Probabilistic summation of fuzzy numbers

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Abstract: *We introduced an alternative look on fuzzy numbers based on two random variables and to their summation. This approach covers triangular norm - based approach, however, it is much more general. We illustrate it on some examples.*

Keywords: *copula, fuzzy arithmetics, fuzzy number, triangle function, triangular norm.*

1 Introduction

Fuzzy numbers and the related arithmetics were deeply studied since 1975 [3]. For overview of most important definitions and results we recommend [8, 1].

Definition 1. Consider a fuzzy subset A of $\mathbb{R} =]-\infty, \infty[$. Let μ_A be its membership function. *A fuzzy number* is the fuzzy subset A whenever

- it is normal, i.e., $\mu_A(x_0) = 1$ for some $x_0 \in \mathbb{R}$,
- function μ_A is upper semi-continuous and convex, i.e.,

$$\bigcup_{\alpha>\beta} \{x \in \mathbb{R} | \, \mu_A \geq \alpha\} = \{x \in \mathbb{R} | \, \mu_A \geq \beta\}$$

  for each $\alpha, \beta \in [0, 1]$, and

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min(\mu_A(x), \mu_A(y))$$

  for each $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,

- it has bounded support, i.e., there are $x_1, x_2 \in \mathbb{R}, x_1 < x_2$ such that

$$\mu_A(x) = 0 \text{ whenever } x < x_1 \text{ or } x > x_2.$$

In any type of fuzzy arithmetics, multiplication by a real constant $c \in \mathbb{R}$ gives either 0 if $c = 0$, or, if $c \neq 0$ then $\mu_{cA}(x) = \mu_A(\frac{x}{c})$. Similarly, concerning $c + A$, we have $\mu_{c+A}(x) = \mu_A(x - c)$.

However, for the processing of proper fuzzy numbers, several approaches have been proposed, so far. They are based on a given (left-continuous) triangular norm (t-norm) $T : [0, 1]^2 \to [0, 1]$ (i.e., an associative, commutative and monotone binary operation on $[0, 1]$ with neutral element $e = 1$; for more details see [5]), modifying the Zadeh extension principle from [14]. We will focus on the summation only, and for a given t-norm T, it is defined as follows:

Definition 2. Let A, B be fuzzy numbers and T a triangular norm. Then the T-sum $C = A \boxplus_T B$ has the membership function

$$\mu_C(z) = \sup \{T(\mu_A(x), \mu_B(z - x)) | \, x \in \mathbb{R}\} . \tag{1}$$

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Note that there are also alternative definition of fuzzy numbers, using the distribution functions, or survival functions, see [4, 7, 10], but still exploiting formula (1).

Inspired by the above mentioned alternative definitions, we offer in this contribution an alternative look on fuzzy numbers and their summation.

The contribution is organized as follows. In the next section, we use a probabilistic look on fuzzy numbers as a pair of random variables $A = (X, Y)$, and propose a probabilistic approach to summation of fuzzy numbers. In Section 3, several examples are introduced. Finally, concluding remarks are added, especially towards the probabilistic approach to multiplication of fuzzy numbers.

2 Probabilistic approach to the summation of fuzzy numbers

Consider a fuzzy number A and a real value x_0 such that $\mu_A(x_0) = 1$. It is evident that the functions $F_A, S_A : \mathbb{R} \to [0, 1]$ given by

$$F_A(x) = \begin{cases} \mu_A(x) & \text{if } x \leq x_0, \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad S_A(x) = \begin{cases} 1 & \text{if } x \leq x_0, \\ \mu_A(x) & \text{otherwise,} \end{cases}$$

are a distribution function and a survival function, respectively. Moreover, considering random variables X and Y defined on some probabilistic space $(\Omega, \mathcal{A}, P)$, and related to F_A and S_A, respectively, it is evident that $X \leq Y$ in strict sense, i.e., $X(\omega_1) \leq Y(\omega_2)$ for any $\omega_1, \omega_2 \in \Omega$. Also the range of X is contained in $[x_1, x_0]$, and the range of Y is contained in $[x_0, x_2]$, compare Definition 1. Moreover, observe that $\mu_A = \min(F_A, S_A)$. We summarize the above facts in the next proposition.

