is composed of Chebyshev polynomials.

Definition 2.8. *[6] Chebyshev polynomials of the first kind T_n are defined by*

$$T_0(x) = 1,\ T_1(x) = x,\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ \textit{for}\ n > 0.$$

Chebyshev polynomials of the second kind U_n are defined by

$$U_0(x) = 1,\ U_1(x) = 2x,\ U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),\ \textit{for}\ n > 0.$$

Theorem 2.9. *[5] Let T_n be the Chebyshev polynomial of the first kind, then for $x \geq 0$, $x \in \mathbb{R}$ and $\alpha = \arccos(x)$ is $T_n(\cos\alpha) = \cos(n\alpha)$.*

Theorem 2.10. *[5] The roots of the polynomial T_n (U_n) are given by*

$$x_k = \cos\left(\frac{\pi}{2}\frac{2k-1}{n}\right), \quad \left(x_k = \cos\left(\pi\frac{k}{n+1}\right)\right), \quad \textit{for}\ k \in \{1, \dots, n\}.$$

Theorem 2.11. *[5] Let T_n be the Chebyshev polynomial of the first kind. Then its derivative is as follows*

$$T_n'(x) = nU_{n-1}(x),$$

where U_{n-1} is Chebyshev polynomial of the second kind .

We will now look for such Chebyshev polynomials which are continuous and increasing on $[0, e]$ and $T_n(0) = 0$, $T_n(e) = e$, for $e \in (0, 1)$ and $T_n(x) \geq x$ for $x \in [0, e]$.

Investigation. From $T_n(0) = 0$ we get $T_n(0) = -T_{n-2}(0) = 0$. More, $4|(n-1)$. From $T_n(e) = e \in (0, 1)$ we get

$$\begin{aligned} e = \cos(n\arccos(e)) &\Leftrightarrow n\arccos(e) = 2k\pi \pm \arccos(e) \Leftrightarrow \\ n = \frac{2k\pi \pm \arccos(e)}{\arccos(e)} &= \frac{2k\pi}{\arccos(e)} \pm 1, \text{for}\ k\ \in \mathbb{Z}. \end{aligned}$$

Therefore

$$\frac{\pi}{\arccos(e)} \in \mathbb{Q}.$$

For fulfillment of other conditions we will look for x_{e_1}, which is the smallest positive nonzero point at which the polynomial T_n attains its local maximum and $x_{e_1} > e$ and $T_n(x_{e_1}) = 1$. Directly from the previous theorem we get:

Theorem 2.12. *Let T_n be Chebyshev polynomial of the first kind. Then the local extremes are in the points x_e which are given by:*

$$x_e = \cos\left(\frac{k\pi}{n}\right), \quad k\ \in \{1, \dots, n\}.$$

Remark: If we consider only polynomials that satisfy the above conditions, then the smallest positive point giving a local maximum is:

$$x_{e_0} = \cos\left(\frac{n-1}{2n}\pi\right).$$

And now we find the smallest point $e \in (0, 1)$ such that $T_n(e) = e$. Then

$$e = \cos(n\arccos(e)) \Leftrightarrow n\arccos(e) = 2k\pi \pm \arccos(e) \Leftrightarrow e = \cos\left(\frac{2k\pi}{n \pm 1}\right).$$

This equality is satisfied for $k \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and for higher k it s the same up to sign. Summarizing the previous we get:

$$\cos\left(\frac{n-1}{2n}\pi\right) > \left|\cos\left(\frac{2k\pi}{n \pm 1}\right)\right|.$$

Since the cosine function is decreasing in the interval $(0, \frac{\pi}{2}]$, we get

$$\left|\frac{\pi}{2} - \frac{n-1}{2n}\pi\right| > \left|\frac{\pi}{2} - \frac{2k\pi}{n \pm 1}\right| \Leftrightarrow \frac{n-1}{2n}\pi < \frac{2k\pi}{n \pm 1}.$$

From the previous investigation we have $k = \lfloor \frac{n}{4} \rfloor$ and

$$\frac{n-1}{2n}\pi < \frac{2\pi}{n+1}\left\lfloor \frac{n}{4} \right\rfloor \Leftrightarrow (n-1)(n+1) < 4n\left\lfloor \frac{n}{4} \right\rfloor = 4n\frac{n-1}{4}.$$

This inequality is satisfied only for $n = 1$. There are no Chebyshev polynomials of the first kind, which would suit our conditions.

