Some remarks
on level dependent capacities based Sugeno integral

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Abstract. *The standard Sugeno integral has several equivalent ways to be introduced. This equivalence fails when generalizing the standard capacities into level dependent capacities. We discuss several possible types of Sugeno integral based on level dependent capacities. Some illustrative examples are added.*

Keywords: *capacity; level dependent capacity; Sugeno integral*

1 Introduction

For a measurable space $(X, \mathcal{A})$, a monotone set function m: $\mathcal{A} \to [0,1]$ is called a capacity whenever m(∅) = 0 and m(X) = 1. Observe that capacities are sometimes called also fuzzy measures [6]. Sugeno has introduced his integral in 1974 [6], considering fuzzy events, i.e. $\mathcal{A}$-measurable functions $f: X \to [0,1]$, as a functional $S_m(f)$: $\mathcal{F} \to [0,1]$, where $\mathcal{F}$ is the class of all fuzzy events on $(X, \mathcal{A})$. S_m was given by

$$S_m(f) = sup\ \{min\ (a, m(A)) \mid a \cdot 1_A\ \leq f\}\ . \tag{1}$$

Equivalently, S_m can be expressed as

$$S_m(f) = sup\ \{min\ (a, m(f\ \geq a)) \mid a\ \in\ [0,1]\}, \tag{2}$$

or

$$S_m(f) = sup\ \{min\ (m(A),\ min(f(x) \mid x\ \in A) \mid A\ \in \mathcal{A})\}. \tag{3}$$

In [2], another equivalent definition of Sugeno integral was introduced, namely

$$S_m(f) = inf\ \{max\ (a, m(f\ \geq a)) \mid a\ \in\ [0,1]\}. \tag{4}$$

Recently, the concept of capacities was extended to level dependent capacities [1], see also [3, 4].
A mapping $M: \mathcal{A}\ \times\ [0,1] \to [0,1]$ such that for each $t \in [0,1]$, $M(\cdot, t) = m_t$ is a capacity, is called a level dependent capacity.
The aim of this paper is to discuss Sugeno integral with respect to level dependent capacities, discussing its different forms based on extension of formulae (1) – (4). The paper is organized as follows. In the next section, we introduce extremal forms of level dependent capacities based Sugeno integral following the approach from [3], and versions of this integral deduced from formulae (1) – (4). In section 3, some examples are given. Finally, some concluding remarks are added.

2 Sugeno integral and level dependent capacities

Sugeno integral, as introduced in [6], is a special instance of universal integrals proposed by Klement et al. in [4]. In the framework of universal integrals, all information contained in a capacity m and a fuzzy event f is summarized into one special function $h_{m,f}: [0,1] \to [0,1]$ given by $h_{m,f}(t) = m(f\ \geq t)$. This function can be seen as generalized survival function (i.e., a complement to distribution function). In the case of universal integrals extended for level dependent capacities, Klement at al. have proposed in [3] to consider the function $h_{M,f}: [0,1] \to [0,1]$ given by $h_{M,f}(t) = M(\{f\ \geq t\}, t)\ = m_t\ (f\ \geq t)$. Observe that while $h_{m,f}$ is a decreasing function (and thus Borel measurable), these properties need not

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be satisfied for $h_{M,f}$. Then, again following [3], two decreasing boundaries $(h_{M,f})_*, (h_{M,f})^*$: $[0,1] \to [0,1]$ of $h_{M,f}$ can be considered,

$$(h_{M,f})_* = sup\ \{h : [0,1] \to [0,1] \mid h\ is\ decreasing, h \le h_{M,f}\} \quad \text{and}$$

$$(h_{M,f})^* = inf\ \{h : [0,1] \to [0,1] \mid h\ is\ decreasing, h \ge h_{M,f}\}.$$

It is not difficult to check that for each $t \in [0,1]$ it holds

$$(h_{M,f})_*(t) = inf\ \{h_{M,f}(u) \mid u \in [0,t]\} \qquad \text{and}$$

$$(h_{M,f})^*(t) = sup\ \{h_{M,f}(v) \mid v \in [t,1]\}.$$

Obviously $(h_{M,f})_* = h_{M,f} = (h_{M,f})^*$ if and only if $h_{M,f}$ is decreasing. Following [3], the smallest Sugeno integral based on level dependent capacities
$(Su_M)_*: \mathcal{F} \to [0,1]$ is given by

$$(Su_M)_*(f) = sup\ \left\{min\left(t,\ (h_{M,f})_*(t)\right) \mid\ t \in [0,1]\right\}. \tag{5}$$

