Construction of copulas by means of measure-preserving transformations

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Abstract: *The aim of the paper is the presentation of a construction of n-copulas which is based on an arbitrary n-copula and some special measure-preserving transformations. We also show an equivalent alternative approach for obtaining such copulas. In the last section, properties of resulting 2-dimensional copulas are investigated.*

Keywords: *Copula, Measure-preserving transformation.*

1 Introduction

Copulas are functions describing the dependence structure of random vectors. By the Sklar theorem [13], copulas join multivariate distribution functions of random vectors to their one-dimensional marginal distribution functions. More precisely, for each random vector $(X_1, X_2, \ldots, X_n)$, $n \in \mathbb{N}$, $n \geq 2$, there exists a copula C such that the joint distribution function H of a random vector $(X_1, X_2, \ldots, X_n)$ and marginal distribution functions $F_1, F_2, \ldots, F_n$ of the random variables $X_1, X_2, \ldots, X_n$, respectively, are related by

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)), \text{for all } x_1, \ldots, x_n \in \overline{\mathbb{R}}.$$

Copulas can be seen as the restrictions to the unit n-box of joint distribution functions with marginals uniformly distributed over the unit interval. From practical point of view, the importance of copulas follows from the fact that they enable to separate the modeling of a complex n-dimensional random process into two parts, namely, into looking for appropriate one-dimensional (marginal) distribution functions and for an appropriate copula. There are plenty of applications of copulas, for instance, in quantitative finance, engineering, medicine, weather and climate research, etc. Mathematically, copulas can be introduced as follows.

Definition 1. *Let $\mathbf{I} = [0, 1]$. A function $C : \mathbf{I}^n \to \mathbf{I}$ is an n-copula if it satisfies the following conditions:*

(C1) $C(x_1, \ldots, x_n) = 0$ *if* $x_i = 0$ *for some* $i \in \{1, \ldots, n\}$, *i.e.,* 0 *is the annihilator of* C,

(C2) $C(x_1, \ldots, x_n) = x_i$ *if* $x_j = 1$ *for all* $j \neq i$, $i, j \in \{1, \ldots, n\}$, *i.e.,* 1 *is a neutral element of* C,

(C3) C *is n-increasing, i.e., the C-volume V_C of each n-box* $\prod_{i=1}^{n} [x_i, y_i] \subseteq \mathbf{I}^n$ *is non-negative:*

$$V_C\Big(\prod_{i=1}^{n}[x_i, y_i]\Big) := \sum_{\mathbf{v} \in \prod_{i=1}^{n}\{x_i, y_i\}} (-1)^{N(\mathbf{v})} C(\mathbf{v}) \geq 0,$$

where $N(\mathbf{v}) = \mathrm{card}(\{j : v_j = x_j\})$.

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Note that for each n-copula C and all $(x_1,\dots,x_n)\in\mathbf{I^n}$ it holds

$$W(x_1,\dots,x_n)\le C(x_1,\dots,x_n)\le M(x_1,\dots,x_n),$$

where $W(x_1,\dots,x_n)=\max\{x_1+\cdots+x_n-n+1,0\}$ and $M(x_1,\dots,x_n)=\min\{x_1,\dots,x_n\}$. The upper bound M is a copula for each number n of arguments, while the lower bound W is a copula only for $n=2$. An important n-copula (for each $n\ge 2$) is the product copula Π, modeling the independence of random variables, given by $\Pi(x_1,\dots,x_n)=x_1\cdots x_n$. For more details on copulas we refer to [11].

In the last period copulas have been studied very intensively. A large number of recent papers have been devoted to constructions of copulas, e.g., [1, 2, 3, 10, 12, 8, 5, 9], among others. Basic methods of constructing copulas are also studied in the monograph [11]. The aim of this paper is to present a construction method for n-copulas by means of measure-preserving transformations. We also show an alternative way for obtaining these copulas. In the last section we prove several results for binary copulas.

