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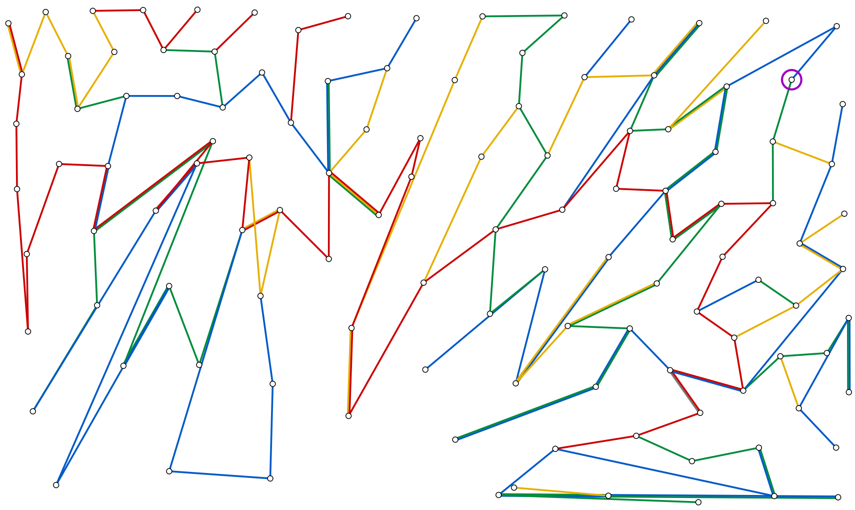

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Invited talks

Representations of distributive residuated lattice expansions

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These two talks will present joint and ongoing work with Claudette Robinson (University of Johannesburg), Peter Jipsen (Chapman University), Wilmari Morton (Wits University), and Jaš Šemrl (University of the West of Scotland).

Over the past few years we have been studying a class of residuated lattices expansions known as quasi relation algebras. These were first described by Galatos and Jipsen [6]. In particular, we are interested in *distributive* quasi relation algebras (i.e. where the lattice reduct is distributive). Our focus has been on obtaining representations of distributive quasi relation algebras (DqRAs) as algebras of binary relations.

In the first talk, we will give some background, describe our main representation construction [4], and give some related results that will be needed later [3]. Then we will show how pregroups (a class of expansions of partially ordered monoids, first described by Lambek [9]) can be used to find representable algebras [5].

In the second talk, we will describe dual frames developed for DqRAs [1, 2]. While these dual representations are themselves interesting, our main focus will be on how they are used for *representation games* for the algebras. Inspired by previous game theoretic approaches to finding representations [7, 8], we present two representation games, to be played on finite DqRAs and their dual frames. These methods yield previously inaccessible representation results.

During the talks we will also highlight some interesting open problems.

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Residuated partially ordered algebras, logic and preclones

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In the first lecture we introduce partially ordered algebras with operations that are monotone or antitone in each argument. The standard examples are algebras of binary relations, residuated po-semigroups and residuated lattices, as well as many po-subvarieties and their connection with substructural logic and algebraic proof theory. The second installment concentrates on the structure theory of residuated po-algebras, considering poset decompositions and po-Plonka sums. In the last lecture we focus on the general theory of S-preclones for a finite monoid S, which generalizes clone theory to the po-algebra setting and beyond.

In more detail, we aim to cover the following topics:

Lecture 1: Tonicities, inequational logic, and partially ordered varieties (based on standard sources and joint work with Marta Bilkova and Melissa Sugimoto). We begin with the universal algebraic theory of partially ordered algebras (τ-po-algebras) specified by a set $\mathcal{F}$ of function symbols and a tonicity $\tau : \mathcal{F} \to \{+, -\}^*$. This framework allows primitive operations to be monotone $(+)$ or antitone $(-)$ at specified argument positions, naturally capturing structures such as left-residuated po-monoids, residuated po-semigroups, residuated lattices, po-groups, tense posets, and algebras of binary relations. In this setting, inequational logic is treated as a 2-dimensional deductive system within Abstract Algebraic Logic (AAL). We consider τ-preorders (the order analogues of congruences) and cover order-theoretic counterparts of the fundamental isomorphism and correspondence theorems, Birkhoff's subdirect representation theorem, and the order H-S-P and S-$\mathsf{L}_{\to}$-P characterization theorems for τ-povarieties and τ-quasivarieties. We also discuss aspects of algebraizability of substructural logics and algebraic proof theory.