Similarly, the greatest Sugeno integral based on level dependent capacities
$(Su_M)^*: \mathcal{F} \to [0,1]$ is given by

$$(Su_M)^*(f) = sup\ \left\{min\left(t,\ (h_{M,f})^*(t)\right) \mid\ t \in [0,1]\right\}. \tag{6}$$

Evidently, it holds

$$(Su_M)^*(f) = sup\ \left\{min\left(t, h_{M,f}(v)\right) \mid 0 \le t \le v \le 1\right\} \tag{7}$$

and

$$(Su_M)_*(f) = inf\ \left\{max\left(t, h_{M,f}(u)\right) \mid 0 \le u \le t \le 1\right\}. \tag{8}$$

Rewriting formulae (1) – (4) for level dependent capacities we get the next possible forms of level dependent capacities based Sugeno integral:

$$Su_M^{(1)}(f) = sup\ \left\{min\left(a, m_a(A)\right) \mid a \cdot 1_A \le f\right\}, \tag{9}$$

$$Su_M^{(2)}(f) = sup\ \left\{min\left(a, m_a(f \ge a)\right) \mid a \in [0,1]\right\}, \tag{10}$$

$$Su_M^{(3)}(f) = sup\ \left\{min\left(t, m_t(A)\right) \mid A \in \mathcal{A}, t = min(f(x) \mid x \in A)\right\}, \tag{11}$$

$$Su_M^{(4)}(f) = inf\ \left\{max\left(a, m_a(f \ge a)\right) \mid a \in [0,1]\right\}. \tag{12}$$

It is not difficult to check that $Su_M^{(2)} = Su_M^{(3)}$ due to the monotonicity of m_a for each fixed $a \in [0,1]$. Moreover, $Su_M^{(1)} \ge Su_M^{(2)}$ because of the fact that $a \cdot 1_{\{f \ge a\}} \le f$ for each $a \in [0,1]$. We have the following Hasse diagram of all introduced versions of Sugeno integral based on level dependent capacities, see Figure 1.

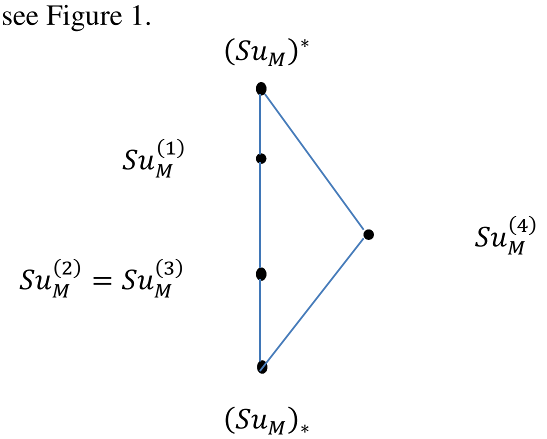

Fig. 1 Hasse diagram of different Sugeno integrals based on level dependent capacities

3 Examples

Obviously, if $M = (m_t)_{t\in[0,1]}$ is a constant level dependent capacity, $m_t = m$ for all $t \in [0,1]$, then all introduced integrals coincide.

Example 1.
For an arbitrary but fixed measurable space $(X, \mathcal{A})$,
i) consider $M = (m_t)_{t\in[0,1]}$, $m_t = m^*$ if $t \in \left[0, \frac{1}{2}\right]$ and $m_t = m_*$ if $t \in \left]\frac{1}{2}, 1\right]$,
where m^*, m_* : $\mathcal{A} \to [0, 1]$ are given by

$$m^*(A) = \begin{cases} 0 \; if \; A = \emptyset, \\ 1 \quad\; else, \end{cases}$$

and

$$m_*(A) = \begin{cases} 1 \; if \; A = X, \\ 0 \quad\; else. \end{cases}$$

Then for all integrals introduced in Section 2 we have the same result, namely

$$med\left(sup\, f, \frac{1}{2}, inf\, f\right).$$

Observe that the coincidence of all integrals from Section 2 appears, whenever the system $M = (m_t)_{t\in[0,1]}$ is decreasing in t, as then the function $h_{M,f}$ is decreasing, independently of f.

ii) Consider now $M = (m_t)_{t\in[0,1]}$ given by $m_t = m_*$ if $t \in \left[0, \frac{1}{2}\right]$, and $m_t = m^*$ if $t \in \left]\frac{1}{2}, 1\right]$.
Then:

- if $sup\, f \le \frac{1}{2}$, all integrals from Section 2 have value $inf\, f$,
- if $inf\, f \ge \frac{1}{2}$, all integrals from Section 2 have value $sup\, f$,
- in the remaining case, $(Su_M)_*(f) = inf\, f = Su_M^{(4)}(f)$,
  $(Su_M)^*(f) = sup\, f = Su_M^{(i)}(f), i = 1,2,3.$

