Properties of aggregation operators extended via extension principle

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Abstract: *Recently an extension of aggregation operators via extension principle was proposed in the literature. This is a tool for aggregation of fuzzy truth values (fuzzy sets in* $[0,1]$*). We study some properties of these extended aggregation operators with respect to the properties of original aggregation operators. We show that basic properties are preserved by the extension: symmetry, idempotency, neutral element and annihilator.*

Keywords: *Aggregation operator, Fuzzy truth values, Extension principle, Type-2 fuzzy sets, Type-2 aggregation operator.*

1 Introduction

The theory of aggregation of real numbers is well established (see e.g. [11], [3], [18]). It is useful in fuzzy logic systems based on fuzzy sets (we will refer to as type-1 fuzzy sets). The concept of type-2 fuzzy sets was introduced by Zadeh [19] as an extension of classical fuzzy sets. The membership grades of type-2 fuzzy sets are classical fuzzy sets in $[0,1]$, we will refer to as fuzzy truth values. The type-2 fuzzy sets are very useful in circumstances where it is difficult to determine an exact membership function for a fuzzy set [7]. This makes them to be an attractive tool in many real problems. However, there is no sufficiently developed theory allowing us to aggregate fuzzy truth values. This is one of several obstacles for applicability of the systems based on type-2 fuzzy sets.

Some particular generalized aggregation operators were studied: type-2 t-norms and type-2 t-conorms in [4], [6], [13], [14]; type-2 implications in [5]; α-level approach to type-1 OWA operator is developed in [20], and it is applied in [2]; an overview of linguistic aggregation operators is given in [17] summarizing results from [1], [8], [9], [12]. Theoretical aspects of aggregation operators for fuzzy truth values are presented in [10] and [16]. The authors of [10] focus on multi-dimensional aggregation of fuzzy numbers, especially with trapezoidal shape. They applied the extension principle to multi-dimensional functions (with certain conditions) and obtained multi-dimensional aggregation functions on the lattice of fuzzy numbers. In [15] is proposed an extension of aggregation operators via convolution. The resulting operator aggregates fuzzy truth values. We study some basic properties of this extended aggregation operators: symmetry, idempotency, neutral element and annihilator based on the properties of original classical aggregation operator.

The paper is organized as follows. Section 2 contains basic definitions and notations that are used in the remaining parts of the paper. Section 3 presents the extension of aggregation operator via convolution and some properties of the extended aggregation operators are studied. Conclusions are drawn in Section 4.

2 Definitions and notations

In this section we present some basic concepts and terminology that will be used throughout the paper.

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A mapping $f : X \to [0,1]$ is called a fuzzy set in a set X, the value $f(x)$ is called a membership grade of x. A fuzzy set f in X is normal if there exists $x \in X$ such that $f(x) = 1$. The crisp set $Ker(f) = \{x \in X \mid f(x) = 1\}$ is called a kernel of f. Let X be a linear space, a fuzzy set f in X is convex if it is satisfied $f(\lambda x_1 + (1-\lambda)x_2) \geq \min(f(x_1), f(x_2))$ for all $\lambda \in [0,1]$, where x_1, x_2 are arbitrary elements of X.

Let $\mathcal{F}$ denotes a class of all fuzzy sets in $[0,1]$. Elements of $\mathcal{F}$ are called fuzzy truth values.[1] Moreover, we denote by $\mathcal{F}_N$, $\mathcal{F}_C$ a class of all normal, convex fuzzy truth values, respectively.

Definition 1. *A function $A : [0,1]^n \to [0,1]$ is called an n-ary aggregation operator on $[0,1]$ if and only if it satisfies the conditions:*

$(A1)$ $A(0,\ldots,0) = 0$;

$(A2)$ $A(1,\ldots,1) = 1$;

$(A3)$ $x_1 \leq y_1, \ldots, x_n \leq y_n$ *implies* $A(x_1,\ldots,x_n) \leq A(y_1,\ldots,y_n)$.

for all $x_1, y_1, \ldots x_n, y_n \in [0,1]$.