Lecture 2: Structure theory via po-Płonka sums of residuated po-semigroups (based on joint work with Stefano Bonzio, José Gil-Férez, Adam Přenosil, Melissa Sugimoto and Giuseppe Zecchini). The second lecture focuses on the structure theory of residuated partially ordered semigroups $\langle A, \leqslant, \cdot, \backslash, / \rangle$, where $x \cdot y \leqslant z \iff x \leqslant z/y \iff y \leqslant x\backslash z$. We investigate the role of positive central idempotents $p \in \mathrm{ZE}^+\mathbf{A}$ ($a \leqslant pa = ap$ and $p^2 = p$) and balanced residuated semigroups (satisfying $x\backslash x \approx x/x =: 1_x$ and $y \leqslant 1_x y$). To reconstruct such algebras from their fibers $\mathbf{A}_p = \{a \in A : 1_a = p\}$, we generalize the classical Płonka sum construction to *semilattice-directed systems of metamorphisms* $\Xi = \{\xi_{pq} : \mathbf{A}_p \looparrowright \mathbf{A}_q : p \preccurlyeq q \in I\}$. Unlike homomorphisms, metamorphisms handle each operation and relation coordinatewise via distinct families of maps $\langle \varphi_{pq}, \psi_{pq} \rangle$. We establish compatibility conditions under which the reconstructed partial order

$$a \leqslant b \iff \varphi_{pr}(a) \leqslant_r \psi_{qr}(b) \quad \text{(where } r = p \vee q\text{)}$$

yields a valid po-algebra. This establishes that every *steady* residuated semigroup decomposes into a po-Płonka sum of integrally closed residuated monoids (and into Brouwerian semilattices in the idempotent commutative case).

Lecture 3: S-preclones, S-relations, and the ${}^S\mathrm{Pol}$–${}^S\mathrm{Inv}$ Galois connection (based on joint work with Erkko Lehtonen and Reinhard Pöschel). In the final lecture, we present the general theory of S-preclones for an arbitrary finite monoid S of signa, generalizing classical clone theory to signed operations. An S-operation $f : A^n \to A$ carries an argument signum $\mathrm{sgn}(f) = (s_1, \ldots, s_n) \in S^n$. S-preclones are sets of S-operations containing the identity id_A and closed under cyclic shifts (ζ), argument transpositions (τ), addition of fictitious signed arguments (∇^s), identification of equal-signum arguments (Δ), and composition ($\circ$). Dually, an m-ary S-relation is a family $\boldsymbol{\varrho} = (\varrho_s)_{s \in S}$ with $\varrho_s \subseteq A^m$. We introduce the S-preservation relation

$$f \rhd^S \boldsymbol{\varrho} \iff \forall s \in S : f(\varrho_{s_1 s}, \ldots, \varrho_{s_n s}) \subseteq \varrho_s$$

and study the induced Galois connection ${}^S\mathrm{Pol}$–${}^S\mathrm{Inv}$. For finite A, we prove the Galois closure theorems: ${}^S\langle F \rangle = {}^S\mathrm{Pol}\,{}^S\mathrm{Inv}\,F$ and ${}^S[Q] = {}^S\mathrm{Inv}\,{}^S\mathrm{Pol}\,Q$. Furthermore, we analyze the lattice ${}^S\mathcal{L}_A$ of all S-preclones on A, showing that it is atomic and coatomic with finitely many atoms and coatoms, explore embeddings of the classical clone lattice $\mathcal{L}_A$, and highlight the motivating case of $\pm$-preclones over the group $S = \{+, -\}$ characterizing monotone and antitone operations on posets.

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Contributed talks

Symmetric lattice identities and medians

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An n-ary lattice term f is symmetric when its value does not change when we permute its variables i.e, $f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots x_{\sigma(n)})$ for all $\sigma \in S_n$. A symmetric lattice identity is an identity where both sides are symmetric terms. An example of a symmetric identity

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z),$$

which is famously equivalent to the distributive law (which itself is a nonsymmetric identity).

Conjecture 1 *Each lattice variety can be axiomatized by symmetric identities.*

In this talk we establish that a stronger version of this conjecture is not true: there is a ternary lattice identity which is not equivalent to any system of ternary symmetric identities.