2 Copulas and measure-preserving transformations

Let us briefly describe a correspondence between copulas and measure-preserving transformations on the unit interval. More details can be found, e.g., in [6], also see the references therein.

Let us denote by $\mathcal{B}(\mathbf{I})$ the system of all Borel subsets of the unit interval $\mathbf{I}$. We say that a mapping $f:\mathbf{I}\to\mathbf{I}$ is a measure-preserving transformation on the unit interval, if for every $B\in\mathcal{B}(\mathbf{I})$, the pre-image $f^{-1}(B)\in\mathcal{B}(\mathbf{I})$ and $\lambda(f^{-1}(B))=\lambda(B)$, where λ is the standard Lebesgue measure on $\mathcal{B}(\mathbf{I})$.

Let us assign to each number $a\in\mathbf{I}$ a function $f_a\colon\mathbf{I}\to\mathbf{I}$ in the following way: for $a=0$ let f_a be the identity function, $f_0(t)=t$, for $a=1$ let $f_1(t)=1-t$, and for any $a\in]0,1[$ let f_a be a piecewise linear function, defined by

$$f_a(t)=\max\left\{1-\frac{t}{a},\frac{t-a}{1-a}\right\},\ \text{ i.e., } f_a(t)=\begin{cases}1-\frac{t}{a} & \text{if } t\in[0,a],\\ \frac{t-a}{1-a} & \text{if } t\in]a,1].\end{cases}\qquad(1)$$

It is easy to see that $f_a^{-1}([0,x])=[a(1-x),x+a(1-x)]$ and $\lambda(f_a^{-1}([0,x]))=\lambda([0,x])=x$, see Fig.1. Clearly, functions f_a, $a\in\mathbf{I}$, are measure-preserving transformations on the unit interval. Note that to simplify the notation, in what follows, instead of the notation $f_a^{-1}([0,x])$ we will write $f_a^{-1}[0,x]$ only.

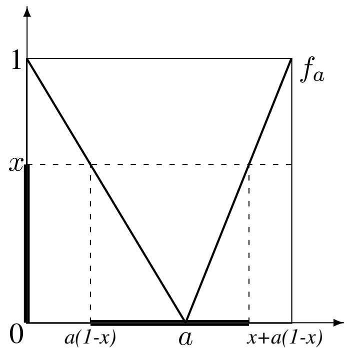

Figure 1: The graph of $f_a, a\in]0,1[$. It holds $f_a^{-1}[0,x]=[a(1-x),x+a(1-x)]$.

The following theorem describes the above mentioned correspondence between copulas and measure-preserving transformations on the unit interval, see, e.g., [6].

Theorem 1. *If $\varphi_1,\varphi_2,\dots,\varphi_n$ are measure-preserving transformations on the unit interval, then the function $C_{\varphi_1,\varphi_2,\dots,\varphi_n}:\mathbf{I^n}\to\mathbf{I}$ defined by*

$$C_{\varphi_1,\varphi_2,\dots,\varphi_n}(x_1,x_2,\dots,x_n):=\lambda(\varphi_1^{-1}[0,x_1]\cap\varphi_2^{-1}[0,x_2]\cap\cdots\cap\varphi_n^{-1}[0,x_n])\qquad(2)$$

is an n-copula. Conversely, for every n-copula C, there exist n measure-preserving transformations $\varphi_1, \varphi_2, \dots, \varphi_n$ such that

$$C = C_{\varphi_1, \varphi_2, \dots, \varphi_n}. \tag{3}$$

Note that the representation of an n-copula C in the form (3) is not unique. If C is determined by measure-preserving transformations $\varphi_1, \varphi_2, \dots, \varphi_n$, and $\varphi\colon \mathbf{I} \to \mathbf{I}$ is any measure-preserving transformation, then it also holds $C = C_{\varphi\circ\varphi_1, \varphi\circ\varphi_2, \dots, \varphi\circ\varphi_n}$.

Relation (2) can be understood as a method for constructing new n-copulas.

