On Constructing Ordinal Sums of Fuzzy Implications

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Abstract: *In this contribution, new ways of constructing of ordinal sum of fuzzy implications are indicated. Sufficient properties of fuzzy implications as summands for obtaining a fuzzy implication as a result are presented.*

Keywords: *fuzzy implication, ordinal sum, R-implication, triangular norm*

1 Introduction

Fuzzy implications are one of the most important fuzzy connectives in many applications such as fuzzy reasoning and fuzzy control. For that reason new families of these connectives are the subject of investigation. One of the directions of such research is considering an ordinal sum of fuzzy implications on the pattern of the ordinal sum of t-norms. Some interesting results connected to representation of the residual implication corresponding to a fuzzy conjunction (for example continuous or at least left-continuous t-norm) given by an ordinal sum were obtained in [2, 3, 6]. In [8] Su et al. introduced a concept of ordinal sum of fuzzy implications similar to the construction of the ordinal sum of t-norms.

In this contribution, some of the ideas are recalled and new possibilities of defining ordinal sums of fuzzy implications are proposed. The operations obtained by the presented methods are not necessarily fuzzy implications. Sufficient properties for fuzzy implications as summands for obtaining a fuzzy implication are presented.

Firstly, in Section 2, we recall basic definitions and results concerning t-norms and fuzzy implications including constructions of ordinal sums of these fuzzy connectives. Then, in Section 3, we indicate new methods of constructing ordinal sums of fuzzy implications. At the end we suggest further research directions for the ordinal sums of fuzzy implications.

2 Preliminaries

Here we recall the notions of a t-norm and a fuzzy implication, as well as some of the constructions of ordinal sums of these fuzzy connectives.

2.1 Triangular norms

First, we put some very basic information about triangular norms (t-norms).

Definition 2.1 ([5], p. 4). *A triangular norm is an increasing, commutative and associative operation* $T : [0, 1]^2 \to [0, 1]$ *with neutral element* 1.

Definition 2.2 ([5], p. 27). *A triangular norm* T *is called Archimedean, if for each* $(x, y) \in (0, 1)^2$ *there is an* $n \in \mathbb{N}$ *such that* $x_T^{(n)} < y$.

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Example 2.3 ([5], p. 4, [4], p. 7). *Here, we list well-known basic t-norms, from which T_M, T_P, T_L are continuous, and T_P, T_L are both continuous and Archimedean.*

$$T_M(x,y) = \min(x,y), \qquad T_P(x,y) = xy,$$

$$T_L(x,y) = \max(x+y-1,0), \qquad T_D(x,y) = \begin{cases} x, & \text{if } y = 1 \\ y, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases},$$

$$T_{nM}(x,y) = \begin{cases} 0, & \text{if } x+y \leq 1 \\ \min(x,y), & \text{otherwise} \end{cases}.$$

Now, let us recall a representation of continuous t-norms by means of ordinal sums.

Theorem 2.4 ([5], p. 128). *For an operation $T : [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) T is a continuous t-norm.

(ii) T is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e., there exists a uniquely determined (finite or countably infinite) index set I, a family of uniquely determined pairwise disjoint open subintervals (a_k, b_k) of $[0,1]$ and a family of uniquely determined continuous Archimedean t-norms $(T_k)_{k \in A}$ such that

$$T(x,y) = \begin{cases} a_k + (b_k - a_k) T_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) & \textit{if } (x,y) \in [a_k, b_k]^2 \\ \min(x,y) & \textit{otherwise} \end{cases}.$$

The above representation is based on the ordinal sum of arbitrary t-norms ([5], p. 82).

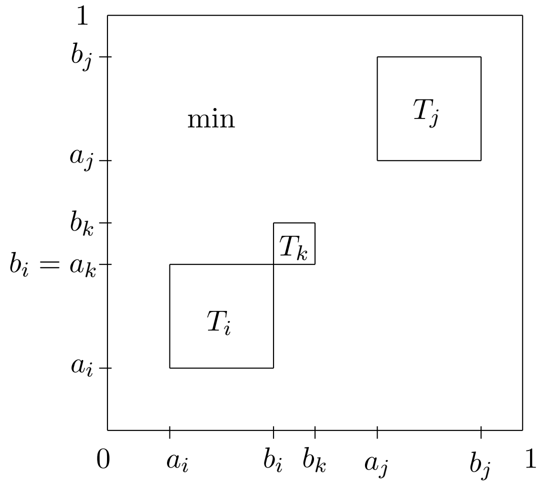

Figure 1: The structure of an ordinal sum of t-norms

2.2 Fuzzy Implications

Now, we focus on fuzzy implications, their possible properties, as well as the class of R-implications.