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Old but NU: Finding Large Monoidal Intervals

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Let A be a finite set and M a transformation monoid on A. For $n \geq 1$, let $\mathcal{O}_A^{(n)}$ denote the set of all n-ary operations on A. The set

$$\{f \in \mathcal{O}_A^{(n)} : f(\mu_1(x), \ldots, \mu_n(x)) \in M \text{ for all } \mu_1, \ldots, \mu_n \in M\}$$

is called the stabilizer of M. The lattice of all clones on a A is naturally partitioned into finitely many intervals, each having as its least element the clone generated by a transformation monoid and as its greatest element the corresponding stabilizer. The problem of classifying transformation monoids according to the cardinalities of the corresponding monoidal intervals was posed by Á. Szendrei in [1].

Combining the results of Krokhin [2] with computer-assisted computations, we show that near-unanimity relations give rise to monoidal intervals of cardinality $|\mathcal{P}(\mathbb{N})|$.

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Monoidal Categories of Effect Algebras

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A categorical description of the tensor product is provided within the context of quantum structures, specifically focusing on effect algebras, partial commutative monoids, and related structures. To address this, the framework of multicategories is adopted, allowing for a more general and natural treatment of these concepts. The main result is a formal demonstration that the aforementioned multicategories are representable. Furthermore, the relationship between these structures is explored through multiadjunctions. These results provide a unified perspective on the construction of tensor products and establish a solid foundation for further research into higher categorical structures of quantum logic.

Exploration of non-classical negation axioms

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We explore the implicational interconnections between different types of lattice based non-classical logic algebras. That is, we consider structures of the form $\mathbf{L} = \langle L; \wedge, \vee, 0, 1, {}^{-} \rangle$ with the background assumption that $\langle L; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice. Our investigation focusses on the additional axioms the 'negation' (or 'complementation') operator $x \mapsto \overline{x}$ may fulfil, and on the implications between those properties. In this respect we consider several generalisations of orthocomplemented lattices (as used in quantum logic) and of pseudocomplemented lattices (as appearing, *e.g.*, in Heyting algebras when dealing with intuitionistic logic). The properties of negation we study are mainly taken from the ones surveyed in [3], without any claims to represent the available literature with any degree of completeness. A different set of negation axioms was examined in [2], but the final picture we get of the situation is richer than both of these works and more along the lines of [1], where again other properties were looked at.

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Some new semidirect-product-type constructions of residuated lattices

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Representation theorems for residuated posets and residuated lattices are well understood in the presence of the classical axioms of *divisibility* and *prelinearity*: ordinal sums, poset products and the Γ-functor all rest on them. As soon as these axioms are dropped, the known representation techniques largely fail. We introduce two new associative product-type constructions — inspired by wreath and semidirect products of semigroups — for arbitrary **integral residuated lattices**, where neither commutativity nor divisibility is assumed.

Products via nuclei. A closure operator γ on an integral residuated lattice $\mathbf{A}$ is a *nucleus* if $\gamma(x)\cdot\gamma(y) \leq \gamma(x\cdot y)$; it is a *strong nucleus* if, in addition, $\gamma(x)\backslash\gamma(y) = \gamma(x\backslash y)$ and $\gamma(x)/\gamma(y) = \gamma(x/y)$ for all $x, y \in A$. We write $\mathrm{SN}\,\mathbf{A}$ for the set of all strong nuclei on $\mathbf{A}$.

Definition (J. Kühr). Let $\mathbf{A}$ and $\mathbf{B}$ be integral residuated lattices. A mapping $\boldsymbol{\alpha}\colon B \to \mathrm{SN}\,\mathbf{A}$ is a *nuclei-morphism* from $\mathbf{B}$ to $\mathbf{A}$, written $\boldsymbol{\alpha}\colon \mathbf{B} \curvearrowright \mathbf{A}$, if

$$\boldsymbol{\alpha}[1] = I_A \qquad \text{and} \qquad \boldsymbol{\alpha}[x]\circ\boldsymbol{\alpha}[y] = \boldsymbol{\alpha}[x\wedge y] = \boldsymbol{\alpha}[x\cdot y] \qquad (x, y \in B).$$

