Examples of Archimedean generators from the Williamson transform and why to use a linear approximation

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Abstract

We illustrate a construction method for obtaining additive generators of Archimedean copulas proposed by McNeil and Nešlehová [7], the so-called Williamson n-transform. Then we use weighted sum of Dirac functions to approximate generators of two-dimensional Archimedean copulas by linear splines to circumvent the problem with the non-existence of explicit inverse.

Keywords: Archimedean copula, Williamson transform, approximation

1 Introduction

Copulas form an important class of multivariate dependence models. They have a lot of practical applications, including multivariate survival modelling. Recall that copulas aggregate 1-dimensional marginal distribution functions into n-dimensional ($n \geq 2$) joint distribution functions. For more details we recommend [12].

We first define a copula. A function $C\colon [0,1]^n \to [0,1]$ is called a (n-dimensional) copula whenever it satisfies the boundary conditions (C1) and it is an n-increasing function, see (C2), where:

(C1) $C(x_1,\dots,x_n) = 0$ whenever $0 \in \{x_1,\dots,x_n\}$, i.e., 0 is an annihilator of C, and $C(x_1,\dots,x_n) = x_i$ whenever $x_j = 1$ for each $j \neq i$ (i.e., 1 is a neutral element of C),

(C2) For any $\mathbf{x},\mathbf{y} \in [0,1]^n$, $\mathbf{x} \leq \mathbf{y}$, it holds

$$V_C([\mathbf{x},\mathbf{y}]) = \sum_{\varepsilon \in \{-1,1\}^n} \left(C(\mathbf{z}_\varepsilon) \prod_{i=1}^{n} \varepsilon_i \right) \geq 0,$$

where $\mathbf{z}_\varepsilon = (z_1^{\varepsilon_1},\dots,z_n^{\varepsilon_n})$, $z_i^1 = y_i$, $z_i^{-1} = x_i$.

Note that $V_C([\mathbf{x},\mathbf{y}])$ is called the C-volume of the n-dimensional interval (n-box) $[\mathbf{x},\mathbf{y}]$.

Due to Sklar's theorem [15] for a random vector $Z = (X_1,\dots,X_n)$, a function $F_Z\colon R^n \to [0,1]$ is a joint distribution function of Z if and only if there is a copula $C\colon [0,1]^n \to [0,1]$ so that

$$F_Z(x_1,\dots,x_n) = C\left(F_{X_1}(x_1),\dots,F_{X_n}(x_n)\right), \tag{1}$$

where $F_{X_i}\colon R \to [0,1]$ is a distribution function related to the random variable X_i, $i = 1,\dots,n$. The copula C in (1) is unique whenever random variables $X_1,\dots,X_n$ are continuous. For some other details on copulas see [4] and [12].

Hereafter we will consider a class of copulas named Archimedean copulas. In the simplest case, Archimedean 2-copulas are characterized by the associativity of C and the diagonal inequality $C(x,x) < x$ for all $x \in]0,1[$. They are necessarily symmetric, i.e., they can model the stochastic dependence of exchangeable random variables (X,Y) only, yet their popularity in practice (hydrology, financial, and other applied areas) is indisputable, mainly due to the representation using one-dimensional functions called generators as shown in the next result, attributed to Moynihan [11].

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Theorem 1 *A function $C\colon [0,1]^2 \to [0,1]$ is an Archimedean copula if and only if there is a convex (i.e. a 2-monotone) continuous strictly decreasing function $f\colon [0,1] \to [0,\infty]$, $f(1) = 0$, so that*

$$C(x,y) = f^{(-1)}(f(x) + f(y)), \tag{2}$$

where the pseudo-inverse $f^{(-1)}\colon [0,\infty] \to [0,1]$ is given by

$$f^{(-1)}(u) = f^{-1}\big(\min(u, f(0))\big).$$

The function f is called an additive generator of the copula C, and it is unique up to a positive multiplicative constant.