An n-ary aggregation operator A is called: symmetric if for each permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ and each $x_1,\ldots,x_n \in [0,1]$ it holds $A(x_1,\ldots,x_n) = A(x_{\sigma(1)},\ldots,x_{\sigma(n)})$; idempotent if for each $x \in [0,1]$ it holds $A(x,\ldots,x) = x$. An element $a \in [0,1]$ is called an annihilator of n-ary aggregation operator A if for each $x_1,\ldots,x_n \in [0,1]$ when $x_k = a$ for some $k = 1,\ldots,n$, then $A(x_1,\ldots,x_n) = a$. An element $e \in [0,1]$ is called a neutral element of n-ary aggregation operator A if for each $k = 1,\ldots,n$ and each $x \in [0,1]$ it holds

$$A(\underbrace{e,\ldots,e}_{(k-1)-times}, x, \underbrace{e,\ldots,e}_{(n-k)-times}) = x.$$

3 Extension of aggregation operators and its properties

3.1 Extension of aggregation operators

According to Zadeh's extension principle [19] n-ary aggregation operator $A : [0,1]^n \to [0,1]$ can be extended by the convolution with respect to minimum $\wedge$ and maximum $\vee$ to n-ary operator $\widetilde{A} : \mathcal{F}^n \to \mathcal{F}$ as follows:

$$\widetilde{A}(f_1,\ldots,f_n)(y) = \sup_{A(x_1,\ldots,x_n)=y} (f_1(x_1) \wedge \ldots \wedge f_n(x_n)), \tag{1}$$

where $y, x_1,\ldots,x_n \in [0,1]$ and $f_1,\ldots,f_n \in \mathcal{F}$.

Clearly, if $f_1,\ldots,f_n$ are crisp values from $[0,1]$ considered as fuzzy subsets of $[0,1]$, the obtained result $\widetilde{A}(f_1,\ldots,f_n)$ is fuzzy truth value corresponding to the crisp aggregation of considered values, i.e., the original classical aggregation is embedded into fuzzy extension (1).

3.2 Symmetry

An extended aggregation operator $\widetilde{A}$ is called symmetric if for each permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ and each $f_1,\ldots,f_n \in \mathcal{F}$ the following holds:

$$\widetilde{A}(f_1,\ldots,f_n) = \widetilde{A}(f_{\sigma(1)},\ldots,f_{\sigma(n)}). \tag{2}$$

Theorem 1. *Let A be an n-ary aggregation operator and $\widetilde{A} : \mathcal{F}^n \to \mathcal{F}$ be an extended aggregation operator on fuzzy truth values given by (1). Then A is symmetric if and only if $\widetilde{A}$ is symmetric.*

[1] The reason is that the elements of $\mathcal{F}$ are grades of type-2 fuzzy sets. Recall that type-2 fuzzy sets are fuzzy sets whose membership grades are fuzzy sets in $[0,1]$, i.e. type-2 fuzzy set is a mapping $\widetilde{f} : X \to \mathcal{F}$.

Proof. Sufficiency: Straightforward from (1).

Necessity: Let A be asymmetric. Then there exist $z_1, \dots, z_n \in [0,1]$ and a permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $A(z_1, \dots, z_n) \neq A(z_{\sigma(1)}, \dots, z_{\sigma(n)})$. Let f_i, for all $i = 1, 2, \dots, n$, be defined by: $f_i(z_i) = 1$ and $f_i(x) = 0$, for $x \in [0,1] - \{z_i\}$. Let $y = A(z_1, \dots, z_n)$. Then

$$\widetilde{A}(f_1, \dots, f_n)(y) = \sup_{A(x_1,\dots,x_n)=y} (f_1(x_1) \wedge \dots \wedge f_n(x_n)) = f_1(z_1) \wedge \dots \wedge f_n(z_n) = 1$$

and from $y \neq A(z_{\sigma(1)}, \dots, z_{\sigma(n)})$ it follows

$$\widetilde{A}(f_{\sigma(1)}, \dots, f_{\sigma(n)})(y) = \sup_{A(x_{\sigma(1)},\dots,x_{\sigma(n)})=y} (f_{\sigma(1)}(x_{\sigma(1)}) \wedge \dots \wedge f_{\sigma(n)}(x_{\sigma(n)})) = 0.$$

So also $\widetilde{A}$ is asymmetric. □

3.3 Idempotency

An extended aggregation operator $\widetilde{A}$ is called idempotent if for each $f \in \mathcal{F}$ the following holds:

$$\widetilde{A}(f, \dots, f) = f. \tag{3}$$

Theorem 2. *Let A be an idempotent n-ary aggregation operator. Then an extended aggregation operator on convex fuzzy truth values $\widetilde{A} : \mathcal{F}_C^n \to \mathcal{F}_C$ given by (1) is idempotent.*