Definition 2.5 ([1], p. 2, [4], p. 21). *A function $I : [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:*
(I1) decreasing in its first variable,
(I2) increasing in its second variable,
(I3) $I(0,0) = 1$,
(I4) $I(1,1) = 1$,
(I5) $I(1,0) = 0$.

There are many potential properties of fuzzy implications (see, e.g., [1], p. 9). We recall here only one which will be important in the sequel.

Definition 2.6 ([7]). *We say that a fuzzy implication I fulfils the consequent boundary property* (CB) *if*

$$I(x,y) \geq y, \quad x,y \in [0,1]. \tag{CB}$$

Example 2.7 ([1], pp. 4,5). *The following are very known examples of fuzzy implications. Almost all of them, except for I_{RS} fulfil property* (CB).

$$I_{\text{ŁK}}(x,y) = \min(1-x+y,1), \qquad I_{\text{GG}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \frac{y}{x}, & \text{if } x > y \end{cases},$$

$$I_{\text{GD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}, \qquad I_{\text{RS}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases},$$

$$I_{\text{RC}}(x,y) = 1-x+xy, \qquad I_{\text{YG}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{if else} \end{cases},$$

$$I_{\text{DN}}(x,y) = \max(1-x,y), \qquad I_{\text{FD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(1-x,y), & \text{if } x > y \end{cases},$$

$$I_{\text{WB}}(x,y) = \begin{cases} 1, & \text{if } x \leq 1 \\ y, & \text{if } x = 1 \end{cases}, \qquad I_{\text{DP}}(x,y) = \begin{cases} y, & \text{if } x = 1 \\ 1-x, & \text{if } y = 0 \\ 1, & \text{if } x < 1,\ y > 0 \end{cases}.$$

Definition 2.8. *A function $I : [0,1]^2 \to [0,1]$ is called a residual implication (an R-implication) if there exists a t-norm T such that*

$$I(x,y) = I_T(x,y) = \sup\{t \in [0,1] : T(x,t) \leq y\}, \quad x,y \in [0,1]. \tag{1}$$

Example 2.9. *Table 1 shows R-implications obtained by formula* (1) *from basic t-norms presented in Example 2.3.*

t-norm T	R-implication I_T
T_M	I_{GD}
T_P	I_{GG}
T_L	I_{LK}
T_D	I_{WB}
T_nM	I_{FD}

Table 1: Examples of basic R-implications

Theorem 2.10 ([1], p. 83). *If T is a continuous t-norm with an ordinal sum structure (see Theorem 2.4), then the corresponding R-implication I_T is given by*

$$I_T(x,y) = \begin{cases} 1, & \textit{if } x \leq y \\ a_k + (b_k - a_k) I_{T_k}\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \textit{if } x,y \in [a_k,b_k],\ x > y \\ y, & \textit{otherwise} \end{cases}. \tag{2}$$

Now, let us recall a recent approach to the construction of ordinal sum of fuzzy implications [8]. This construction method is based on the construction of the ordinal sum of t-norms.

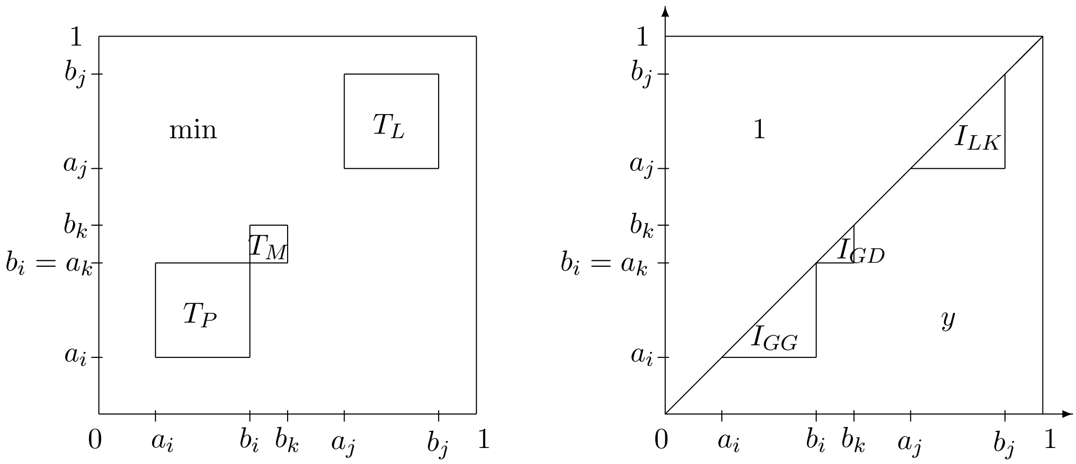

Figure 2: The structures of an ordinal sum of t-norms and R-implication I_T given by (2)