2.2 Function iteration

Other particular solution of functional equation (1) is closely related to the iteration of functions [4]. In the following text we denote by $\mathcal{F}$ the set of all nondecreasing functions $f : [0, e] \to [0, e]$ satisfying the conditions $f(x) \geq x$, $f(0) = 0$ and $f(e) = e$.

The following lemma and corollaries explain methods of construction permutable functions.

Lemma 2.13. *[2] Let g and $f : X \to X$ be permutable functions (i.e. $f \circ g(x) = g \circ f(x)$ for all $x \in X$). We further assume nondecreasing (nonincreasing) surjective function $\lambda : X \to X$. Then the functions*

$$\Phi(x) = \lambda^{(-1)} \circ f \circ \lambda(x) \quad \textit{and} \quad \Psi(x) = \lambda^{(-1)} \circ g \circ \lambda(x),$$

where $\lambda^{(-1)}$ is pseudoinverse function to λ, form a pair of permutable functions.

Proof. Since the function λ is a nondecreasing (nonincreasing) surjection, the equality $\lambda \circ \lambda^{(-1)}(x) = x$ is satisfied. Which means that

$$\begin{aligned}
\Phi \circ \Psi(x) &= \lambda^{(-1)} \circ f \circ \lambda \circ \lambda^{(-1)} \circ g \circ \lambda(x) = \lambda^{(-1)} \circ f \circ g \circ \lambda(x) \\
&= \lambda^{(-1)} \circ g \circ f \circ \lambda(x) = \lambda^{(-1)} \circ g \circ \lambda \circ \lambda^{(-1)} \circ f \circ \lambda(x) = \Psi \circ \Phi(x).
\end{aligned}$$

Note. Although the function λ can be in general nonincreasing as well, in the following text we consider only the nondecreasing case due to our restrictions to permutable functions. □

Corollary 2.14. *[2] Let f and g be permutable functions and moreover $f, g \in \mathcal{F}$. Further we assume a nondecreasing surjective function $\lambda : [0, e] \to [0, e]$. Then the functions*

$$\Phi(x) = \lambda^{(-1)} \circ f \circ \lambda(x), \quad \Psi(x) = \lambda^{(-1)} \circ g \circ \lambda(x)$$

form a pair of permutable functions, and moreover $\Phi, \Psi \in \mathcal{F}$.

Corollary 2.15. *[2] Let f be a function such that $f \in \mathcal{F}$. We further assume a nondecreasing surjective function $\lambda : [0, e] \to [0, e]$, and functions $\Phi_n(x) = \lambda^{(-1)} \circ f^n \circ \lambda(x)$ for $n \in \mathbb{N}_0$. Then the functions Φ_i and Φ_j, for $i, j \in \mathbb{N}_0$ form a pair of permutable functions and moreover $\Phi_i, \Phi_j \in \mathcal{F}$.*

The proof of the current and previous corollary is based on certain properties of function iteration and on properties of pseudoinverse functions.

In a search for permutable functions we can as well draw from existing functions as it is shown in the following example.

Example 2.16. *Consider a t-conorm restricted to the set* $[0, e]^2$*, i.e.*

$$S_e(x, y) = eS\left(\frac{x}{e}, \frac{y}{e}\right), \quad \textit{for } (x, y) \in [0, e]^2,$$

and its diagonal functions δ_n^**. Then* $\delta_m^* \circ \delta_n^* = \delta_n^* \circ \delta_m^*$ *for* $m, n \in \mathbb{N}$ *[1]. More specifically, consider restriction of t-conorm probabilistic sum*

$$S_e(x, y) = x + y - \frac{xy}{e}, \quad \textit{for } (x, y) \in [0, e]^2,$$

and the diagonal functions δ_2^* *and* δ_3^* *given by*

$$\delta_2^*(x) = x\left(2 - \frac{x}{e}\right), \quad \delta_3^*(x) = x\left(3 - \frac{3x}{e} + \frac{x^2}{e^2}\right).$$

Then $\delta_2^*(x) \circ \delta_3^*(x) = \delta_3^*(x) \circ \delta_2^*(x)$ *for all* $x \in [0, e]$.

3 Invariant transformation of t-norms

As mentioned before, uninorms are generalizations of t-norms. Hence in this section we deal with an invariant transformation of t-norms. Before we introduce the necessary condition for invariant transformations, we demonstrate a φ-transfomation via the diagonal function δ_n of Frank t-norms.

