The role of meager elements in homogeneous effect algebras

Jan Paseka

Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, CZ-611 37 Brno, Czech Republic
paseka@math.muni.cz

Generalizations of Boolean algebras as carriers of probability measures are (lattice) effect algebras. They are a common generalization of MV-algebras and orthomodular lattices ([1], [2], [3], [7]). In the present paper, we continue the study of homogeneous effect algebras started in [5]. This class of effect algebras includes orthoalgebras, lattice ordered effect algebras and effect algebras satisfying the Riesz decomposition property.

In [5] it was proved that every homogeneous effect algebra is a union of its blocks, which are defined as maximal sub-effect algebras satisfying the Riesz decomposition property. In [8] Tkadlec introduced the so-called property (W+) as a common generalization of orthocomplete and lattice effect algebras.

The aim of our paper is to show that every block of an Archimedean homogeneous effect algebra satisfying the property (W+) is lattice ordered. Therefore, any Archimedean homogeneous effect algebra satisfying the property (W+) is covered by MV-algebras. As a corollary, this yields that every block of a homogeneous orthocomplete effect algebra is lattice ordered.

As a by-product of our study we extend the results on sharp and meager elements of [6] into the realm of Archimedean homogeneous effect algebras satisfying the property (W+).

List of selected results and definitions

Definition 1 A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinct elements, called the *zero* and the *unit* element, and $\oplus$ is a partially defined binary operation called the *orthosummation* on E which satisfy the following conditions for any $x, y, z \in E$:

(Ei) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(Eii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$),

(Eiv) if $1 \oplus x$ is defined then $x = 0$.

$(E; \oplus, 0, 1)$ is called an orthoalgebra *if $x \oplus x$ exists implies that $x = 0$.*

An effect algebra E satisfies the Riesz decomposition property *(or RDP) if, for all $u, v_1, v_2 \in E$ such that $u \le v_1 \oplus v_2$, there are u_1, u_2 such that $u_1 \le v_1, u_2 \le v_2$ and $u = u_1 \oplus u_2$.*

An effect algebra E is called homogeneous *if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2 \leq u'$, there are u_1, u_2 such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = u_1 \oplus u_2$ (see [5]).*

A subset B of E is called a block *of E if B is a maximal sub-effect algebra of E with the Riesz decomposition property.*

An element x of an effect algebra E is called sharp *if $x \wedge x' = 0$. The set $\mathrm{S}(E) = \{x \in E \mid x \wedge x' = 0\}$ is called a* set of all sharp elements *of E (see [4]).*

In what follows set (see [6])

$$\mathrm{M}(E) = \{x \in E \mid \textit{ if } v \in \mathrm{S}(E) \textit{ satisfies } v \leq x \textit{ then } v = 0\}.$$

We also define

$$\mathrm{HM}(E) = \{x \in E \mid \textit{ there is } y \in E \textit{ such that } x \leq y \textit{ and } x \leq y'\}.$$

An element $x \in \mathrm{HM}(E)$ is called hypermeager.

Lemma 2 *Let E be an effect algebra. Then $\mathrm{HM}(E) \subseteq \mathrm{M}(E)$. Moreover, for all $x \in E$, $x \in \mathrm{HM}(E)$ iff $x \oplus x$ exists and, for all $y \in \mathrm{M}(E)$, $y \neq 0$ there is $h \in \mathrm{HM}(E)$, $h \neq 0$ such that $h \leq y$.*

Definition 3 For an element x of an effect algebra E we write $\mathrm{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n-times) exists for every positive integer n and we write $\mathrm{ord}(x) = n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E. An effect algebra E is *Archimedean* if $\mathrm{ord}(x) < \infty$ for all $x \in E$.