Definition 2.11 ([8]). *Let $\{I_k\}_{k\in A}$ be a family of implications and $\{[a_k, b_k]\}_{k\in A}$ be a family of pairwise disjoint close subintervals of $[0,1]$ with $0 < a_k < b_k$ for all $k \in A$, where A is a finite or infinite index set. The mapping $I : [0,1]^2 \to [0,1]$ given by*

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \text{if } x, y \in [a_k, b_k] \\ I_{GD}(x,y), & \text{otherwise} \end{cases} \tag{3}$$

we call an ordinal sum of fuzzy implications $\{I_k\}_{k\in A}$.

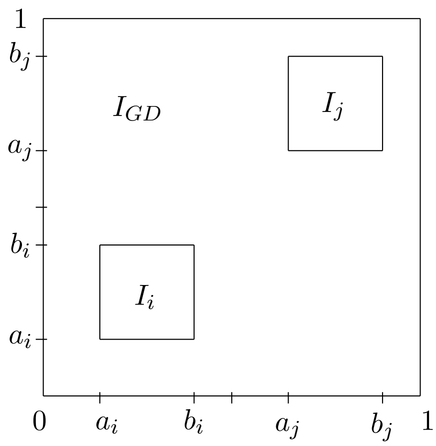

Figure 3: The structure of an ordinal sum of fuzzy implications given by (3)

It may be that I given by (3) is not an implication.

Example 2.12 ([8]). *Let*

$$I(x,y) = \begin{cases} \frac{1}{4} + \left(\frac{1}{2} - \frac{1}{4}\right) I_{RS}\left(\frac{x-\frac{1}{4}}{\frac{1}{2}-\frac{1}{4}}, \frac{x-\frac{1}{4}}{\frac{1}{2}-\frac{1}{4}}\right) & \text{if } (x,y) \in [\frac{1}{4}, \frac{1}{2}]^2, \\ I_{GD}(x,y) & \text{otherwise.} \end{cases}$$

It is easy to see that $I\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{4} < \frac{1}{3} = I\left(\frac{3}{4}, \frac{1}{3}\right)$, i.e. I does not satisfy (I1).

The next theorem gives out the conditions that I given by (3) satisfies (I1).

Theorem 2.13 ([8]). *Let $\{I_k\}_{k\in A}$ be a family of implications. Then ordinal sum of implication given by* (3) *satisfies (I1) if and only if I_k satisfies* (CB) *whenever $k \in A$ and $b_k < 1$.*

Let us notice, that the construction in Definition 2.11 involves intervals $[a_i, b_i]$ which are necessarily disjoint. However, in the construction of t-norms the intervals can have a common point. It is still an open problem, whether we can add some additional assumptions on the construction for the intervals do not have to be disjoint.

3 Main results

Here, we propose tree ways of generating a new fuzzy implication from given ones. Let start with the first method, which is a kind of generalization of the results obtained e.g. in [6] for residual implications.

Let $\{I_k\}_{k\in A}$ be a family of implications and $\{(a_k, b_k)\}_{k\in A}$ be a family of pairwise disjoint subintervals of $[0, 1]$ with $a_k < b_k$ for all $k \in A$, where A is a finite or infinite index set. Let us consider an operation $I : [0, 1]^2 \to [0, 1]$ given by the following formula

$$I(x,y) = \begin{cases} 1, & \text{if } x \le y \\ a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \text{if } x, y \in [a_k, b_k], x > y \\ y, & \text{otherwise} \end{cases} . \tag{4}$$