Proposition 1. *A fuzzy subset A of $\mathbb{R}$ is a fuzzy number (with membership function μ_A) if and only if there is a pair of random variables (X, Y) with non-overlapping ranges, $X \leq Y$, related to a distribution function F_X and survival function S_Y, respectively, so that $\mu_A = \min(F_X, S_Y)$.*

Consider fuzzy numbers $A \sim (X_A, Y_A)$ and $B \sim (X_B, Y_B)$. Our aim is to introduce a sum C of A and B. If we consider only functions F_A, S_A, F_B, S_B, one can apply any triangle function τ [11, 12]. Observe that triangle functions are defined on distance distribution functions, i.e., those with support in $[0, \infty]$, and thus one should consider F_{A,x_1} given by $F_{A,x_1} = F_A(x + x_1)$, F_{S_A,x_0} given by $F_{S_A,x_0}(x) = 1 - S_A(x + x_0)$, etc.

If one considers triangle function τ_T defined by means of a t-norm T,

$$\tau_T(F_{X_1}, F_{X_2})(z) = \sup \{T(F_{X_1}(x), F_{X_2}(y)) | \, x + y = z\} ,$$

we recover the formula (1).

However, considering random variables X_A, Y_A, X_B, Y_B, it is natural to look for the sum $X_C = X_A + X_B$ (and then to the related distribution function F_C), and to the sum $Y_C = Y_A + Y_B$ (and then to the related survival function S_C). It is well-known that the distribution function F_{X_C} of a random variable $X_C = X_A + X_B$ depends on the joint distribution function $F_{X_A,X_B} : \mathbb{R}^2 \to [0, 1]$,

$$F_{X_C}(z) = P(\{(x, y) \in \mathbb{R}^2 | \, x + y \leq z\}), \tag{2}$$

where the probability P is introduced by the joint distribution function F_{X_A,X_B}. Since Sklar [13] we know that for each couple of random variables X_A and X_B there is a copula $K : [0, 1]^2 \to [0, 1]$ so that

$$F_{X_A,X_B}(x, y) = K(F_{X_A}(x), F_{X_B}(y)).$$

For more details on copulas we recommend lecture notes [9]. Now, we are ready to introduce probabilistic approach to summation of fuzzy quantities.

Definition 3. Let $K : [0, 1]^2 \to [0, 1]$ be a fixed copula and $(X_A, Y_A), (X_B, Y_B)$ be couples of random variables linked to fuzzy numbers A and B, respectively. Then the K-sum of A and B, $C = A \oplus_K B$, is linked to a couple (X_C, Y_C) of random variables, $X_C = X_A + X_B, Y_C = Y_A + Y_B$, with distribution functions F_{X_C} and F_{Y_C}, respectively, given by (2) and considering respectively join distribution functions $F_{X_A,X_B} = K(F_{X_A}, F_{X_B})$ and $F_{Y_A,Y_B} = K(F_{Y_A}, F_{Y_B})$.

Observe that formula (1) cannot decrease the uncertainty characterized be the spreads of incoming fuzzy numbers (it is always between the maximal incoming spread, and the sum of incoming spreads), see [1], what is not the case of our approach. Note that this phenomenon was till now obtained only when the constraint fuzzy arithmetics was considered [6, 2].

3 Examples

For the sake of simplicity, we will consider only random variables uniformly distributed over a given interval. We denote by $X_{[a,b]}$ ($Y_{[a,b]}$) a random variable uniformly distributed over $[a, b]$.