Example 2.
Fix $X = [0,1]$ and $\mathcal{A} = \mathcal{B}([0,1])$ (Borel subsets of $[0,1]$). Consider now $M = (m_t)_{t\in[0,1]}$ given by $m_t = m^*$ if $t \in \left[0, \frac{1}{4}\right]$, $m_t = m_*$ if $t \in \left]\frac{1}{4}, \frac{1}{2}\right[$, $m_t(A) = \sqrt{\lambda(A)}$ for $A \in \mathcal{A}$, where λ is the standard Lebesgue measure on $\mathcal{B}([0,1])$ if $t \in \left[\frac{1}{2}, 1\right]$, and $f_1(x) = x$. Then we have

$$h_{M,f_1}(t) = \begin{cases} 1 \; if \; t \in \left[0, \dfrac{1}{4}\right], \\ 0 \; if \; t \in \left]\dfrac{1}{4}, \dfrac{1}{2}\right[, \\ \sqrt{1-t} \; if \; t \in \left[\dfrac{1}{2}, 1\right], \end{cases}$$

$$\left(h_{M,f_1}\right)^*(t) = \begin{cases} 1 \; if \; t \in \left[0, \dfrac{1}{4}\right], \\ \sqrt{0.5} \; if \; t \in \left]\dfrac{1}{4}, \dfrac{1}{2}\right[, \\ \sqrt{1-t} \; if \; t \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

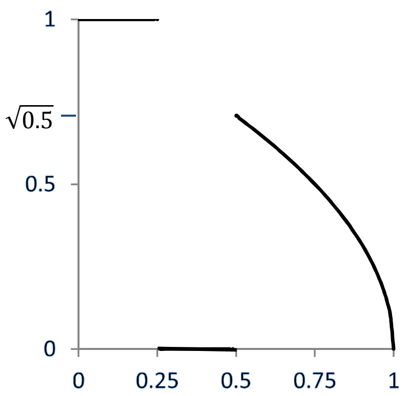

Fig. 2 Function h_{M,f_1}

$$\left(h_{M,f_1}\right)_*(t) = \begin{cases} 1 \; if \; t \in \left[0, \frac{1}{4}\right], \\ 0 \; if \; t \in]\frac{1}{4}, 1]. \end{cases}$$

Then introduced integrals are :

$$(\mathrm{Su_M})_*(f_1) = \frac{1}{4} = Su_M^{(4)}(f_1)\,,\; (\mathrm{Su_M})^*(f_1) = \frac{\sqrt{5}}{2} - \frac{1}{2} = Su_M^{(i)}(f_1), i = 1,2,3.$$

Example 3.
Fix $= [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$, $f(x) = x$. For measurable space $(X, \mathcal{A})$ consider

i) $M = (m_t)_{t \in [0,1]}$ given by

$$m_{\frac{1}{3}} = m_*, \qquad m_{\frac{2}{3}} = m^*, \qquad else\; m_t(A) = \lambda(A)$$

for $A \in \mathcal{A}$, where λ is the standard Lebesgue measure on $\mathcal{B}([0,1])$.
Then introduced integrals are :

$$(\mathrm{Su_M})_*(f) = \frac{1}{3} = Su_M^{(4)}(f), (\mathrm{Su_M})^*(f) = \frac{2}{3} = Su_M^{(i)}(f), i = 1,2,3.$$

ii) Consider now $M = (m_t)_{t \in [0,1]}$ given by

$$m_0 = m_*,\; m_{\frac{1}{3}} = m^*,\; m_{\frac{2}{3}} = m_*\,, m_1 = m^*\,, else\; m_t(A) = t \; if \; A \notin \{\emptyset, X\}.$$

Then introduced integrals are :

$$(\mathrm{Su_M})_*(f) = 0 = Su_M^{(4)}(f), (\mathrm{Su_M})^*(f) = 1 = Su_M^{(i)}(f), i = 1,2,3.$$

4 Concluding remarks

We have introduce several versions of Sugeno integral for level dependent capacities, following the ideas from [3] . Our examples suggest $(Su_M)_* = Su_M^{(4)}$ and $(Su_M)^* = Su_M^{(i)}, i = 1,2,3$. This problem is an open problem for our further investigation. In our next study, we will also focus on copula-based Sugeno integral for level dependent capacities, and study the properties of all introduced integrals. A special focus will be put on the (comonotone) maxitivity of introduced functionals, where we aim to relate our results to those presented in [5] .

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