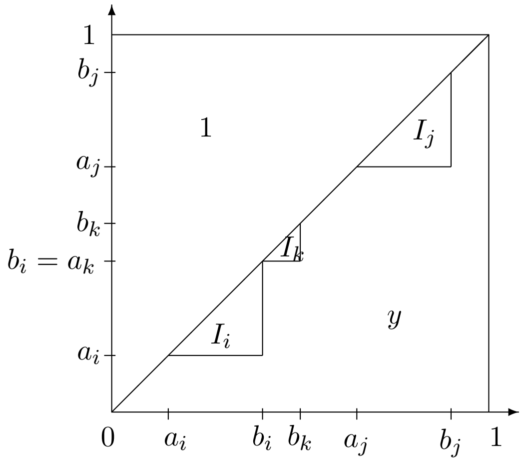

Figure 4: The structure of an operation given by (4)

Remark 3.1. *Let us observe, that the operation I given by* (4) *can be noted as*

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \text{if } x, y \in [a_k, b_k], y < x \\ I_{GD}(x,y), & \text{otherwise} \end{cases} .$$

Lemma 3.2. *Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications. Then I given by* (4) *satisfies (I2), (I3), (I4) and (I5).*

Proof. First, let us consider the condition (I2). Let $y_1 < y_2, x, y_1, y_2 \in [0, 1]$.
If $x \in [a_k, b_k]$ for some $k \in A$, then we obtain the following cases
1. $y_2 < a_k$ or $x \le y_1$ or both $y_1 < a_k$ and $x \le y_2$. Then $I(x, y_1) = I_{GD}(x, y_1) \le I_{GD}(x, y_2) = I(x, y_2)$.
2. $y_1 < a_k \le y_2 \le x$. Then $I(x, y_1) = y_1 < a \le a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y_2-a_k}{b_k-a_k}\right) = I(x, y_2)$.
3. $a_k \le y_1 \le y_2 \le x$. Then using monotonicity of I_k we have $I(x, y_1) = a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y_1-a_k}{b_k-a_k}\right) \le a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y_2-a_k}{b_k-a_k}\right) = I(x, y_2)$.
4. $a_k \le y_1 < x \le y_2$. Then $I(x, y_1) = a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y_1-a_k}{b_k-a_k}\right) \le 1 = I(x, y_2)$.
In other cases we have similar situation as in 1.
Directly from (4) we have $I(0, 0) = I(1, 1) = 1$. So I fulfils (I3) and (I4). To prove (I5) let us consider two cases. If there exists $k \in A$ such that $[a_k, b_k] = [0, 1]$, then $I(1, 0) = I_k(1, 0) = 0$. Otherwise $I(1, 0) = y = 0$. □

Example 3.3. *Let*

$$I(x,y) = \begin{cases} 1, & \text{if } x \le y \\ 0.5 I_{RS}(2x, 2y), & \text{if } x, y \in [0, 0.5] \\ y, & \text{otherwise} \end{cases}$$

I does not fulfill (I1).

The following result can be proved in a similar way to Theorem 2.13.

Theorem 3.4. *The operation I given by* (4) *satisfies (I1) if and only if I_k satisfies* (CB) *whenever $k \in A$ and $b_k < 1$.*

As we can see, not every fuzzy implication can be used in constructions (3) and (4). Below we present a structure in which any fuzzy implications can be used.

Now, let $\{I_k\}_{k \in A}$ be a family of implications and $\{[a_k, b_k]\}_{k \in A}$ be a family of pairwise disjoint close subintervals of $[0, 1]$ with $0 < a_k < b_k$ for all $k \in A$, where A is a finite or infinite index set. Let us consider an operation $I : [0, 1]^2 \to [0, 1]$ given by the following formula

$$I(x, y) = \begin{cases} a_k + (b_k - a_k) I_k \left(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k} \right), & \text{if } x, y \in [a_k, b_k] \\ I_{RS}(x, y), & \text{otherwise} \end{cases} . \tag{5}$$

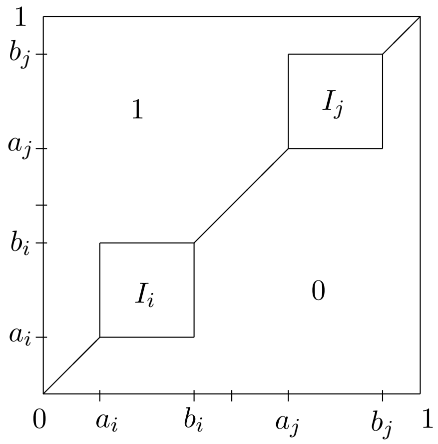

Figure 5: The structure of an operation given by (5)