Example 1. A pair $(X_{[a,b]}, Y_{[c,d]})$ is linked to a fuzzy number A if and only if $b \leq c$, and then

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } x \in [a,b], \\ 1 & \text{if } x \in [b,c], \\ \frac{d-x}{d-c} & \text{if } x \in [c,d], \\ 0 & \text{otherwise,} \end{cases}$$

i.e., A is a trapezoidal fuzzy number, $A = TPFN(a, b, c, d)$. If $b = c$, then A is triangular fuzzy number, $A = TFN(a, b, d)$, see [1].

Example 2. Consider three basic copulas $W, \Pi, M : [0, 1]^2 \to [0, 1]$

$$\begin{aligned} W(x,y) &= \max(0, x + y - 1), \\ \Pi(x,y) &= xy, \\ M(x,y) &= \min(x,y). \end{aligned}$$

Then:

i) The copula M models the total positive dependence. Thus, if we consider $X_{[a_1,b_1]}$ and $X_{[a_2,b_2]}$, necessarily $X_{[a_2,b_2]} = a_2 + (b_2 - a_2) \cdot \frac{X_{[a_1,b_1]}-a_1}{b_1-a_1}$, and thus

$$X_{[a_1,b_1]} + X_{[a_2,b_2]} = X_{[a_1+a_2,b_1+b_2]}.$$

Then, if $A = TPFN(a_1, b_1, c_1, d_1)$ and $B = TPFN(a_2, b_2, c_2, d_2)$, it holds

$$C = A \oplus_M B = TPFN(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2),$$

recovering the traditional min-based sum of trapezoidal fuzzy numbers given by formula (1) (in general, $\oplus_M = \boxplus_M$).

ii) The copula W models the total negative dependence, and then

$$X_{[a_2,b_2]} = b_2 - (b_2 - a_2) \cdot \frac{X_{[a_1,b_1]} - a_1}{b_1 - a_1}.$$

Consequently,

$$X_{[a_1,b_1]} + X_{[a_2,b_2]} = X_{[\min(a_1+b_2,a_2+b_1),\max(a_1+b_2,a_2+b_1)]},$$

which in the case $a_1 + b_2 = a_2 + b_1$ (i.e., the intervals $[a_1, b_1]$ and $[a_2, b_2]$ have the same length), yields $X_{[a_1,b_1]} + X_{[a_2,b_2]} = a_1 + b_2$ (i.e., the Dirac distribution in point $a_1 + b_2$ is obtained). Thus

$$\begin{aligned} A \oplus_W B = TPFN(&\min(a_1 + b_2, a_2 + b_1), \max(a_1 + b_2, a_2 + b_1), \\ &\min(c_1 + d_2, c_2 + d_1), \max(c_1 + d_2, c_2 + d_1)). \end{aligned}$$

Consider $A = B = TFN(0, 1, 2)$. Then $A \oplus_W B = 1$ is a crisp real number.

iii) The copula Π models the independence, and thus the random vector $(X_{[a_1,b_1]}, X_{[a_2,b_2]})$ is uniformly distributed over the rectangle $[a_1, b_1] \times [a_2, b_2]$ (and its density is constant $\frac{1}{(b_1-a_1)(b_2-a_2)}$).
For the simplicity consider $A = B = TFN(0, 1, 2)$, i.e., $b_1 - a_1 = b_2 - a_2 = 1$. Then, for $X_C = X_A + X_B$ its distribution function $F_{X_C} = F_{X_A} * F_{X_B}$ is the convolution of distribution function F_{X_A} and F_{X_B}, i.e.,

$$F_{X_C}(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{x^2}{2} & \text{if } x \in [0,1], \\ \frac{4x-2-x^2}{2} & \text{if } x \in [1,2], \\ 1 & \text{otherwise.} \end{cases}$$