Let $\mathcal{F}_2$ be the class of all additive generators of binary copulas characterized in the above theorem. More details about the generators can be found in [4, 5, 12] and about construction methods for additive generators in [1, 2, 3, 6, 10].

Before we review several known facts for additive generators of copulas, let us briefly recall a link between copula C and Spearman's correlation coefficient ρ,

$$\rho = 12E[UV] - 3 = 12 \iint_{[0,1]^2} uv dC(u,v) - 3 = 12 \iint_{[0,1]^2} C(u,v) du dv - 3 \tag{3}$$

as well as Kendall's correlation coefficient τ,

$$\tau = 4E[C(U,V)] - 1 = 4 \iint_{[0,1]^2} C(u,v) dC(u,v) - 1 \tag{4}$$

where $U = F_X(X)$ and $V = F_Y(Y)$ are uniformly distributed random variables, that are connected by the same copula as are X and Y. Alternatively, Kendall's tau can be computed directly from copula generator,

$$\tau = 1 + 4\int_0^1 \frac{f(t)}{f'(t)} dt = 1 - 4\int_0^\infty t\left(f^{(-1)\prime}(t)\right)^2 dt$$

which is far more convenient.

Any binary Archimedean copula $C\colon [0,1]^2 \to [0,1]$ generated by an additive generator $f\colon [0,1] \to [0,\infty]$, is also a triangular norm [5, 14] and thus, it can be univocally extended to an n-ary function (we keep the original notation also for this extension) $C\colon [0,1]^n \to [0,1]$ given by

$$C(x_1,\dots,x_n) = f^{(-1)}\left(\sum_{i=1}^n f(x_i)\right). \tag{5}$$

Obviously, for any $n \geq 2$, C satisfies the boundary conditions (C1). However, for $n > 2$, (C2) may fail. For example, the smallest binary copula $W\colon [0,1]^2 \to [0,1]$ given by $W(x,y) = \max(0, x+y-1)$ is generated by the additive generator $f_W\colon [0,1] \to [0,\infty]$, $f_W(x) = 1 - x$. Its n-ary extension is given by

$$W(x_1,\dots,x_n) = 1 - \min\left(1, \sum_{i=1}^n (1-x_i)\right) = \max\left(0, \sum_{i=1}^n x_i - (n-1)\right).$$

Consider $\mathbf{x},\mathbf{y} \in [0,1]^n$, $\mathbf{x} = (\frac{1}{2},\dots,\frac{1}{2})$, $\mathbf{y} = (1,\dots,1)$. Then $V_W([\mathbf{x},\mathbf{y}]) = 1 - \frac{n}{2}$, i.e., this volume is negative whenever $n > 2$, which shows that W is a copula only for $n = 2$. A complete description of additive generators of binary copulas such that the corresponding generated n-ary function is also an n-ary copula, $n > 2$, was given by McNeil and Nešlehová in [7] and is recalled in the next theorem.

Theorem 2 *Let $f\colon [0,1] \to [0,\infty]$ be a continuous strictly decreasing function such that $f(1) = 0$ (i.e., f is an additive generator of a continuous Archimedean t-norm, see [5]). Then the n-ary function $C\colon [0,1]^n \to [0,1]$ given by (5) is an n-ary copula if and only if the function $g\colon [-\infty,0] \to [0,1]$ given by $g(u) = f^{(-1)}(-u)$ is $(n-2)$-times differentiable with non-negative derivatives $g',\dots,g^{(n-2)}$ on $]-\infty,0[$ (or equivalently, $(-1)^n(f^{(-1)})^{(n)}(u) \geq 0$), and $g^{(n-2)}$ is a convex function.*

We denote by $\mathcal{F}_n$ the class of all additive generators that generate n-ary copulas as characterized in Theorem 2.