Theorem. If $\boldsymbol{\alpha}\colon \mathbf{B} \curvearrowright \mathbf{A}$, then the algebra $\mathbf{A} \ltimes_{\boldsymbol{\alpha}} \mathbf{B}$ with universe $\sum_{x\in B} \boldsymbol{\alpha}[x]A$, unit $(1,1)$, componentwise lattice operations (taken in the nucleus images) and

$$\begin{aligned}(a_1,b_1)\cdot(a_2,b_2) &= \big(\boldsymbol{\alpha}[b_1\cdot b_2](a_1\cdot a_2),\, b_1\cdot b_2\big),\\ (a_1,b_1)\backslash(a_2,b_2) &= \big(a_1\backslash\boldsymbol{\alpha}[b_1](a_2),\, b_1\backslash b_2\big),\\ (a_2,b_2)/(a_1,b_1) &= \big(\boldsymbol{\alpha}[b_1](a_2)/a_1,\, b_2/b_1\big),\end{aligned}$$

is again an integral residuated lattice.

Theorem (Associativity). Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be integral residuated lattices.

(i) If $\boldsymbol{\alpha}\colon \mathbf{B} \curvearrowright \mathbf{A}$ and $\boldsymbol{\beta}\colon \mathbf{C} \curvearrowright \mathbf{A}\ltimes_{\boldsymbol{\alpha}} \mathbf{B}$, then there are nuclei-morphisms $\overline{\boldsymbol{\beta}}\colon \mathbf{C} \curvearrowright \mathbf{B}$ and $\overline{\boldsymbol{\alpha}}\colon \mathbf{B}\ltimes_{\overline{\boldsymbol{\beta}}} \mathbf{C} \curvearrowright \mathbf{A}$ such that

$$(\mathbf{A}\ltimes_{\boldsymbol{\alpha}}\mathbf{B})\ltimes_{\boldsymbol{\beta}}\mathbf{C} \;\cong\; \mathbf{A}\ltimes_{\overline{\boldsymbol{\alpha}}}(\mathbf{B}\ltimes_{\overline{\boldsymbol{\beta}}}\mathbf{C}).$$

(ii) Conversely, every such pair $(\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}})$ arises from a unique pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as in (i), and the two assignments are mutually inverse.

Products via product morphisms. A more general, purely order-theoretic construction avoids nuclei altogether.

> **Definition.** A mapping $\phi\colon B \to A$ between integral residuated lattices $\mathbf{B}$ and $\mathbf{A}$ is a *product morphism* if
>
> $$\phi(1) = 1, \qquad \phi(x) \wedge \phi(y) = \phi(x \wedge y), \qquad \phi(x) \cdot \phi(y) \leq \phi(x \cdot y)$$
>
> for all $x, y \in B$.
>
> **Theorem.** For a product morphism $\phi\colon \mathbf{B} \to \mathbf{A}$ the algebra
>
> $$\mathbf{A} \ltimes_\phi \mathbf{B} = \Big(\textstyle\sum_{x \in B} (\phi(x)],\, \wedge,\, \vee,\, \cdot,\, \backslash,\, /,\, (1,1) \Big),$$
>
> with componentwise $\wedge, \vee, \cdot$ and
>
> $$(a, x) \bullet (b, y) = \big((a \bullet b) \wedge \phi(x \bullet y),\, x \bullet y\big) \qquad \text{for } \bullet \in \{\backslash, /\},$$
>
> is again an (integral) residuated lattice.
>
> **Theorem (Associativity).** Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be integral residuated lattices. If $\phi\colon \mathbf{B} \to \mathbf{A}$ and $\psi\colon \mathbf{C} \to \mathbf{A} \ltimes_\phi \mathbf{B}$ are product morphisms, then there are product morphisms $\overline{\psi}\colon \mathbf{C} \to \mathbf{B}$ and $\overline{\phi}\colon \mathbf{B} \ltimes_{\overline{\psi}} \mathbf{C} \to \mathbf{A}$ such that
>
> $$(\mathbf{A} \ltimes_\phi \mathbf{B}) \ltimes_\psi \mathbf{C} \;\cong\; \mathbf{A} \ltimes_{\overline{\phi}} (\mathbf{B} \ltimes_{\overline{\psi}} \mathbf{C}),$$
>
> and, as before, every right-associated pair $(\overline{\phi}, \overline{\psi})$ arises from a unique left-associated pair (ϕ, ψ) in this way.