Example 1. *The mappings $\varphi_1,\ \varphi_2 : [0,1] \to [0,1]$*

$$\varphi_1(t) = \begin{cases} 2t & \text{if } t \in [0, 1/2[, \\ 2t-1 & \text{if } t \in [1/2, 1] \end{cases}, \qquad \varphi_2(t) = t,$$

are measure-preserving transformations. The copula C_{φ_1,φ_2} is given in Fig.2.

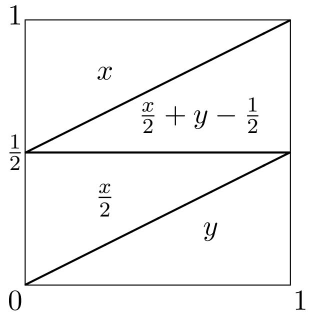

Figure 2: Copula C_{φ_1,φ_2} from Example 1

Given an n-copula C generated by measure-preserving transformations $\varphi_1, \varphi_2, \dots, \varphi_n$, and measure-preserving transformations f_{a_i}, $a_i \in \mathbf{I}$, $i = 1, \dots, n$, introduced above, we can construct a new n-copula in the following way.

Definition 2. *Let $C\colon \mathbf{I^n} \to \mathbf{I}$ be an n-copula generated by measure preserving transformations $\varphi_1, \varphi_2, \dots, \varphi_n\colon \mathbf{I} \to \mathbf{I}$, i.e., $C = C_{\varphi_1,\varphi_2,\dots,\varphi_n}$. For each $i = 1, \dots, n$, let a_i be a number in $\mathbf{I}$ and $f_{a_i}\colon \mathbf{I} \to \mathbf{I}$ the corresponding measure-preserving transformation defined by (1). We define the function $Ca_1, \dots, a_n\colon \mathbf{I^n} \to \mathbf{I}$ as follows*

$$Ca_1, \dots, a_n(x_1, \dots, x_n) = C_{f_{a_1}\circ\varphi_1, \dots, f_{a_n}\circ\varphi_n}(x_1, \dots, x_n). \tag{4}$$

Clearly, the function $Ca_1, \dots, a_n$ defined by (4) is an n-copula. We can write

$$C_{a_1,\dots,a_n}(x_1, \dots, x_n) = \lambda\left(\bigcap_{i=1}^{n} \varphi_i^{-1} \circ f_{a_i}^{-1}[0, x_i]\right), \tag{5}$$

too.

Copulas $Ca_1, \dots, a_n$ can also be obtained consecutively. As f_0 is an identity mapping on $\mathbf{I}$, it is easy to show that for each $a_1 \in \mathbf{I}$, it holds

$$C_{a_1,0,\dots,0} = C_{f_{a_1}\circ\varphi_1, \varphi_2, \dots, \varphi_n}.$$

Similarly, for all $a_1, a_2 \in \mathbf{I}$,

$$(C_{a_1,0,\dots,0})_{0,a_2,0,\dots,0} = C_{f_{a_1}\circ\varphi_1, f_{a_2}\circ\varphi_2, \varphi_3, \dots, \varphi_n},$$

etc., and finally, for all $a_1, \ldots, a_n \in \mathbf{I}$, it holds

$$(\cdots((C_{a_1,0,\ldots,0})_{0,a_2,0,\ldots,0})\cdots)_{0,\ldots,0,a_n} = C_{f_{a_1}\circ\varphi_1, f_{a_2}\circ\varphi_2,\ldots,f_{a_n}\circ\varphi_n} \stackrel{def}{=} C_{a_1,\ldots,a_n}.$$

For most copulas it is not easy to determine measure-preserving transformations generating them. An alternative formula for copulas $C_{a_1,\ldots,a_n}$ is given in Theorem 2 whose proof is based on the previous property and the following lemma which is–for simplicity–formulated for the first step of the previous approach only.