Additive generators, which generate an n-ary copula for any $n \geq 2$, are called universal generators. The class of all universal additive generators will be denoted by $\mathcal{F}_\infty$. It is not difficult to check that $\mathcal{F}_2 \supset \mathcal{F}_3 \supset \ldots \supset \mathcal{F}_\infty$.

The n-monotone Archimedean copula generators may be characterized using a little known integral transform introduced by Williamson in 1956, see [17]. In McNeil and Nešlehová [7] there is a description of this transform, which, for a fixed $n \geq 2$, will be called the Williamson n-transform. In what follows, we discuss the Williamson n-transform and illustrate it by examples.

2 The Williamson n-transform

An interesting link between additive generators of copulas and positive distance functions [8], i.e., distribution functions with support in $]0, \infty[$, was described in details in [7]. Based on the results of Williamson [17], we recall the next important result.

Theorem 3 (McNeil & Nešlehová [7], Corollary 3.1.) *The following claims are equivalent for an arbitrary $n \in \{2, 3, \ldots\}$:*

(i) $f \in \mathcal{F}_n$

(ii) Under the notation of Theorem 2, the function $F\colon]-\infty, \infty[\to [0,1]$ given by $F(x) = 0$ if $x \leq 0$, and for $x > 0$,

$$F(x) = 1 - \sum_{k=0}^{n-2} \frac{(-1)^k x^k (f^{(-1)})^{(k)}(x)}{k!} - \frac{(-1)^{n-1} x^{n-1} (f^{(-1)})_+^{(n-1)}(x)}{(n-1)!} \tag{6}$$

is a distribution function of a positive random variable X (i.e., $P(X \leq 0) = 0$), where $\cdot_+^{(n-1)}$ denotes the right-derivative of order $n-1$.

Note that due to [17], if F is a positive distance function, i.e., a distribution function of a positive random variable X, then for a fixed $n \in \{2, 3, \ldots\}$ the Williamson n-transform provides an inverse transformation to (6),

$$f^{(-1)}(x) = \int_x^\infty \left(1 - \frac{x}{t}\right)^{n-1} dF(t) = \begin{cases} \max\left(0, E\left[1 - \frac{x}{X}\right]^{n-1}\right), & x > 0 \\ 1 - F(0), & x = 0, \end{cases} \tag{7}$$

where $x \in [0, \infty[$ and $f^{(-1)}(\infty) = 0$.

Note that a similar relationship can be shown between additive generators from $\mathcal{F}_\infty$ and positive distance functions, based on the Laplace transform, i.e

$$f^{(-1)}(x) = \int_0^\infty e^{-xt} dF(t). \tag{8}$$

For more and interesting details we recommend [7].

Let F be a distance function related to a positive random variable X. For any $c > 0$, the random variable $c.X$ possesses the distance function F_c given by $F_c(x) = F\left(\frac{x}{c}\right)$. Then, for any $n \in \{2, 3, \ldots\}$,

$$f_c^{(-1)}(x) = \int_x^\infty \left(1 - \frac{x}{t}\right)^{n-1} dF_c(t) = \int_x^\infty \left(1 - \frac{x}{t}\right)^{n-1} dF\left(\frac{t}{c}\right) = \int_{\frac{x}{c}}^\infty \left(1 - \frac{x}{cu}\right)^{n-1} dF(u) = f^{(-1)}\left(\frac{x}{c}\right).$$

Obviously, for the related additive generators it holds that $f_c = c.f$, i.e., they generate the same copula. Vice versa, clearly from (6) it follows that if two generators generate the same (n-ary) Archimedean copula, the corresponding positive random variables differ only in a positive multiplicative constant. The next result follows.

Theorem 4 *For each $n \in \{2, 3, \ldots\}$, there is an one-to-one correspondence between the class $\mathcal{F}_n$ and the class $\mathcal{H}$ of all factor classes of positive distance functions related to the equivalence $F \sim G$ if and only if $G(x) = F\left(\frac{x}{c}\right)$ for some $c > 0$.*

In the following, we illustrate the construction method by few examples.