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Simultaneous Extensions of States on Quantum Logics

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Abstract

Suppose that L is a quantum logic (= an orthomodular poset) and A, B are Boolean subalgebras of L. Suppose further that s_1(resp. s_2) is a state on A (resp. on B). We ask when both s_1 and s_2 allow for a simultaneous extension over L. Under a natural necessity condition on A, B, s_1, s_2 we find a sufficient condition for the existence of the extension. This condition is the requirement of an abundance of two-valued states on L. We then show by the Greechie pasting technique that this condition is far from necessary even for state-space-rich logics.

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Maximal Set of Non-Isomorphic Lattices Represented by Δ-Incidence Matrices Preserving $\{0, 1\}$-Aggregation

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It is perfectly natural that nowadays we want to perform all computations using computers. However, especially in mathematics, we may encounter structures that are difficult to describe in a way that a computer can understand. One such structure is a lattice, which is an ordered set in which every pair of elements has a well-defined minimum and maximum.

The aim of this paper is therefore to represent lattices using matrices, which computers are very good at working with, and then perform aggregations with them. For example, given two binary matrices of the same dimensions, we define their minimum and maximum as new matrices whose entries are the element-wise minimum or maximum – corresponding to logical conjunction (AND) and disjunction (OR), respectively. We apply such aggregation operations – particularly MIN and MAX – to matrices representing lattices of increasing size, starting with two-element lattices and progressing to larger ones. We will demonstrate that this task is not as straightforward as it might initially seem, as a number of challenges arise that even modern computational tools struggle to overcome. On the other hand, this approach opens up a completely new possibility: until now, we have only been able to perform aggregation within a single, specific lattice. But now, we can aggregate across different lattices themselves. We prove that for both the minimum and the maximum, there exist maximal closed sets of lattices, from which it follows that the "lattice of all equally-sized finite lattices" does not exist.

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Mal'tsev conditions arising from functional relations

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We call a relation $\mathbf{R} \subseteq A^k$ *functional* if it is the graph of an operation i.e. if its first $k-1$ coordinates uniquely determine its last. If $\mathcal{V}$ is a variety, the *free functional relation* generated by $\mathbf{R}$ in $\mathcal{V}$ is obtained by factoring out the free relational structure generated by $\mathbf{R}$ with the smallest congruence θ such that $\mathbf{R}/\theta$ is functional.

The free functional relation can be trivial. For a given $\mathbf{R}$, the triviality of the free functional relation generated by $\mathbf{R}$ in $\mathcal{V}$ can be naturally described by a Mal'tsev condition. In this talk, we explore some of these Mal'tsev conditions, and the Galois connection between relations and varieties induced by the triviality of the free functional relation.

On algebras with easy direct limits

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The direct limit construction belongs to basic tools used in Universal algebra. Moreover, G. Grätzer [1], page 161 specified problems for the direct limit operator that have not yet been solved. The question of what we can get by a direct limit construction from one algebra is very basic in this context and little explored.

Let $\mathcal{A}$ be an algebra. We denote by $\underrightarrow{\mathbf{L}}\mathcal{A}$ the class of all isomorphic copies of direct limits which can be obtained from $\mathcal{A}$ and we denote by $\mathbf{R}\mathcal{A}$ the set of all retracts of $\mathcal{A}$. Then $\mathbf{R}\mathcal{A} \subseteq \underrightarrow{\mathbf{L}}\mathcal{A}$. In general, an algebra from $\underrightarrow{\mathbf{L}}\mathcal{A}$ need not be a retract of $\mathcal{A}$, [2].

We say that $\mathcal{A}$ is *with easy direct limits* if every algebra from $\underrightarrow{\mathbf{L}}\mathcal{A}$ is isomorphic to a retract of $\mathcal{A}$. Retracts of an algebra with easy direct limits form a direct limit closed class and it is the next motivation to deal with direct limit families in which only one algebra occurs.

The goal of the talk is to give a results overview on algebras with easy direct limits. General algebras as well as several specific algebraic structures as monounary algebras, vector spaces and groups will be mentioned.

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Dual spaces of ortholattices

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We describe digraphs with topology, which give dual representations of ortholattices. This is done via so-called dual Ploščica spaces of lattices. We define the dual space of a general ortholattice as the dual Ploščica space of the lattice reduct of the ortholattice equipped with a map representing the orthocomplement operation. We introduce an abstract ortho-Ploščica space capturing the properties of the dual space of an ortholattice, and we present dual representation theorems between general ortholattices and the ortho-Ploščica spaces. We illustrate our dual representations by examples and present an open problem.