Lemma 1. *For each $a_1 \in \mathbf{I}$ and all $(x_1, \ldots, x_n) \in \mathbf{I}^n$ it holds*

$$C_{a_1,0,\ldots,0}(x_1, \ldots, x_n) = V_C(f_{a_1}^{-1}[0, x_1] \times [0, x_2] \times \ldots \times [0, x_n]). \tag{6}$$

Proof. On the one hand, as $f_{a_1}^{-1}[0, x_1] = [a_1(1-x_1), x_1 + a_1(1-x_1)]$, due to the properties of the transformation φ_1 and the Lebesgue measure, it holds

$$\begin{aligned}
&C_{a_1,0,\ldots,0}(x_1, \ldots, x_n) \\
&= \lambda\left(\varphi_1^{-1} \circ f_{a_1}^{-1}[0, x_1] \cap \varphi_2^{-1}[0, x_2] \cap \ldots \cap \varphi_n^{-1}[0, x_n]\right) \\
&= \lambda\left(\varphi_1^{-1}([0, x_1 + a_1(1-x_1)] \setminus [0, a_1(1-x_1)]) \cap \varphi_2^{-1}[0, x_2] \cap \ldots \cap \varphi_n^{-1}[0, x_n]\right) \\
&= \lambda\left(\varphi_1^{-1}[0, x_1 + a_1(1-x_1)] \cap \varphi_2^{-1}[0, x_2] \cap \ldots \cap \varphi_n^{-1}[0, x_n]\right) \\
&\quad -\lambda\left(\varphi_1^{-1}[0, a_1(1-x_1)] \cap \varphi_2^{-1}[0, x_2] \cap \ldots \cap \varphi_n^{-1}[0, x_n]\right) \\
&= C(x_1 + a_1(1-x_1), x_2, \ldots, x_n) - C(a_1(1-x_1), x_2, \ldots, x_n).
\end{aligned}$$

On the other hand, by definition of V_C and the fact that zero is the annihilator of C, we get

$$\begin{aligned}
&V_C(f_{a_1}^{-1}[0, x_1] \times [0, x_2] \times \ldots \times [0, x_n]) \\
&= V_C([a_1(1-x_1), x_1 + a_1(1-x_1)] \times [0, x_2] \times \ldots \times [0, x_n]) \\
&= C(x_1 + a_1(1-x_1), x_2, \ldots, x_n) - C(a_1(1-x_1), x_2, \ldots, x_n),
\end{aligned}$$

and the claim follows.

$\square$

Theorem 2. *Let $C_{a_1,\ldots,a_n}$ be an n-copula introduced by (4). Then it holds*

$$C_{a_1,\ldots,a_n}(x_1, \ldots, x_n) = V_C\left(\prod_{i=1}^{n} f_{a_i}^{-1}[0, x_i]\right). \tag{7}$$

Equation (7) can also be written as

$$C_{a_1,\ldots,a_n}(x_1, \ldots, x_n) = V_C\left(\prod_{i=1}^{n} [a_i(1-x_i), x_i + a_i(1-x_i)]\right). \tag{8}$$

Note that for $n = 2$, copulas defined by formula (8) have already been mentioned in [11]. Special cases which are often of interest are, e.g., copulas

$$\begin{aligned}
C_{0,1}(x, y) &= x - C(x, 1-y), \\
C_{1,0}(x, y) &= y - C(1-x, y), \\
C_{1,1}(x, y) &= x + y - 1 + C(1-x, 1-y).
\end{aligned}$$