Example 1 *Let F be equal to a Dirac function*[1] *focused at point $x_0 = 1$,*

$$F(x) = \delta_1(x) = \begin{cases} 0 & x < 1 \\ 1 & 1 \le x \end{cases},$$

then, as is also shown in [7], by the Williamson n-transform we get generator $f_n(x) = 1 - x^{\frac{1}{n-1}}$ of the weakest n-dimensional Archimedean copula, i.e., the non-strict Clayton copula with parameter $\lambda = \frac{-1}{n-1}$, see Figure 1. By rescaling generator to $\tilde{f}_n(x) = \frac{f(x)}{f(1/2)}$, $x \in [0, 1]$, the copula would not change, yet such a generator is fixed to the value $\tilde{f}_n(\frac{1}{2}) = 1$, which we will use later to show convergence.

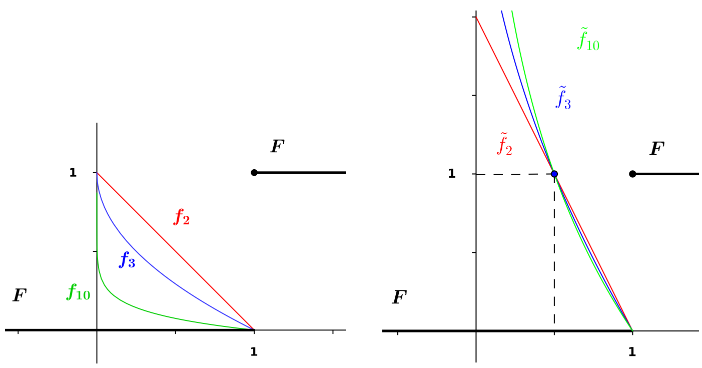


Figure 1: Dirac function F, the corresponding generators f_n for different n and rescaled generators $\tilde{f}_n$.

Example 2 *Let F be a uniform probability distribution function*

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x < b \\ 1 & b \le x \end{cases} \quad \textit{with } 0 \le a < b.$$

Then for dimension $n = 2$ we get

$$f_2^{(-1)}(x) = \int_x^\infty \left(1 - \frac{x}{t}\right)^{2-1} F'(t)dt = \begin{cases} \int_a^b \left(1 - \frac{x}{t}\right) \frac{1}{b-a} dt & x < a \\ \int_x^b \left(1 - \frac{x}{t}\right) \frac{1}{b-a} dt & a \le x < b = \\ \int_x^\infty \left(1 - \frac{x}{t}\right) 0 dt & b \le x \end{cases}$$

$$= \begin{cases} \frac{1}{b-a} \left[t - x \log t\right]_a^b = \frac{1}{b-a}(b - x \log b - a + x \log a) = 1 - \frac{x \log\left(\frac{b}{a}\right)}{b-a} & x < a \\ \frac{1}{b-a} \left[t - x \log t\right]_x^b = \frac{1}{b-a}(b - x \log b - x + x \log x) = \frac{b}{b-a} - \frac{x + x \log\left(\frac{b}{x}\right)}{b-a} & a \le x < b \\ 0 & b \le x \end{cases}$$

[1] Dirac function is defined as $\delta_{x_0}(x) = \begin{cases} 0 & x < x_0 \\ 1 & x \ge x_0 \end{cases}$

(where F' denotes a first derivative of F) from which the corresponding generator can be obtained only numerically, and so is the case also with the higher dimensions, e.g.,

$$f_3^{(-1)}(x) = \begin{cases} 1 - \frac{2x\log\left(\frac{b}{a}\right)}{b-a} + \frac{x^2}{ab} & x < a \\ \frac{b}{b-a} - 2x\log\left(\frac{b}{x}\right) - \frac{x^2}{(b-a)b} & a \le x < b \\ 0 & b \le x \end{cases},$$

displayed in Figure 2.