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Operators **Max** L and **Min** U and duals of Boolean posets

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When working with posets which are not necessarily lattices, one has a lack of lattice operations which causes problems in algebraic constructions. This is the reason why we use the operators Max L and Min U substituting infimum and supremum, respectively. We axiomatize these operators. Two more operators, namely the so-called symmetric difference and the Sheffer operator, are introduced and studied in complemented posets by using the operators Max L and Min U. In Boolean algebras, the symmetric difference is used to construct its dual structure, the corresponding unitary Boolean ring. By generalizing this idea, we assign to each Boolean poset a so-called dual and prove that also, conversely, a Boolean poset can be derived from its dual.

Reducts of degenerate solutions of Yang-Baxter equation

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One of the most important notions when studying non-degenerate solutions of the Yang-Baxter equation is the notion of the retract congruence. When passing to degenerate solutions, the naïve approach fails since we do not obtain a congruence in general. We present here a working generalization of the retract congruence along with some examples that illustrate the choice of the definition.

On representations of double Boolean algebras

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Double Boolean algebras are abstract structures that arise in Formal Concept Analysis from the need to develop a Boolean Concept Logic, based on concepts as units of thought. Several representations have been addressed including contextual (pre-, semi-, proto-concept algebras)[1, 5], topological and logical[2, 3, 4] representations. We will review some of these in our presentation. This is an effort of the presenter to understand what has been done so far, avoiding reinventing the wheel.

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Stability of Mapping Spaces in ε-Simplicial Sets

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A familiar stability phenomenon in simplicial homotopy theory is that horn-filling conditions characterizing categories and groupoids are preserved under mapping-space constructions. For instance, simplicial sets satisfying the inner horn-filling conditions retain this property after passing to suitable mapping spaces, and an analogous statement holds for Kan complexes.

In this talk, I will discuss a related phenomenon for ε-simplicial sets. These were introduced in the simplicial study of Frobenius monoids in the category **Rel** of sets and relations [1], [2]. Within this framework, we consider a distinguished class of ε-fibrant objects, characterized by appropriate filling conditions.

A key ingredient is the close relationship between ε-simplicial sets and paracyclic sets. This connection provides a useful way to reinterpret certain simplicial filling problems and plays an essential role in the analysis of ε-fibrancy.

The main result is a stability theorem showing that, under suitable hypotheses, the relevant ε-fibrancy conditions are preserved under mapping-space constructions. I will explain the analogy with the classical cases of categories and groupoids, the role of paracyclic structures, and the main ideas of the proof.

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Critical elements in algebras of numerical events

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The probability $p(s)$ of the occurrence of an event pertaining to a physical system which is observed in different states s determines a function p from the set S of states of the system to $[0, 1]$. The function p is called a numerical event or more precisely an S-probability. When appropriately structured, sets P of numerical events form so-called algebras of S-probabilities, which are orthomodular posets that can serve as quantum logics. If one deals with a classical physical system, then P will be a Boolean algebra. Starting with a supposed quantum logic or logics obtained by individual measurements, newly added numerical events may turn out to be crucial for the assumption of classicality or non-classicality of the physical system. We will call those numerical events critical and will study their impacts among various classes of algebras of numerical events. Moreover, we will consider the situation of enlarging S by new states and will describe how elements of an algebra of S-probabilities contribute to the character of a logic when there exists a certain relation between them and the element that is newly added.

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Completely Join-preserving Maps between Commutator Lattices as Abstractions for Congruence Extensions

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Commutator lattices [7] are complete lattices endowed with an additional binary operation, called *commutator*, which is commutative, smaller than its arguments and completely distributive w.r.t. the join. They serve as abstractions for congruence lattices of algebras whose term–condition commutators are commutative and distributive w.r.t. arbitrary joins, in particular for congruence lattices of members of congruence–modular varieties endowed with the modular commutator.

The *(minimal) prime spectrum* of a commutator lattice is its set of (minimal) prime elements w.r.t. the commutator, and its topological structure turns out to be particularly useful. In [2, 3], we have constructed and studied the reticulation of a universal algebra by using the *Stone topology* on the prime spectrum of its congruence lattice. In [4] we have studied the Stone, as well as the *flat topology* on the minimal prime spectrum of congruences, then used these topologies to investigate congruences in extensions of universal algebras, generalizing results from [1] on ideals in ring extensions.