Note that $C_{0,0} = C$. The copula $C_{1,1} = \hat{C}$ is the so-called survival copula of a copula C and the copulas $C_{1,0}$ and $C_{0,1}$ are flipped copulas. Particularly, for the basic copulas M, $M(x, y) = \min\{x, y\}$, and W, $W(x, y) = \max\{x + y - 1, 0\}$, it holds:

$$M_{0,1} = M_{1,0} = W \text{ and } M_{1,1} = M,$$

$$W_{0,1} = W_{1,0} = M \text{ and } W_{1,1} = W.$$

3 Several results for binary copulas

A deeper study of binary copulas (copulas, for short) obtained from a copula $C = C_{\varphi_1,\varphi_2}$ by (5) or equivalently by (7), i.e., copulas given by

$$C_{a_1,a_2}(x,y) = \lambda\left(\varphi_1^{-1} \circ f_{a_1}^{-1}[0,x] \cap \varphi_2^{-1} \circ f_{a_2}^{-1}[0,y]\right),$$

or

$$C_{a_1,a_2}(x,y) = V_C\left([a_1(1-x), x + a_1(1-x)] \times [a_2(1-y), y + a_2(1-y)]\right),$$

can be found in [4]. Note that for binary copulas the 2-increasing property means that

$$V_C([x_1,x_2] \times [y_1,y_2]) = C(x_2,y_2) - C(x_1,y_2) + C(x_1,y_1) - C(x_2,y_1) \geq 0$$

for each rectangle $[x_1,x_2] \times [y_1,y_2] \subseteq [0,1]^2$.

In [4] we have also proved that by repeating the construction in binary case we obtain a copula

$$\left(C_{a_1,a_2}\right)_{b_1,b_2}(x,y) = \lambda\left(\varphi_1^{-1} \circ f_{a_1,b_1}^{-1}[0,x] \cap \varphi_2^{-1} \circ f_{a_2,b_2}^{-1}[0,y]\right),$$

where $f_{a_i,b_i} = f_{b_i} \circ f_{a_i}$, $i = 1,2$, or equivalently,

$$\left(C_{a_1,a_2}\right)_{b_1,b_2}(x,y) = V_C\left(f_{a_1}^{-1} \circ f_{b_1}^{-1}[0,x] \times f_{a_2}^{-1} \circ f_{b_2}^{-1}[0,y]\right),$$

and moreover, a geometrical interpretation of this result has also been shown.

Let us show several properties of copulas M_{a_1,a_2}, W_{a_1,a_2} obtained from the basic copulas M and W. First of all, let us mention that as the simplest measure-preserving transformations generating the minimum copula M we can take identity functions on $\mathbf{I}$, i.e., $\varphi_1(t) = \varphi_2(t) = t$, as it can be seen from

$$\lambda\left(\varphi_1^{-1}[0,x] \cap \varphi_2^{-1}[0,y]\right) = \lambda\left([0,x] \cap [0,y]\right) = \min\{x,y\} = M(x,y).$$

Similarly, functions φ_1, φ_2, where $\varphi_1(t) = 1-t$ and $\varphi_2(t) = t$, are measure-preserving transformations generating the copula W because

$$\begin{aligned} \lambda\left(\varphi_1^{-1}[0,x] \cap \varphi_2^{-1}[0,y]\right) &= \lambda\left([1-x,1] \cap [0,y]\right) \\ &= \left\{\begin{array}{ll} 0 & \text{if } x+y \leq 1 \\ x+y-1 & \text{if } x+y \geq 1 \end{array}\right\} = W(x,y). \end{aligned}$$

Now, consider measure-preserving transformations $f_a\colon \mathbf{I} \to \mathbf{I}$, given for any $a \in]0,1[$ by (1) and $f_0(t) = t$, $f_1(t) = 1-t$. As for each $x \in \mathbf{I}$,

$$\begin{aligned} f_1^{-1} \circ f_a^{-1}[0,x] &= f_1^{-1}[a(1-x), x + a(1-x)] \\ &= [1 - x - a(1-x), 1 - a(1-x)] \\ &= [(1-a)(1-x), 1 - a(1-x)] \end{aligned}$$

and

$$f_{1-a}^{-1}[0,x] = [(1-a)(1-x), x + (1-a)(1-x)] = [(1-a)(1-x), 1 - a(1-x)],$$

we get $f_1^{-1} \circ f_a^{-1} = f_{1-a}^{-1}$. Similarly, $f_0^{-1} \circ f_a^{-1} = f_a^{-1}$.

