Duality of quantum states from basic logic

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Symmetry and duality in basic logic

In basic logic one proves a symmetry result, which states that $\Gamma \vdash \Delta$ is provable if and only if its symmetric $\Delta^s \vdash \Gamma^s$ is provable. The symmetric A^s of a formula A is defined putting $p^s \equiv p$ for every literal p in A, and switching any connective into its *dual* copy. One has logical connectives in dual pairs, since one puts the definitory equations, where the connectives are defined from suitable metalinguistic links, in symmetric pairs.

The quantifiers $\forall$ and $\exists$ are given by considering assertions linked by the metalinguistic link *forall*. For the universal quantifier, we consider the assertions: *forall* $z \in D$, $\quad \Gamma \vdash A(z)$, that are written $\Gamma, z \in D \vdash A(z)$ importing $z \in D$ as a further premise in the sequent. Then one puts the definitory equation of $\forall$:

$$\Gamma \vdash (\forall x \in D)A(x) \quad \textit{if and only if} \quad \Gamma, z \in D \vdash A(z)$$

Symmetrically, one defines $\exists$ putting:

$$(\exists x \in D)A(x) \vdash \Delta \quad \textit{if and only if} \quad A(z) \vdash \Delta, z \notin D$$

In logic, the symmetry result is meaningful when literals are considered in dual pairs: $p, p^\perp$, where $.^\perp$ is a primitive negation (Girard's negation). The question is: is there an interpretation for symmetry itself? Logical connectives arise in dual pairs, are there symmetric connectives somewhere? We find an interpretation by means of quantum states.

Representation of quantum states and qubits

A discrete random variable Z yields a set

$$D_Z \equiv \{z = (s(z), p\{Z = s(z)\})\}$$

where $s(z)$ is the outcome and $p\{Z = s(z)\} > 0$ is its frequency. We term D_Z *random first order domain.*

We say that a random first order domain D_Z is *focused* w.r.t. an equality predicate $=$ if and only if it holds

$$z \in D_Z \vdash (z = t_1) \vee \cdots \vee (z = t_m)$$

where the terms $t_i = (s(t_i), p\{Z = s(t_i)\})$, $i = 1, \ldots m$, denote the outcomes of the random variable with their probabilities. Otherwise, D_Z is *unfocused.*

A discrete observable gives a finite random first order domain $D_Z = \{t_1, \ldots, t_m\}$, where $t_i = (s(t_i), p\{Z = s(t_i)\})$. One can write $A(t_i)$ for the proposition "The particle $\mathcal{A}$ is found in state $s(t_i)$ with probability $p\{Z = s(t_i)\}$". Let us summarize all the hypothesis concerning the preparation of the state and its measurement into a set of premises Γ. One has that Γ *yield* $A(t_i)$, for $i = 1 \ldots m$. Such m assertions are written $\Gamma \vdash A(t_i)$ as sequents. Then one has equivalently $\Gamma \vdash A(t_1) \& \ldots \& A(t_m)$, where & is the additive conjunction. The proposition $A(t_1) \& \ldots \& A(t_m)$ represents our knowledge of the state after measurement, namely the probability distribution of the outcomes.

In order to describe the quantum state prior to measurement, one drops the identification of the states, namely, the equality that renders D_Z focused. In such a case, describing our knowledge is possible only considering the predicate $z \in D_Z$ and the proposition $A(z)$: "The particle is in state $s(z)$ with probability $p\{Z = s(z)\}$" *forall* $z \in D_Z$. If the measurement hypothesis are denoted by Γ, we apply the definitory equation of the universal quantifier given above. So we describe the quantum state by the proposition

$$(\forall x \in D_Z) A(x)$$

By the rules $\forall r$, $\& f$ and by a *substitution of the variable by closed terms*, one derives the sequent $(\forall x \in D_Z) A(x) \vdash A(t_1) \& \ldots \& A(t_m)$. In our terms, it says that the probability distribution follows from the state, after a quantum measurement. A measurement is represented by a substitution of the free variable z by the closed terms t_i in our model. The converse sequent holds if and only if D_Z is focused, as one can prove. This enables us to characterize quantum states predicatively. Moreover, formally, we have an interpretation via the existential quantifier.

In order to represent qubits, we consider the measurement of the spin w.r.t. the z axis. The outcome of a measurement of a qubit q is "spin down" $\downarrow$ with probability α^2 and "spin up" $\uparrow$ with probability β^2, $\alpha^2 + \beta^2 = 1$. Then the random first order domain is the set $D_Z = \{(\downarrow, \alpha^2), (\uparrow, \beta^2)\}$ and the state of q is represented by the predicative formula $(\forall x \in \{(\downarrow, \alpha^2), (\uparrow, \beta^2)\}) A(x)$.

