$F : \mathbf{Set} \longrightarrow \mathbf{Rel}$ zabudlivý funktor

$F(X) = X \qquad x\, F(f)\, y \quad \text{iff} \quad f(x) = y$

$G : \mathbf{Rel} \longrightarrow \mathbf{Set}$

$\Theta \subseteq A \times B$

$G(\Theta) : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$

$$G(\Theta)(U) = \{ b \in B : a\, \Theta\, b \text{ pre nejaké } a \in U \}$$

$\eta : 1_{\mathbf{Set}} \longrightarrow GF$

$\eta_X : X \longrightarrow GF(X)$ $\quad$ v Set

$\eta_X(x) = \{x\}$

$\varepsilon : FG \longrightarrow 1_{\mathbf{Rel}}$

$\varepsilon_A : FG(A) \longrightarrow A \qquad \varepsilon_A \subseteq \underset{\overset{\parallel}{\mathcal{P}(A)}}{FG(A)} \times A$

$U\, \varepsilon_A\, a \quad \text{iff} \quad a \in U$

Prirodzenosť $\eta, \varepsilon$ bola dokázaná dávnejšie.

Treba ešte overiť trojuholníkové rovnosti:

$$
\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, "\mathrm{id}_{F(X)}"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
& F(X)
\end{tikzcd}
$$

v Rel

$$
\begin{tikzcd}
x \arrow[r, mapsto, "F(\eta_X)"] \arrow[dr, mapsto, "\cap"'] & \{x\} \arrow[d, mapsto, "\psi"] \\
& x
\end{tikzcd}
$$

jediný taký, že $x F(\eta_x)$

$$
\begin{tikzcd}
G(A) \arrow[r, "\eta_{G(A)}"] \arrow[dr, "\mathrm{id}_{G(A)}"'] & GFG(A) \arrow[d, "G\varepsilon_A"] \\
& G(A)
\end{tikzcd}
$$

v Set

vezmime $U \in G(A) = P(A)$

$$
\begin{tikzcd}
U \arrow[r, mapsto] \arrow[dr, mapsto, "\mathrm{id}"'] & \{U\} \arrow[d, mapsto] \\
& U
\end{tikzcd}
$$

prečo?

$$P(A) = FG(A) \xrightarrow{\varepsilon_A} A$$

$U \varepsilon_A a$ iff $a \in U$

$$G(\varepsilon_A) : GFG(A) \longrightarrow G(A)$$

$G(\varepsilon_A)(\{U\}) = \{a \in A : U \varepsilon_A a\} =$

$= \{a \in A : a \in U\} = U$

# ADJUNKCIA MEDZI POSETMI

$$(P, \leq) \underset{G}{\overset{F}{\rightleftarrows}} (Q, \leq)$$

- funktory = izotónne zobrazenia

- hľadáme prirodzené transformácie

$$id_P \xrightarrow{\eta} GF$$

$$FG \xrightarrow{\varepsilon} id_Q$$

- ale ak sú $P, Q$ posety, toto proste znamená, že

$$\forall x \in P \quad x \leq GF(x)$$
$$\forall y \in Q \quad FG(y) \leq y$$

- trojuholníky kolabujú na

$$F(x) \leq FGF(x) \leq F(x);$$

čiže $F(x) = FGF(x)$

podobne $G(y) = GFG(y)$

$F : \mathbb{Z} \longrightarrow \mathbb{R}$ vnorenie $F(x) = x$

$G : \mathbb{R} \longrightarrow \mathbb{Z}$ $G(x) = \lfloor x \rfloor$ dolná celá časť

$x \leq GF(x) = \lfloor x \rfloor \quad x \in \mathbb{Z}$

$FG(y) = \lfloor y \rfloor \leq y \quad y \in \mathbb{R}$

$FGF(x) = F(x)$ ✓

$GFG(y) = G(y)$ ✓

$X$ – topologický priestor

$\sigma(X)$ – otvorené množiny

$C(X)$ – uzavreté množiny

$cl : \sigma(X) \longrightarrow C(X)$ uzáver $\quad cl(A) = \bigcap\limits_{\substack{B \in C(X) \\ B \supseteq A}} B$

$int : C(X) \longrightarrow \sigma(X)$ vnútro $\quad int(B) = \bigcup\limits_{\substack{A \in \sigma(X) \\ A \subseteq B}} A$

$A \in \sigma(X)$

$A \subseteq int(cl(A))$

$cl(int(B)) \subseteq B$

$cl(int(cl(A))) = cl(A)$ D. Ú.

$int(cl(int(B))) = int(B)$

Definícia

$F, G$ sú adjungovaná dvojica funktorov, ak $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$

$$\underbrace{\mathcal{D}(F(A), B)}_{\substack{\downarrow \\ \text{šípky v } \mathcal{D} \\ \text{z } F(A) \text{ do } B}} \cong \underbrace{\mathcal{C}(A, G(B))}_{\substack{\downarrow \\ \text{šípky v } \mathcal{C} \\ \text{z } A \text{ do } G(B)}}$$

prirodzene v $A, B$

Pozrime znovu $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Mon}$

$F$ – voľný monoid
$U$ – zábudlivý

$$\mathbf{Mon}(F(A), B) \cong \mathbf{Set}(A, U(B))$$

$A$ je množina
$B$ je monoid

„transpozícia"

zúžme na $A$

$$F(A) \xrightarrow{f} B \quad \rightsquigarrow \quad A \xrightarrow{\bar{f}} U(B)$$

$$A \xrightarrow{g} U(B) \quad \rightsquigarrow \quad F(A) \xrightarrow{\bar{g}} B$$

dá sa jednoznačne rozšíriť

transpozícia

$f = \bar{\bar{f}} \qquad g = \bar{\bar{g}} \longrightarrow$ teda $x \mapsto \bar{x}$ je bijekcia