Theorem 3.5. *The operation I given by* (5) *is a fuzzy implication.*

Proof. First, let us consider the condition (I1). Let $x_1 < x_2$, $x_1, x_2, y \in [0, 1]$.
If $y \in [a_k, b_k]$ for some $k \in A$, then we consider the following cases
1. $x_1 < a_k$. Then $I(x_1, y) = I_{RS}(x, y_1) = 1 \geq I(x_2, y)$.
2. $x_1, x_2 \in [a_k, b_k]$. Then using monotonicity of I_k we have $I(x_1, y) = a_k + (b_k - a_k) I_k \left(\frac{x_1 - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k} \right) \geq a_k + (b_k - a_k) I_k \left(\frac{x_2 - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k} \right) = I(x_2, y)$.
3. $b_k < x_2$. Then $I(x_1, y) \geq 0 = I(x_2, y)$.
In other cases values of I are the same as values of I_{RS}, which give the condition (I1).
To prove (I2) let us take $y_1 < y_2$, $x, y_1, y_2 \in [0, 1]$.
If $x \in [a_k, b_k]$ for some $k \in A$, then we obtain the following cases
1. $y_1 < a_k$. Then $I(x, y_1) = I_{RS}(x, y_1) = 0 \leq I(x, y_2)$.
2. $y_1, y_2 \in [a_k, b_k]$. Then using monotonicity of I_k we have $I(x, y_1) = a_k + (b_k - a_k) I_k \left(\frac{x - a_k}{b_k - a_k}, \frac{y_1 - a_k}{b_k - a_k} \right) \leq a_k + (b_k - a_k) I_k \left(\frac{x - a_k}{b_k - a_k}, \frac{y_2 - a_k}{b_k - a_k} \right) = I(x, y_2)$.
3. $b_k < y_2$. Then $I(x, y_1) \leq 1 = I(x, y_2)$.
In other cases values of I are the same as values of I_{RS}. So, we obtain (I2).
Directly from (6) we have $I(0, 0) = I(1, 1) = 1$. So I fulfils (I3) and (I4). To prove (I5) let us consider two cases. If there exists $k \in A$ such that $[a_k, b_k] = [0, 1]$, then $I(1, 0) = I_k(1, 0) = 0$. Otherwise $I(1, 0) = 0$. So, operation given by (6) is an implication. $\square$

In both constructions (3) and (5) the intervals $[a_k, b_k]$ must be separable. This means that we are unable to construct fuzzy implications in which the values $I(x, x)$ for $x \in (0, 1)$ depend on the component implications I_k. Below we present a construction that solves this problem.

Let $\{I_k\}_{k\in A}$ be a family of implications and $\{(a_k, b_k)\}_{k\in A}$ be a family of pairwise disjoint subintervals of $[0, 1]$ with $a_k < b_k$ for all $k \in A$, where A is a finite or infinite index set. Let us consider an operation $I : [0, 1]^2 \to [0, 1]$ given by the following formula

$$I(x,y) = \begin{cases} a_k + (b_k - a_k)I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & \text{if } x, y \in (a_k, b_k] \\ 1, & \text{if } x \le y \\ 0, & \text{otherwise} \end{cases} . \tag{6}$$

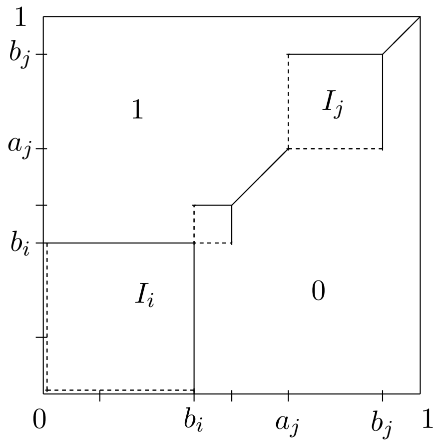

Figure 6: The structure of an operation given by (6)

Example 3.6. *Let*

$$I(x,y) = \begin{cases} 1, & \text{if } x \le y \\ 0.5 I_{RC}(2x, 2y), & \text{if } x, y \in (0, 0.5] \\ 0.5 + 0.1 I_{LK}(10x-5, 10y-5), & \text{if } x, y \in (0.5, 0.6] \\ 0, & \text{otherwise} \end{cases}$$

I is an implication.

The following result can be proved in a similar way to Theorem 3.5.

Theorem 3.7. *The operation I given by* (6) *is a fuzzy implication.*

4 Conclusion

In this paper we indicate three methods of constructing ordinal sum of fuzzy implications. In future research, it would be useful to examine the properties of these ordinal sums. Another problem is whether the proposed ordinal sums preserve properties of its summands.

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