Example 3.1. *Frank t-norms are defined by [3]:*

$$T_p^F(x, y) = \begin{cases} T_M(x, y) & \textit{if } p = 0, \\ T_P(x, y) & \textit{if } p = 1, \\ T_L(x, y) & \textit{if } p = +\infty, \\ \log_p\left(1 + \frac{(p^x-1)(p^y-1)}{p-1}\right) & \textit{otherwise.} \end{cases}$$

The diagonal function for minimum t-norm is given by $\delta_{n,0}(x) = x$*. Invariance is thus apparent in this case. The diagonal function for product t-norm* $T_P(x, y) = xy$ *is defined by* $\delta_{n,1}(x) = x^n$*. After transformation we obtain* $(xy)^n = x^n y^n$*. Invariance is thus again maintained.*

The diagonal function for Łukasiewicz t-norm $T_L(x, y) = \max\{0, x + y - 1\}$ *is* $\delta_{n,\infty}$ *given by* $\delta_{n,\infty}(x) = \varphi(x) = \max\{0, nx - n + 1\}$*. Invariance is again maintained, as can be seen by substitution.*

For the other cases the diagonal functions $\delta_{n,p}$ *are as follows:*

$$\delta_{n,p}(x) = \varphi(x) = \log_p\left(1 + \frac{(p^x - 1)^n}{(p - 1)^{n-1}}\right).$$

Then the transformation looks as follows

$$T_p^F(\varphi(x), \varphi(y)) = \log_p\left(1 + \frac{(p^x - 1)^n (p^y - 1)^n}{(p - 1)^{2n-1}}\right),$$

$$\varphi(T_p^F(x, y)) = \log_p\left(1 + \frac{\left(\frac{(p^x-1)(p^y-1)}{p-1}\right)^n}{(p - 1)^{n-1}}\right) = \log_p\left(1 + \frac{(p^x - 1)^n (p^y - 1)^n}{(p - 1)^{2n-1}}\right),$$

and thus $T_p^F(\varphi(x), \varphi(y)) = \varphi(T_p^F(x, y))$*. This altgother means, that the invariance towards transformation by the diagonal functions, is maintained for the class of Frank t-norms.*

Now we can introduce the aforementioned necessary condition of invariance.

Theorem 3.2. *[2]* (Necessary condition of invariance) *Let* $T : [0, 1]^2 \to [0, 1]$ *be a t-norm,* δ_n *be diagonal functions of* T *and* $\varphi : [0, 1] \to [0, 1]$ *be a nondecreasing surjective function. If* φ *is an invariant of the transformation of the t-norm* T*, then* $\varphi \circ \delta_n(x) = \delta_n \circ \varphi(x)$ *for all* $x \in [0, 1], n \in \mathbb{N}$.

Proof. Since the function φ is a nondecreasing surjection, the original t-norm is formed by the transformation iff $\varphi(T(x,y)) = T(\varphi(x), \varphi(y))$. Hence $\varphi \circ \delta_n(x) = \delta_n \circ \varphi(x)$ for all $x \in [0,1]$ and $n \in \mathbb{N}$. □

The following theorems show a further relation between diagonal functions, actually additive generators of t-norms, and invariant transformation.

Theorem 3.3. *[2] Let $T : [0,1]^2 \to [0,1]$ be a strict t-norm. We further assume a function $\varphi : [0,1] \to [0,1]$. If $\varphi \in \Delta_T$, then the original t-norm is formed by the φ-transformation.*

Proof. Since the function φ is bijective, equation $\varphi(T_\varphi(x,y)) = T(\varphi(x), \varphi(y))$ is fulfilled. By the assumption $\varphi \in \Delta_T$, we will further write only $\delta_n(T_\varphi(x,y)) = T(\delta_n(x), \delta_n(y))$, for $n \in \mathbb{N}$. The proof of the equation $T_\varphi = T$ will proceed by induction on n.

1. For $n = 1$, the equation holds trivially. For $n = 2$, we assume that there exists some $(x_0, y_0) \in [0,1]^2$ such that $T(x_0, y_0) \neq T_\varphi(x_0, y_0)$. However, then

$$T(T(x_0,y_0), T(x_0,y_0)) = \delta_2(T(x_0,y_0)) \neq T(\delta_2(x_0), \delta_2(y_0)) = T(T(x_0,y_0), T(x_0,y_0)),$$

   which is a contradiction (in the previous step we use associativity of T and the fact that the function δ_n^{-1} is increasing). Thus $T_\varphi = T$ for the transformation by the function δ_2.