In the Hilbert space C^2 we consider the orthonormal basis $\{|\downarrow\rangle, |\uparrow\rangle\}$. We write the state evidentiating its *relative phase* ϕ:$|x\rangle = \alpha|\downarrow\rangle + e^{i\phi}\beta|\uparrow\rangle$. Different qubits yielding the same probability distribution can be characterized by ϕ. So an unfocused domain $D_Z = \{(\downarrow, \alpha^2), (\uparrow, \beta^2)\}$, corresponds to a family of vectors $\alpha|\downarrow\rangle + e^{i\phi}\beta|\uparrow\rangle$, $\phi \in [0, 2\pi)$. Two qubits in the same family can be distinguished by measurement if and only if they are orthogonal. This forces $\alpha^2 = \beta^2 = 1/2$ and $\phi' - \phi = \pi$. We consider the phases $\phi = 0$ and $\phi = \pi$, which give real factors, and characterize the couple of orthogonal vectors $|+\rangle = 1/\sqrt{2}|\downarrow\rangle + 1/\sqrt{2}|\uparrow\rangle$ and $|-\rangle = 1/\sqrt{2}|\downarrow\rangle - 1/\sqrt{2}|\uparrow\rangle$.

Duality and phase duality

So, a fixed measurement basis $|\downarrow\rangle$ and $|\uparrow\rangle$ determines two unfocused copies of the domain $D_U = \{(\downarrow, 1/2), (\uparrow, 1/2)\}$, relative to the uniform distribution U of the outcomes. We shall label them D^+ and D^-. They are equal as sets, from an extensional point of view. The labels $+$ and $-$ give an "intensional" characterization, to represent qubits in states $|+\rangle$ and $|-\rangle$.

On the other side, we have the singletons $D_\uparrow = \{(\uparrow, 1)\}$ and $D_\downarrow = \{(\downarrow, 1)\}$, relative to the measurement of qubits in states described by vectors $|\uparrow\rangle$ and $|\downarrow\rangle$.

A qubit in state $\downarrow$ is represented by the proposition $(\forall x \in D_\downarrow)A(x)$ and in state $\uparrow$ by the proposition $(\forall x \in D_\uparrow)A(x)$. A qubit in state $|+\rangle$ is represented by the proposition $(\forall x \in D^+)A(x)$, and in state $|-\rangle$ is represented by $(\forall x \in D^-)A(x)$. So, for different qubits, we have two different lists of pairs of propositions.

Symmetrically we have two lists of pairs obtained representing quantum states by $\exists$. It is clear that $(\forall x \in D_\downarrow)A(x) = (\exists x \in D_\downarrow)A(x)$ and $(\forall x \in D_\uparrow)A(x) = (\exists x \in D_\uparrow)A(x)$.

Then we make a unique list of pairs, shorthanded $A_\downarrow, A_\uparrow$. They are like pairs of literals and we apply Girard duality to them:

$$A_\downarrow^\perp \equiv A_\uparrow \qquad A_\uparrow^\perp \equiv A_\downarrow$$

Similarly, one has $(\forall x \in D^+)A(x) = (\exists x \in D^+)A(x)$, $(\forall x \in D^-)A(x) = (\exists x \in D^-)A(x)$ if and only if one considers the following axiom (phase axiom) for unfocused domains:

$$A(y), z \in D \vdash A(z), y \in D^\top$$

where y, z are first order variables. $\top$ exchanges $+$ and $-$: we term it *phase duality*. One can see that phase axioms are inconsistent on focused domains and characterize singletons *when substitution is allowed*, namely when measurement is considered. So our logic can be equipped with a second list of pairs of literals, A^+, A^-, switched by phase duality:

$$A^{+\top} \equiv A^- \qquad A^{-\top} \equiv A^+$$

which makes a sense only prior to measurement. It is read as the identity after measurement.

Actually, duality $\perp$ describes the action of the Pauli matrix σ_X (the *NOT* gate) while phase duality $\top$ describes the action of the Pauli matrix σ_Z (the phase gate). Then $\perp$ is extended to the literals A^+, A^- by the identity, analogously for $\top$.

A similarity with the behaviour of singletons can be exploited also to represent the four Bell's states by means of the unfocused domains D^+ and D^-. Phase duality is extended to the representation.

References

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