Proposition 1. *Let* $a_1, a_2 \in \mathbf{I}$. *Then*

(i) $M_{a_1,a_2} = M_{1-a_1,1-a_2}$,

(ii) $W_{a_1,a_2} = M_{1-a_1,a_2}$, $W_{a_1,a_2} = M_{a_1,1-a_2}$.

Proof.
(i) As M is generated by identity transformations, $M_{1,1} = M$ and for each $a \in \mathbf{I}$, $f_1^{-1} \circ f_a^{-1} = f_{1-a}^{-1}$; for each $(x, y) \in \mathbf{I}^2$, we can write

$$\begin{aligned} M_{a_1,a_2}(x,y) &= (M_{1,1})_{a_1,a_2}(x,y) = \lambda\left(f_1^{-1} \circ f_{a_1}[0,x] \cap f_1^{-1} \circ f_{a_2}[0,y]\right) \\ &= \lambda\left(f_{1-a_1}^{-1}[0,x] \cap f_{1-a_2}^{-1}[0,y]\right) = M_{1-a_1,1-a_2}(x,y). \end{aligned}$$

(ii) Clearly, $M_{1,0} = W$. Thus

$$\begin{aligned} W_{a_1,a_2}(x,y) &= (M_{1,0})_{a_1,a_2}(x,y) = \lambda\left(f_1^{-1} \circ f_{a_1}[0,x] \cap f_0^{-1} \circ f_{a_2}[0,y]\right) \\ &= \lambda\left(f_{1-a_1}^{-1}[0,x] \cap f_{a_2}^{-1}[0,y]\right) = M_{1-a_1,a_2}(x,y). \end{aligned}$$

Moreover, by (i), $M_{1-a_1,a_2} = M_{a_1,1-a_2}$.

□

Note that the product copula Π is invariant with respect to our construction, $\Pi_{a_1,a_2} = \Pi$ for all $a_1, a_2 \in \mathbf{I}$, as can easily be shown by (8).

Example 2. *Consider the minimum copula M and any $a_1, a_2 \in \mathbf{I}$. If $a_1 = a_2$ then $M_{a_1,a_2} = M$. Suppose that $a_1 < a_2$. Then*

$$M_{a_1,a_2}(x,y) = \min\{x, y, \max\{0, (1-a_1)x + a_2 y + a_1 - a_2\}\},$$

see Fig.3(left). M_{a_1,a_2} is a singular copula with support uniformly distributed over the segments connecting the vertices $(0, (a_2 - a_1)/a_2)$ and $((a_2 - a_1)/(1 - a_1), 0)$, next $(0, (a_2 - a_1)/a_2)$ and $(1, 1)$, and finally, $((a_2 - a_1)/(1 - a_1), 0)$ and $(1, 1)$. For $a_1 > a_2$ we can use property (i) in Proposition 1 and the previous formula, see Fig.3(right). Note that by using (ii) in Proposition 1, the formulas for W_{a_1,a_2} can be obtained.