Pozrime znovu $\quad Set \underset{P}{\overset{U}{\rightleftarrows}} Rel$

$$Rel(U(A), B) \cong Set(A, P(B))$$

$$U(A) \xrightarrow{\theta} B \rightsquigarrow A \xrightarrow{\bar{\theta}} P(B)$$

$$A \xrightarrow{f} P(B) \rightsquigarrow U(A) \xrightarrow{\bar{f}} B$$

$$\bar{\bar{\theta}} = \theta$$

$$\bar{\bar{f}} = f$$

Pre posetové adjunkcie dostaneme

$$F(a) \leq b \iff a \leq G(B)$$

GALOIS CONNECTION

(máme vždy len 0 alebo 1 šípku)

„PRIRODZENÉ" čo to znamená

B fixné

$$\left.\begin{array}{l} \mathcal{D}(F(-), B) : \mathcal{C} \longrightarrow Set \\ \mathcal{C}(-, G(B)) : \mathcal{C} \longrightarrow Set \end{array}\right\} \text{funktory!}$$

A fixné

$$\left.\begin{array}{l} \mathcal{D}(F(A), -) : \mathcal{D}^{op} \longrightarrow Set \\ \mathcal{C}(A, G(-)) : \mathcal{D}^{op} \longrightarrow Set \end{array}\right\} \text{funktory!}$$

Tvrdíme, že prvé dva aj druhé dva sú izomorfné vo svojej kategórii funktorov.

# INTERMEZZO: REPREZENTOVATEĽNÉ FUNKTORY

Fixnime $A$ v kategórii $\mathcal{C}$

$H^A : \mathcal{C} \longrightarrow \mathcal{S}et$

$H^A(X) = \mathcal{C}(A, X)$;

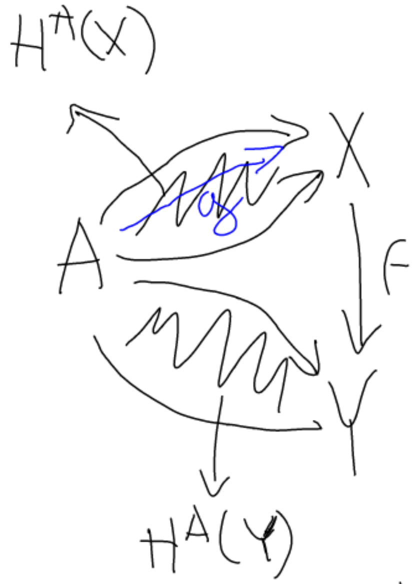
$f : X \longrightarrow Y$

$H^A(f) : H^A(X) \longrightarrow H^A(Y)$ ako?

$H_A : \mathcal{C}^{op} \longrightarrow \mathcal{S}et$

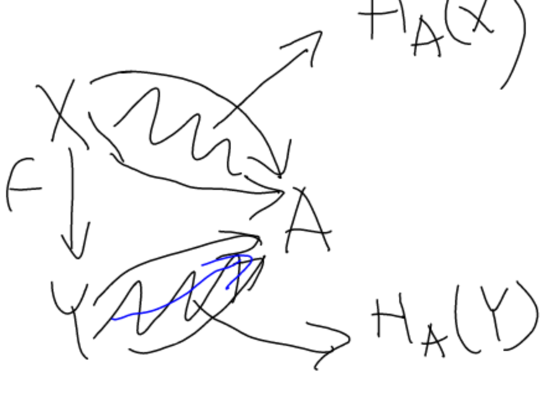
$H_A(X) = \mathcal{C}(X, A)$ $\quad f : X \longrightarrow Y$

$H_A(f) : H^A(Y) \longrightarrow H^A(X)$ ako?




$g \in H^A(X)$

$H^A(f)(g) = f \circ g$




$H_A(f)(g) = g \circ f$

$$\mathcal{D}(F(A), B) \cong \mathcal{C}(A, G(B))$$

$$\mathcal{D}(F(A), \_) \cong \mathcal{C}(A, G(\_)) \quad v \ [\mathcal{D}, Set]$$

$$\begin{tikzcd}
\mathcal{D}(F(A), B_1) \arrow[r, "g \mapsto \bar{g}", "\cong"'] \arrow[d, "f \circ _"'] & \mathcal{C}(A, G(B_1)) \arrow[d, "G(f) \circ _"] \\
\mathcal{D}(F(A), B_2) \arrow[r, "\cong", "g \mapsto \bar{g}"'] & \mathcal{C}(A, G(B_2))
\end{tikzcd}$$

– komutativny diagram v Set

$$f : B_1 \to B_2 \ v \ \mathcal{D}$$

$$\mathcal{D}(F(\_), B) \cong \mathcal{C}(\_, G(B)) \quad v \quad [\mathcal{C}^{op}, Set]$$

$$h : A_1 \to A_2 \ v \ \mathcal{C}$$

$$\begin{tikzcd}
\mathcal{D}(F(A_1), B) \arrow[r, phantom, "\cong"] & \mathcal{C}(A_1, G(B)) \\
\mathcal{D}(F(A_2), B) \arrow[u, "_ \circ h"] \arrow[r, phantom, "\cong"] & \mathcal{C}(A_2, G(B)) \arrow[u, "_ \circ G(h)"']
\end{tikzcd}$$