My talk will focus on the abstract case for this theory, which we have developed in [5, 6] by using complete join–semilattice morphisms between commutator lattices as an abstraction for the action on congruences of morphisms between similar algebras. The right adjoint of such a join–semilattice morphism turns out to be continuous and even a homeomorphism between the minimal prime spectra of the two commutator lattices (its domain and codomain) w.r.t. the Stone and flat topologies under certain conditions regarding the prime spectra and compact elements of these lattices.

While this abstraction has allowed us to treat the general case of arbitrary morphisms between similar algebras, it would be interesting to retrieve from it the particular case in [4] of extensions, so that of algebra embeddings. Another correspondence between this abstract case and the one of congruence lattices that remains to be investigated is identifying, among the compact elements of various kinds of algebraic commutator lattices, those that correspond to principal congruences in this concrete case of congruence lattices. Approaching these

open problems would obviously rely on obtaining representations for certain algebraic commutator lattices as congruence lattices endowed with the term–condition commutator.
Keywords: (algebraic) commutator lattice; (minimal) prime element; (Zariski/Stone/spectral, flat/inverse) topology; (algebra, ring) extension.
MSC 2010: 08A30; 08B10; 06B10; 06D22; 06E15.

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Direct Product of Directed Graphs of Algebraic Length 1

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Let $\mathbb{A}$ and $\mathbb{C}$ be finite directed graphs. Then $\mathbb{A}^{\mathbb{C}}$ denotes the directed graph with vertex set A^C (the set of functions from C to A), and edges defined as follows: for $f, g \in A^C$, $f \to g$ in $\mathbb{A}^{\mathbb{C}}$ iff $f(c_1) \to g(c_2)$ holds in $\mathbb{A}$ whenever $c_1 \to c_2$ holds in $\mathbb{C}$. We then call $\mathbb{C}$ injective if $\mathbb{A}^{\mathbb{C}} \cong \mathbb{B}^{\mathbb{C}}$ implies $\mathbb{A} \cong \mathbb{B}$ for arbitrary finite directed graphs $\mathbb{A}, \mathbb{B}$.

In [1] László Lovász proved that a directed graph which has n vertices and contains a loop in every vertex but no other edges is injective. A variant of this injectivity problem for posets was solved by Ralph McKenzie in [2].

Examples of non-injective directed graphs are the bipartite ones and their natural generalizations, directed graphs which have algebraic length greater than 1, that is, there exists a $k > 1$ such that the directed cycle of length k is a homomorphic image of it. We do not know any other connected examples of non-injective graphs, so the question arises: Could directed graphs of algebraic length 1 and injective ones be the same among finite, connected directed graphs?

The class of injective directed graphs is closed under direct product, so we examined if the direct product of connected directed graphs of algebraic length 1 also has a component of the same algebraic length. This problem was set as a task at the Schweitzer competition in 2025.

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Star-Free Modules

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Dynamic algebra provides an algebraic framework for expressing weakest preconditions of programs and has been widely used in formal program verification. When one wishes to verify *quantum* programs, it is natural to extend dynamic algebra to settings where the underlying propositional logic is not classical but quantum—that is, where the lattice of propositions is an *orthomodular lattice* rather than a Boolean algebra.

Orthomodular lattices arise naturally in quantum mechanics: the closed subspaces of any Hilbert space form a complete orthomodular lattice, the so-called Hilbert lattice. Crucially, orthomodular lattices are in general non-distributive, which distinguishes quantum logic from classical propositional logic and forces a careful re-examination of algebraic constructions that are routine in the Boolean setting.

A *star-free dynamic orthomodular lattice* (introduced in [2]) is an orthomodular lattice equipped with a necessity operator $\Box(a, p)$—read as "p holds after executing program a"—for programs built from sequential composition, non-deterministic choice, and tests, but *without* the Kleene star. The Kleene star is excluded because its treatment in the orthomodular/quantum setting raises substantial additional difficulties.

The main result of [2] is a *Stone-type representation theorem*: every star-free dynamic orthomodular lattice embeds into its canonical extension, a complete star-free dynamic orthomodular lattice constructed from the proper filters of the original one. This mirrors the classical Stone representation of Boolean algebras by set algebras, and it establishes one half of a discrete duality between star-free dynamic orthomodular lattices and the corresponding Kripke-style frames (star-free dynamic orthomodular frames).