Using a similar reasoning for random variables Y_A, Y_B, Y_C, we see that $F_{Y_C}(x) = F_{X_C}(x-2)$, and thus

$$S_{Y_C}(x) = \begin{cases} 1 & \text{if } x \leq 2, \\ \frac{4x-2-x^2}{2} & \text{if } x \in [2,3], \\ \frac{(4-x)^2}{2} & \text{if } x \in [3,4], \\ 0 & \text{otherwise,} \end{cases}$$

and thus the fuzzy number $C = A \oplus_\Pi B$ has a membership function μ_C given by

$$\mu_C(x) = \begin{cases} \frac{x^2}{2} & \text{if } x \in [0,1], \\ \frac{4x-2-x^2}{2} & \text{if } x \in [1,3], \\ \frac{(4-x)^2}{2} & \text{if } x \in [3,4], \\ 0 & \text{otherwise.} \end{cases}$$

Observe that, considering $A = B = TFN(0, 1, 2)$, it holds:

- $A \oplus_M B = TFN(0, 2, 4) = A \boxplus_M B$;
- $A \oplus_W B = 1$ but $A \boxplus_W B = TFN(1, 2, 3)$, i.e., $A \oplus_W B$ is a fuzzy subset of $A \boxplus_W B$;
- for $D = A \boxplus_\Pi B$,

$$\mu_D(x) = \begin{cases} (\frac{x}{2})^2 & \text{if } 0 \leq x \leq 2, \\ (\frac{4-x}{2})^2 & \text{if } 2 \leq x \leq 4, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $A \boxplus_\Pi B$ is a fuzzy subset of $A \oplus_\Pi B$.

4 Concluding remarks

We have introduced a new look on fuzzy numbers by means of random variables. This fact has opened the door to a new type of probabilistic fuzzy arithmetics. In this paper, only the probabilistic summation of fuzzy numbers was considered and exemplified. The other parts of probabilistic fuzzy arithmetics will be the topic of our further investigations.

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References

[1] D. Dubois, E. Kerre, R. Mesiar, H. Prade: Fuzzy interval analysis. In: D. Dubois, H. Prade (Eds.), Fundamentals of Fuzzy Sets, The Handbooks of Fuzzy Sets Series, Kluwer, Boston, Mass (2000), pp. 483-581.

[2] R.E. Giachetti, R.E. Young: A parametric representation of fuzzy numbers and their arithmetic operators. Fuzzy Set and Systems 91 (2) , 1997, pp. 185-202.

[3] A. Kaufmann, M.M. Gupta: Introduction to Fuzzy Arithmetics. Theory and Applications. International Thomson Computer Press, London 1991.

[4] E.P. Klement: Strong law of large numbers for random variables with values in the fuzzy real line. Communications IFSA Mathematics Chapter 1, 1987, pp.7-11.

[5] E.-P. Klement, R. Mesiar, E. Pap: Triangular Norms. Kluwer Academic Publishers, Dortrecht, 2000.

[6] G.J. Klir: Fuzzy arithmetic with requisite constraints. Fuzzy Set and Systems 91 (2), 1997, pp. 165-175.

[7] R. Lowen: On $(\mathbb{R}(L), \oplus)$. Fuzzy Set and Systems 10, 1983, pp. 203-209.

[8] M. Mareš: Computation over fuzzy quantities. CRC Boca Raton FL, 1994.

[9] R.B. Nelsen: Copulas and quasi-copulas. An introduction to their properties and applications (Chapter B.) Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms, 2005, pp. 391-413.

[10] S.E. Rodabaugh: Fuzzy addition in L-fuzzy real line. Fuzzy Set and Systems 8, 1982, pp. 39-51.

[11] S. Saminger-Platz , C. Sempi: A primer on triangle functions I. Aequationes Mathematicae 76 (3), 2008, pp. 201-240.

[12] S.Saminger-Platz, C. Sempi: A primer on triangle functions II. Aequationes Mathematicae 80 (3), 2010, pp. 239-268.

[13] A. Sklar: Fonctions de répartition à n dimensions et leurs marges. Publ. Inst. Statist. Univ. Paris 8, 1959, pp. 229-231.

[14] L.A. Zadeh : The concept of a linguistic variable and its application to approximate reasoning-I,II and III. Information Sciences 8 (3), pp. 199-249; 8 (4) , pp. 301-357; 9 (1) , pp. 43-80, 1975.