We say that a finite system $F = (x_k)_{k=1}^n$ of not necessarily different elements of an effect algebra E is *orthogonal* if $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ (written $\bigoplus_{k=1}^{n} x_k$ or $\bigoplus F$) exists in E. An arbitrary system $G = (x_\kappa)_{\kappa \in H}$ of not necessarily different elements of E is called *orthogonal* if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a orthogonal system $G = (x_\kappa)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K \mid K \subseteq G \text{ is finite}\}$ exists in E and then we put $\bigoplus G = \bigvee\{\bigoplus K \mid K \subseteq G \text{ is finite}\}$. We say that $\bigoplus G$ is the *orthogonal sum* of G and G is *orthosummable.* (Here we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (x_\kappa)_{\kappa \in H_1}$). We denote $G^{\oplus} := \{\bigoplus K \mid K \subseteq G \text{ is finite}\}$.

E is called *orthocomplete* if every orthogonal system is orthosummable. *E fulfills the condition (W+)* [8] if for each orthogonal subset $A \subseteq E$ and each two upper bounds u, v of $A^{\oplus}$ there exists an upper bound w of $A^{\oplus}$ below u, v.

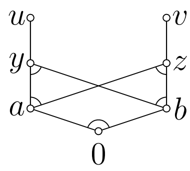
Every orthocomplete effect algebra is Archimedean.

Statement 4 [8, Theorem 2.2] *Lattice effect algebras and orthocomplete effect algebras fulfill the condition (W+).*

Proposition 5 *Let E be an Archimedean effect algebra fulfilling the condition (W+). Then every meager element of E is the orthosum of a system of hypermeager elements.*

Lemma 6 (Shifting lemma) *Let E be an Archimedean effect algebra fulfilling the condition (W+), let $u, v \in E$, and let a_1, b_1 be two maximal lower bounds of u, v. There exist elements y, z and two maximal lower bounds a, b of y, z for which $y \leq u$, $z \leq v$, $a \leq a_1$, $b \leq b_1$, $a \wedge b = 0$, a, b are maximal lower bounds of y, z and y, z are minimal upper bounds of a, b. Furthemore, $(y \ominus a) \wedge (z \ominus a) = 0$, $(y \ominus b) \wedge (z \ominus b) = 0$, $(y \ominus a) \wedge (y \ominus b) = 0$, $(z \ominus a) \wedge (z \ominus b) = 0$.*

The Shifting lemma provides the following *minimax structure*.



Theorem 7 *Let E be an Archimedean homogeneous effect algebra fulfilling the condition (W+). Then every block in E is a lattice and E can be covered by MV-algebras.*

Corollary 8 *Let E be an orthocomplete homogeneous effect algebra. Then E can be covered by MV-algebras.*

Acknowledgment
Financial Support of the Ministry of Education of the Czech Republic under the project MSM0021622409 is gratefully acknowledged.

References

[1] C.C. Chang, Algebraic analysis of many-valued logics, Trans. Amer. Math. Soc. 88 (1958) 467–490.

[2] A. Dvurečenskij, S. Pulmannová: New Trends in Quantum Structures, Kluwer Acad. Publ., Dordrecht/Ister Science, Bratislava 2000.

[3] D.J. Foulis, M.K. Bennett, Effect algebras and unsharp quantum logics, Found. Phys. 24 (1994), 1325–1346.

[4] S. P. Gudder, Sharply dominating effect algebras, Tatra Mt. Math. Publ. 15 (1998), 23–30.

[5] G. Jenča, Blocks of homogeneous effect algebras, Bulletin of the Australian Mathematical Society 64 (2001), 81–98.

[6] G. Jenča, Sharp and Meager Elements in Orthocomplete Homogeneous Effect Algebras, Order 27 (2010), 41-61.

[7] F. Kôpka, F. Chovanec, D-Posets, Math. Slovaca 44 (1994), 21–34.

[8] J. Tkadlec, Common generalizations of orthocomplete and lattice effect algebras, Inter. J. Theor. Phys., 49 (2010), 3279–3285.