2. Now we assume that the equation holds for $\delta_1, \ldots, \delta_n$ and we prove that it holds also for δ_{n+1}. We get

$$T_\varphi(x,y) = \delta_{n+1}^{-1}(T(\delta_{n+1}(x), \delta_{n+1}(y))) \Rightarrow \delta_{n+1}(T_\varphi(x,y)) = T(\delta_{n+1}(x), \delta_{n+1}(y)) \Rightarrow$$
$$T(\delta_n(T_\varphi(x,y)), T_\varphi(x,y)) = T(T(\delta_n(x), x), T(\delta_n(y), y)).$$

   From the induction assumption and associativity of the t-norm T it follows

$$T(T(\delta_n(x), \delta_n(y)), T_\varphi(x,y)) = T(T(\delta_n(x), \delta_n(y)), T(x,y)).$$

   Since the t-norm T is strict, equation $T_\varphi = T$ holds true.

The original t-norm is thus formed by the transformation via diagonal functions. □

Theorem 3.4. *[2] Let $T : [0,1]^2 \to [0,1]$ be a continuous Archimedean t-norm and $f : [0,1] \to [0,\infty]$ be additive the generator of this t-norm. Further let us consider a bijective function $\varphi : [0,1] \to [0,1]$. Then the original t-norm is formed by the φ-transformation iff there exists $\alpha > 0$ such as $\alpha f(x) = f \circ \varphi(x)$ (Schröder's equation).*

Proof. ($\Leftarrow$) The transformed t-norm T_φ is given by

$$T_\varphi(x,y) = \varphi^{-1}[T(\varphi(x), \varphi(y))] = \varphi^{-1} \circ f^{-1}(\min\{f \circ \varphi(x) + f \circ \varphi(y), f(0)\}).$$

Since the t-norm T_φ is a continuous Archimedean t-norm, its additive generator g is given by $g(x) = f \circ \varphi(x)$. There exists $\alpha > 0$, such that $g(x) = \alpha f(x)$, and hence f and g differ only by a positive multiplicative constant. The generator g is thus also a generator of the t-norm T, and consequently $T_\varphi = T$.

($\Rightarrow$) Now we assume $T_\varphi(x,y) = T(x,y)$ for all $(x,y) \in [0,1]^2$, thus

$$T_\varphi(x,y) = \varphi^{-1}[T(\varphi(x), \varphi(y))] = T(x,y).$$

The additive generator of the t-norm T_φ is given by $g(x) = f \circ \varphi(x)$, but since both the t-norms are equal, there exists some $\alpha > 0$ such that $f \circ \varphi(x) = \alpha f(x)$. □

All bijective functions φ on the unit interval, whose transformation form the original t-norm, determine a group of automorphisms $\mathbf{Aut}(T)$. This group for archimedean t-norms is described by Theorem 3.4.

4 Conclusion

This paper shows some conditions under which the φ-transformations of the t-norms and uninorms are invariant. Due to restricted space we skip most of the proofs. But we plan to generalize these results and write a more detailed article.

References

[1] C. Alsina, B. Schweizer, and M. Frank. *Associative Functions: Triangular Norms and Copulas*. World Scientific, 2006.

[2] V. Havlena. Uninorm transformation (in czech). In *SVOČ 2014 – Soutěž studentů vysokých škol ve vědecké činnosti v matematice*, page 9, Ústí nad Labem, 2014. Univerzita J. E. Purkyně.

[3] E. Klement, R. Mesiar, and E. Pap. *Triangular Norms*. Trends in logic, Studia logica library. Springer, 2000.

[4] M. Kuczma, B. Choczewski, and R. Ger. *Iterative Functional Equations*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1990.

[5] T. Rivlin. *The Chebyshev polynomials*. Pure and applied mathematics. Wiley, 1974.

[6] G. J. Tee. Permutable polynomials and rational functions. January 2007.

[7] R. R. Yager and A. Rybalov. Uninorm aggregation operators. *Fuzzy Sets and Systems*, 80:111 – 120, 1996. Fuzzy Modeling.