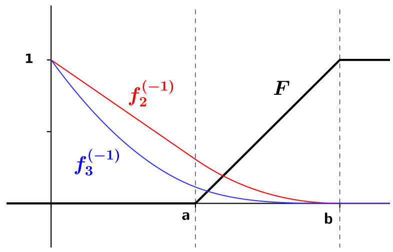


Figure 2: Uniform U(a,b) probability distribution function F and pseudo-inverses of the corresponding generators f_n.

Example 3 *Consider a positive distance function $F(x) = min(1, x^2)$ and the corresponding density $F'(x) = 2x$ on $[0,1]$. Then*

$$f_2^{(-1)}(x) = \int_x^\infty \left(1 - \frac{x}{t}\right)^{2-1} dF(t) = \begin{cases} \int_x^1 (t-x)\frac{2t}{t}dt = \left[(t-x)^2\right]_x^1 = (1-x)^2 & 0 \le x \le 1 \\ 0 & 1 < x \end{cases} = \max(1-x, 0)^2.$$

Then the generator $f_2(x) = 1 - \sqrt{x}$, $x \in [0,1]$, is the generator of Clayton copula for parameter $\lambda = -\frac{1}{2}$. Nevertheless, in higher dimensions, $n \ge 3$, the generator has no closed form, e.g., $f_3^{(-1)}(x) = 1 - 4x + x^2(3 - 2\log x)$ for $x \in [0,1]$ and 0 otherwise.

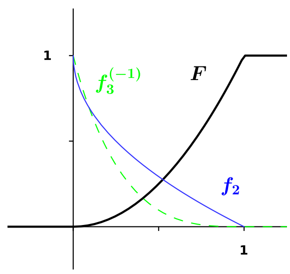


Figure 3: Illustration of Example 3 with non-invertible case $n = 3$.

It is interesting to illustrate also the inverse Williamson n-transform.

Example 4 *Take a generator of*

- *the Ali-Mikhail-Haq copula $f(x) = \frac{1}{x} - 1$ corresponding to the parameter $\lambda = 1$ and denote by F_n, $n = 2, 3, \ldots$, a positive distance function related to f through (6). Then $F_n(x) = 1 - \frac{1}{1+x} - \frac{x}{(1+x)^2} - \ldots - \frac{x^{n-1}}{(1+x)^n} = \left(\frac{x}{1+x}\right)^n$ which can be viewed as a parametric subfamily of all positive valued distribution functions F_p with any positive parameter p.*

- *the product copula $f(x) = -\frac{1}{p}\log x$ with constant $p > 0$ and inverse $f^{-1}(x) = \exp(-px)$. From* (6) *for $n = 2$ we get $F(x) = 1 - \exp(-px)(1 - px)$. By comparing the density $\frac{\partial F(x)}{\partial x} = p^2 x \exp(-px)$ and the convolution of two exponential distribution $\mathcal{D}_\lambda$ densities with parameter $\lambda > 0$, $\int_0^x \lambda \exp(-\lambda t)\lambda \exp(-\lambda(x-t))dt = \lambda^2 x \exp(-\lambda x)$ it becomes clear that the resulting distribution is a distribution of the random variable $Y = X_1 + X_2$, where $X_1, X_2 \sim \mathcal{D}_\lambda$ are independent (and identically distributed) random variables. The relation holds for any $n \geq 2$, thus* (6) *yields a cumulative distribution function of the sum of i.i.d. random variables $X_1, \ldots, X_n \sim \mathcal{D}_p$, $F_{X_1+\ldots+X_n}(x) = 1 - \exp(-px)\sum_{i=1}^{n} \frac{(px)^{i-1}}{(i-1)!}$ with $p > 0$.*

To complete the examples, let us illustrate also the Laplace transform.