Proof. Let $f \in \mathcal{F}_C$. Then for each $y \in [0,1]$:

$$\widetilde{A}(f, \dots, f)(y) = \sup_{A(x_1,\dots,x_n)=y} (f(x_1) \wedge \dots \wedge f(x_n)) \geq f(y), \tag{4}$$

due to the idempotency of A. Now we need to prove the opposite inequality. Suppose that there exist $x_1, \dots, x_n$ with $A(x_1, \dots, x_n) = y$ such that $f(x_i) > f(y)$ for all $i \in \{1, \dots, n\}$. From the convexity of f it follows $x_i > y$ for all $i \in \{1, \dots, n\}$, or $x_i < y$ for all $i \in \{1, \dots, n\}$. If the former is true (the proof of the latter one is similar): let $x_0 = \min\{x_1, \dots, x_n\}$, then $A(x_1, \dots, x_n) \geq A(x_0, \dots, x_0) = x_0 > y$ and we have a contradiction with $A(x_1, \dots, x_n) = y$. □

3.4 Neutral element

A function $g : [0,1] \to [0,1]$ is called a neutral element of extended aggregation operator $\widetilde{A}$ if for each $k = 1, \dots, n$ and each $f \in \mathcal{F}$ it holds that:

$$\widetilde{A}(\underbrace{g, \dots, g}_{(k-1)-times}, f, \underbrace{g, \dots, g}_{(n-k)-times}) = f.$$

Theorem 3. *Let A be an n-ary aggregation operator with neutral element e. Then a function $g : [0,1] \to [0,1]$ is a neutral element of extended aggregation operator on fuzzy truth values $\widetilde{A} : \mathcal{F}^n \to \mathcal{F}$ given by (1) if and only if*

$$g(x) = \begin{cases} 1 & , \text{if } x = e, \\ 0 & , \text{otherwise}. \end{cases} \tag{5}$$

Proof. Let g be given by (5). Then for all $f \in \mathcal{F}$, $k \in \{1, \dots, n\}$ it holds:

$$\widetilde{A}(\underbrace{g, \dots, g}_{(k-1)-times}, f, \underbrace{g, \dots, g}_{(n-k)-times})(y) =$$

$$= \sup_{A(x_1,\dots,x_n)=y} (g(x_1) \wedge \dots \wedge g(x_{k-1}) \wedge f(x_k) \wedge g(x_{k+1}) \wedge \dots \wedge g(x_n)) = f(y).$$

The uniqueness follows from the uniqueness of neutral element of aggregation operators in general. □

3.5 Annihilator

A function $g : [0,1] \to [0,1]$ is called an annihilator of extended aggregation operator $\widetilde{A}$ if for each $f_1, \ldots, f_n \in \mathcal{F}$ when $f_k = g$ for some $k = 1, \ldots, n$, then

$$\widetilde{A}(f_1, \ldots, f_n) = g.$$

Theorem 4. *Let A be an n-ary aggregation operator with annihilator a. Then a function g is an annihilator of extended aggregation operator on normal fuzzy truth values $\widetilde{A} : \mathcal{F}_N^n \to \mathcal{F}_N$ given by (1) if and only if*

$$g(x) = \begin{cases} 1 & , \textit{if } x = a, \\ 0 & , \textit{otherwise}. \end{cases} \tag{6}$$

Proof. Let g be given by (6). Then for all $f_1, \ldots, f_n$, where $f_k = g$ for some $k \in \{1, \ldots, n\}$, it holds:

$$\widetilde{A}(f_1, \ldots, f_n)(a) =$$

$$= \sup_{A(x_1,\ldots,x_n)=a} (f_1(x_1) \wedge \ldots \wedge f_{k-1}(x_{k-1}) \wedge g(x_k) \wedge f_{k+1}(x_{k+1}) \wedge \ldots \wedge f_n(x_n)) = 1,$$

because for $x_k = a$ and $x_i \in Ker(f_i)$, for $i \neq k$, it holds $f_1(x_1) = \ldots = f_n(x_n) = 1$. Moreover, for $y \neq a$ we have:

$$\widetilde{A}(f_1, \ldots, f_n)(y) = 0,$$

because $A(x_1, \ldots, x_n) \neq a$ implies $x_i \neq a$ for all $i = 1, \ldots, n$ and consequently $g(x_k \neq a) = 0$.

The uniqueness follows from the uniqueness of annihilator of aggregation operators in general. □

4 Conclusions

Recently an extension of aggregation operators via extension principle was proposed in the literature. This approach leads to the following constructing method: we consider a type-1 aggregation operator with well known properties and extend it via convolution (extension principle) to operator that aggregate functions. In this paper we showed that the extended operator have similar properties as original aggregation operator. More precisely, we proved that the extension preserves symmetry, idempotency, neutral element and annihilator.

Acknowledgement

This work was supported in part by a Grant VEGA 1/0419/13.

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