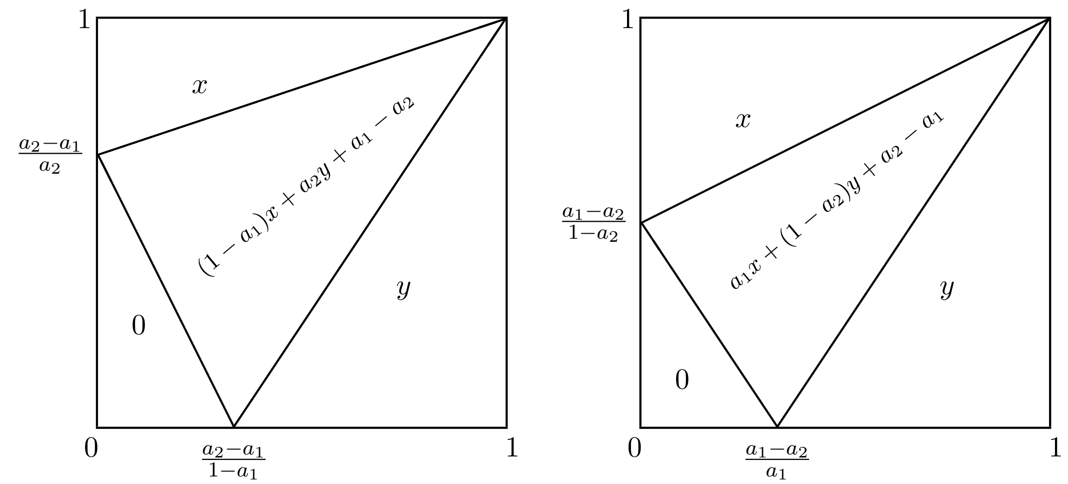

Figure 3: Copulas M_{a_1,a_2} for $a_1 < a_2$ (left) and for $a_1 > a_2$ (right)

Now, consider the family of copulas $\{M_{a_1,a_2}\}_{a_1,a_2\in[0,1]}$ and observe the tail dependence coefficients of its members. Recall that if (X, Y) is a random vector with continuous marginal distribution functions F_X, F_Y and a copula C, then the upper tail dependence coefficient is a number $\lambda_U \in [0, 1]$ given by

$$\lambda_U := \lim_{u\to 1^-} P\left(Y > F_Y^{-1}(u) | X > F_X^{-1}(u)\right) = \lim_{u\to 1^-} \frac{1 - 2u + C(u,u)}{1-u}$$

(if the limit exits). Similarly, the lower tail dependence coefficient is a number $\lambda_L \in [0, 1]$ given by

$$\lambda_L := \lim_{u\to 0^+} P\left(Y \le F_Y^{-1}(u) | X \le F_X^{-1}(u)\right) = \lim_{u\to 0^+} \frac{C(u,u)}{u}.$$

As tail dependence is a copula property, we will write $\lambda_U(C)$ and $\lambda_L(C)$.

While for the minimum copula M it holds $\lambda_U(M) = 1$, $\lambda_U(M_{a_1,a_2})$ attains the value in $[0,1]$, which depends on a_1, a_2 as follows.

Proposition 2. *Let* $a_1, a_2 \in \mathbf{I}$. *Then* $\lambda_U(M_{a_1,a_2}) = 1 - |a_1 - a_2|$.

Proof. Let $a_1 \leq a_2$. Then

$$\begin{aligned}\lambda_U(M_{a_1,a_2}) &= \lim_{u\to 1^-} \frac{1-2u+(1-a_1)u+a_2u+a_1-a_2}{1-u} \\ &= \lim_{u\to 1^-} \frac{(1+a_1-a_2)(1-u)}{1-u} = 1+a_1-a_2.\end{aligned}$$

Similarly, if $a_1 > a_2$, then $\lambda_U(M_{a_1,a_2}) = 1 - a_1 + a_2$.

□

On the other hand, note that $\lambda_L(M) = 1$, but for each $a_1 \neq a_2$, $\lambda_L(M_{a_1,a_2}) = 0$.

Proposition 3. *Let* $C\colon \mathbf{I^2} \to \mathbf{I}$ *be a copula,* $a_1, a_2 \in \mathbf{I}$. *Then*

$$(\hat{C})_{a_1,a_2} = C_{1-a_1,1-a_2}.$$

Proof. Let $C = C_{\varphi_1,\varphi_2}$. As $\hat{C} = C_{1,1}$, for all $(x,y) \in [0,1]^2$ it holds

$$\begin{aligned}(\hat{C})_{a_1,a_2}(x,y) &= (C_{1,1})_{a_1,a_2}(x,y) = \lambda\left(\varphi_1^{-1}\circ f_{1,a_1}^{-1}[0,x] \cap \varphi_2^{-1}\circ f_{1,a_2}^{-1}[0,y]\right) \\ &= \lambda\left(\varphi_1^{-1}\circ f_1^{-1}\circ f_{a_1}^{-1}[0,x] \cap \varphi_2^{-1}\circ f_1^{-1}\circ f_{a_2}^{-1}[0,y]\right) \\ &= \lambda\left(\varphi_1^{-1}\circ f_{1-a_1}^{-1}[0,x] \cap \varphi_2^{-1}\circ f_{1-a_2}^{-1}[0,y]\right) = C_{1-a_1,1-a_2}(x,y).\end{aligned}$$