Example 5 *Starting with positive distance function of*

- *discrete random variable with probability mass concentrated in $\lambda > 0$, i.e. Dirac function $F(x) = 0$ for $x < \lambda$ and 1 otherwise, then the Laplace transform leads through $g(x) = \exp(\lambda x)$ to the product copula Π.*
- *exponential distribution $F(x) = 1 - \exp(-\lambda x)$, $\lambda > 0$, we get $f^{-1}(x) = (\frac{\lambda}{x+\lambda})$ and $f(x) = \lambda(\frac{1}{x} - 1)$ which generates the same copula (Clayton copula with parameter equal to 1) regardless of the choice of λ.*

Now we focus on the Dirac function since it can be viewed as a building block for distribution functions of a random variable with probability mass concentrated in l discrete points.

3 Approximation

In this section we are interested mainly in $(n = 2)$-dimensional case, since it is of most benefit in practice. Therefore hereafter the subscript with generator f gains a different meaning: the number of pieces f is approximated by.

Example 6 *Let $F(x) = \min(1, x^2)$ be the positive distance function from the Example 3 and function*

$$F_2(x) = F\left(\frac{1}{2}\right)\delta_{\frac{1}{2}}(x) + \left(F(1) - F\left(\frac{1}{2}\right)\right)\delta_1(x) = \begin{cases} 0 & x < \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

approximates F by means of a sum of $m = 2$ Dirac functions concentrated in respective points $\left(\frac{1}{2}, \frac{1}{4}\right)$, $\left(1, \frac{3}{4}\right)$. Then the Williamson transform with $n = 2$ yields

$$f_2^{(-1)}(x) = \frac{1}{4}\max\left(0, 1 - \frac{x}{\frac{1}{2}}\right) + \frac{3}{4}\max\left(0, 1 - \frac{x}{1}\right) = \begin{cases} 1 - \frac{5}{4}x & x < \frac{1}{2} \\ \frac{3}{4} - \frac{3}{4}x & \frac{1}{2} \leq x < 1 \\ 0 & 1 \leq x \end{cases}$$

From Example 6 illustrated on Figure 4 we see that for $n = 2$ the additive generator inverse $f_2^{(-1)}$ is piecewise linear and does not coincide with $f^{(-1)}$ in the interval $]0, 1[$.

Dividing an interval $[a_0, a_m]$ by points $\{a_i\}_{i=1,\ldots m}$, $a_0 < a_1 < \ldots a_m$, with concentration of probability given by some probability mass function $p(x)$, the approximate positive distance function

$$F_m(x) = \sum_{i=1}^{m} p(a_i)\delta_{a_i}(x)$$

is then transformed by (7) to the generator inverse (related to some n-dimensional Archimedean copula)

$$f_m^{(-1)}(x) = \sum_{x < a_i} p(a_i)\left(1 - \frac{x}{a_i}\right)^{n-1} = \sum_{i=1}^{m} p(a_i)\max\left(0, 1 - \frac{x}{a_i}\right)^{n-1}. \quad (9)$$

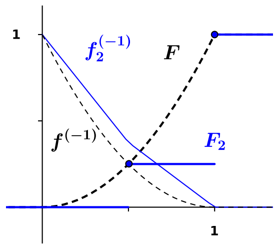


Figure 4: Approximation by the sum of $m = 2$ Dirac functions

Observe that the function $f_m^{(-1)}$ (9) is a $(n-1)$-dimensional spline. For $n = 2$, both $f_m^{(-1)}$ and the corresponding additive generator f_m are linear splines, and the related Archimedean copula C_m is piece-wise linear, as shown in Example 8. In the opposite direction, denote $b_i = f_m^{(-1)}(a_i)$ and $p_i = p(a_i)$ for $i = 1, 2 \ldots m$ with $b_0 = 1$ corresponding to $a_0 = 0$ and, clearly, $b_m = 0$. Having points $\{(a_i, b_i)\}_{i=1,\ldots m}$, their corresponding probabilities can be found by solving exquations (9) with $x = a_1, \ldots, a_{m-1}$ written in the form (for $n = 2$)

$$\begin{pmatrix} 1-\frac{a_1}{a_2} & 1-\frac{a_1}{a_3} & \cdots & 1-\frac{a_1}{a_m} \\ 0 & 1-\frac{a_2}{a_3} & \cdots & 1-\frac{a_2}{a_m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-\frac{a_{m-1}}{a_m} \end{pmatrix} \begin{pmatrix} p_2 \\ p_3 \\ \vdots \\ p_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{m-1} \end{pmatrix}.$$