□

Corollary 1. *Let* $C\colon \mathbf{I^2} \to \mathbf{I}$ *be a radially symmetric copula, i.e., a copula satisfying the property* $C = \hat{C}$. *Then* $C_{a_1,a_2} = C_{1-a_1,1-a_2}$.

Note that property (i) in Proposition 1 is covered by this claim because $M = \hat{M}$.

The following property concerns copulas constructed from absolutely continuous copulas. Recall that a copula C is *absolutely continuous*, if for all $(x,y) \in \mathbf{I^2}$,

$$C(x,y) = \int_0^x \int_0^y \frac{\partial^2 C(x,y)}{\partial x \partial y}\, \mathrm{d}x\, \mathrm{d}y,$$

where $\frac{\partial^2 C(x,y)}{\partial x \partial y}$ is a joint density of C considered as a joint distribution function (restricted to $\mathbf{I^2}$).

Proposition 4. *Let* $C\colon \mathbf{I^2} \to \mathbf{I}$ *be an absolutely continuous copula with joint density* φ *and let* $a_1, a_2 \in \mathbf{I}$. *Then the copula* C_{a_1,a_2} *is absolutely continuous with joint density* φ_{a_1,a_2}, *whose value at each point* (x,y) *is equal to a convex combination of the values of* φ *at vertices of the rectangle* $f_{a_1}^{-1}[0,x] \times f_{a_2}^{-1}[0,y]$.

Proof. Applying formula (7) for C_{a_1,a_2} and a formula for a partial derivative of a function composition we get

$$\begin{aligned}\varphi_{a_1,a_2}(x,y) &= \frac{\partial^2 C_{a_1,a_2}(x,y)}{\partial x \partial y} \\ =\ & \frac{\partial^2}{\partial x \partial y}\left(C(x+a_1(1-x), y+a_2(1-x)) - C(x+a_1(1-x), a_2(1-x))\right. \\ -\ & \left. C(a_1(1-x), y+a_2(1-x)) + C(a_1(1-x), a_2(1-x))\right) \\ =\ & \varphi(x+a_1(1-x), y+a_2(1-x))(1-a_1)(1-a_2) \\ +\ & \varphi(x+a_1(1-x), a_2(1-x))(1-a_1)a_2 \\ +\ & \varphi(a_1(1-x), y+a_2(1-x))a_1(1-a_2) \\ +\ & \varphi(a_1(1-x), a_2(1-x))a_1a_2\end{aligned}$$

Since $(1-a_1)(1-a_2)+(1-a_1)a_2+a_1(1-a_2)+a_1a_2=1$, the above combination is convex, and the claim follows.

□

4 Concluding remarks

We have shown that starting from any n-copula C and any numbers $a_1,\ldots,a_n \in [0,1]$, we can construct another n-copula $C_{a_1,\ldots,a_n}$ by using measure-preserving transformations corresponding to the considered copula C and to the numbers $a_1,\ldots,a_n$. However, because practically it is often not easy to determine measure-preserving transformations generating the copula C, it is important that the same n-copula can be constructed by means of C-volumes V_C of special n-boxes depending on numbers $a_1,\ldots,a_n$. The fact, which of these two equivalent approaches is used, depends on the problem to be solved. In our future work we intend to study, e.g., the relationship between the studied construction and some other constructions of copulas, e.g., ordinal sums, but also the properties of resulting n-copulas for $n>2$.

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