The solution is $p_1 = 1 - (p_2 + \ldots + p_m)$ and

$$p_i = \frac{a_i\left[b_{i-1}(a_{i+1}-a_i) - b_i(a_{i+1}-a_{i-1}+b_{i+1}(a_i-a_{i-1}))\right]}{(a_{i+1}-a_i)(a_i-a_{i-1})} \qquad \text{for } i = 2, \ldots m,$$

with auxiliary point (a_{m+1}, b_{m+1}), where $a_{m+1} \geq a_m$ and thus $b_{m+1} = 0$.

In the following examples we exercise pointwise convergence and show a piecewise linear copula corresponding to the simplest non-trivial case $n = m = 2$.

Example 7 *For the simplest case, $n = 2$, $a_i = \frac{i}{m}$ and $p(a_i) = \frac{1}{m}$, $i = 1, \ldots m$ (evenly spaced and uniformly distributed), we get $f_m^{(-1)}(x) = \sum_{i=1}^m \frac{1}{m} \max\left(0, 1 - \frac{mx}{i}\right)$. If $f_m^{(-1)}(x)$ is to converge to $f^{(-1)}(x) = 1 - x + x\log x$ for $x < 1$ and 0 elsewhere, it needs to converge in any point $x \in]0, 1[$. Let us examine the convergence, say, in $x = \frac{1}{2}$, where*

$$f_m^{(-1)}\left(\frac{1}{2}\right) = \frac{1}{m}\sum_{i=1}^{m} \max\left(0, 1 - \frac{m\frac{1}{2}}{i}\right) = \frac{1}{m}\sum_{i=\lfloor\frac{m}{2}\rfloor+1}^{m}\left(1 - \frac{m}{2i}\right) = \frac{1}{m}\sum_{i=1}^{\frac{m}{2}} \frac{i}{i+\frac{m}{2}} =$$

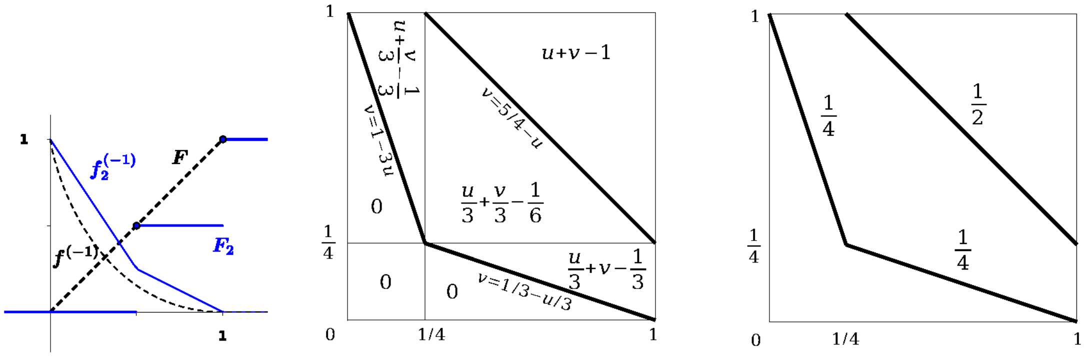
$$\frac{1}{m}\sum_{i=1}^{\frac{m}{2}}\left(1 - \frac{\frac{m}{2}}{i+\frac{m}{2}}\right) = \frac{1}{2} - \frac{1}{2}\sum_{i=\lfloor\frac{m}{2}\rfloor+1}^{m}\frac{1}{i}.$$

Then indeed

$$\lim_{m\to\infty} f_m^{(-1)}\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2}\int_{\frac{m}{2}}^{m}\frac{1}{x}dx = \frac{1}{2} - \frac{1}{2}\left[\ln x\right]_{\frac{m}{2}}^{m} = \frac{1}{2} - \frac{1}{2}\ln 2 = f^{(-1)}\left(\frac{1}{2}\right).$$

Example 8 *Following Example 7, it might help to picture the approximation copula on a simple setting. Due to Example 1 we already know that the trivial case $m = 1$ leads to the weakest copula W. With $m = 2$ we get*

$$F_2(x) = \begin{cases} 0 & x < \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \leq x < 1 \\ 1 & 1 \leq x \end{cases}, \text{ thus } f_2^{(-1)}(x) = \begin{cases} 1 - \frac{3}{2}x & x < \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}x & \frac{1}{2} \leq x < 1 \\ 0 & 1 \leq x \end{cases} \text{ and } f_2(x) = \begin{cases} 1 - 2x & 0 \leq x \leq \frac{1}{4} \\ \frac{2}{3}(1-x) & \frac{1}{4} < x \leq 1 \end{cases}$$



a) b) c)

Figure 5: a) Distance function, generator (inverse) and b) copula, that correspond to uniform distribution approximated in $m = 2$ equally spaced points. c) Probability mass concentrated on copula support.

shown on Figure 5 a), which leads to copula C_2 expressed on Figure 5 b).

To compute measures of dependence (concordance) such as Spearman's rho and Kendall's tau corresponding to singular copula it is generally a challenge, yet for this simple settings it might be an interesting exercise. Since the copula C_2 is piecewise linear, the whole probability mass is concentrated on its support, thus to evaluate the expected values (especially in (4)) one need to find out distribution of the probability. In our case, it is depicted on Figure 5 c). By expressing variable v in terms of u the double integral reduces to one-dimensional integral, then

$$E[UV] = 2\int_0^{1/4} u(1-3u)\frac{\frac{1}{4}}{\frac{1}{4}}du + \int_{1/4}^{1} u\left(\frac{5}{4}-u\right)\frac{\frac{1}{2}}{\frac{3}{4}}du = \frac{2}{64} + \frac{11}{64} = -\frac{13}{64}$$

and

$$E[C(U,V)] = 2\int_0^{1/4} \max\left(0, u + \frac{1-3u-1}{3}\right)\frac{\frac{1}{4}}{\frac{1}{4}}du +$$
$$+ \int_{1/4}^{1} \max\left(\frac{1}{3}(u+\frac{5}{4}-u-\frac{1}{2}), u+\frac{5}{4}-u-1\right)\frac{\frac{1}{2}}{\frac{3}{4}}du = 0 + \frac{1}{8}$$

thus $\rho_2 = 12\frac{13}{64} - 3 = -\frac{9}{16}$ and $\tau_2 = 4\frac{1}{8} - 1 = -\frac{1}{2}$, where the subscript 2 conforms the notation of generator. Although we cannot find explicit form of the original generator f (that corresponds to uniform distribution U[0,1]) and analytically calculate ρ, we still can get $\tau = 1 - \int_0^1 t((1-t+x\ln t)')^2dt = 1 - 4\int_0^1 t\ln^2 t dt = 0$ to measure accuracy of our $m = 2$ approximation.

4 Conclusion

We have discussed a new construction method for obtaining additive generators proposed by McNeil and Nešlehová [7], the so-called Williamson n-transform, and illustrated it by some examples. Some of the generators were shown to not have an explicit form due to non-invertability. Thus a natural approach to utilize any such parametric family is to approximate it by piecewise linear functions with sufficiently dense breakpoints. We showed some simple examples, including calculation of correlation coefficients related to a singular copula.

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