The present paper develops a *module-theoretic* perspective on this structure. Rather than the necessity operator $\Box(a, p)$ of [2] (which corresponds to the *weakest precondition*), we work with a dual action $\ell(a, p)$ that propagates information *forward* along programs. The program terms form a term algebra $QA[\Pi, L]$ which, after quotienting by the natural behavioural equivalence, yields a unital involutive m-semilattice $S_{\mathbf{Dl}}$ acting on the orthomodular lattice L (see [1]). This algebraic structure is the module-theoretic analogue of the complex algebras studied in [2].

A central subtlety—illustrated by a detailed example —is that the involution on programs does not descend to the naive quotient by behavioural equivalence of forward actions alone. One must simultaneously track the behaviour of the dual program a^*, leading to the strengthened equivalence $\approx_{\mathbf{Dl}}$. This phenomenon has a direct counterpart in the frame semantics of [2], where the self-adjointness axiom for test relations plays an analogous role.

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Generalized quasiorders for maximal clones

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It is well-known that a function f preserves an equivalence or a quasiorder relation ϱ if (and only if) each unary polynomial function preserves ϱ which can be obtained from f by substituting constants at all but one arguments. Thus the polymorphism clone $\operatorname{Pol}\varrho$ is completely determined by the endomorphism monoid $\operatorname{End}\varrho$. *Generalized quasiorders* are higher-ary relations with the same property. We describe the generalized quasiorders of (almost all) maximal clones on a finite set.

Jauch–Pironness

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Abstract

Let L be a quantum logic (=orthomodular poset) and let $s : L \to [0, 1]$ be a state on L. Then s is said to be Jauch–Piron if $s(a) = s(b) = 1$ implies that there exists some c, $c \leq a$, $c \leq b$ such that $s(c) = 1$. If each state on L is Jauch–Piron, then L is called a Jauch–Piron logic.

In this talk, we first comment on the open questions as brought about in the literature (see the references below). Then we present a new result that links Jauch–Piron states with the hidden variable hypothesis.

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Clinker equivalences

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We study the maximal relations $\varrho \subseteq U \times U$ that define atomic Boolean concept lattices $\mathfrak{B}(U, U, \varrho)$. The complement $R = (U \times U) \setminus \varrho$ of such a relation ϱ can be characterized as a reflexive relation with the property that the sets $R^{-1}(x) := \{y \in U \mid (y, x) \in R\}$ form an irredundant covering of U. It was proved by J. Järvinen and S. Radeleczki that the rough sets induced by a relation R with this property form an algebraic regular double Stone lattice, i.e. they have the same structure as in the case of an equivalence relation. Therefore, such a reflexive relation R can be viewed as a new generalization of the notion of an equivalence relation, however, in the case when R is symmetric or transitive, it becomes an ordinary equivalence. The term "clinker equivalence" was chosen as its name, because the geometric form of the covering $U = \{R^{-1}(x) \mid x \in U\}$ resembles the sail of Viking ship, specifically referencing the historical clinker-built method of Norse shipbuilding. Here, we present some examples and applications of these relations. We show that they can be interpreted as special directed similarity relations, and can serve as useful tools in classification problems and concept approximations.

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Quasi-equational bases for directed paths

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For a finite relational structure P, let $Q(P)$ and be the class of all relational structures isomorphic to an induced substructure of a power of P. The class $Q(P)$ is axiomatizable by *quasi-identities*, i.e., sentences of the form

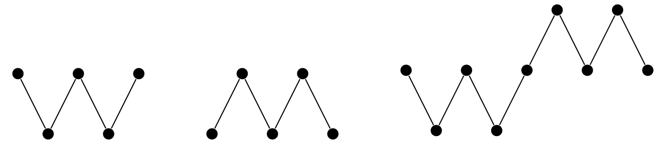
$$(\forall \bar{x})\ [\varphi_1(\bar{x}) \wedge \cdots \wedge \varphi_n(\bar{x}) \Rightarrow \varphi(\bar{x})],$$

where φ_i, φ are atomic formulas. In digraphs these are of the form $x \to y$ (there is an edge from x to y) and $x = y$. Then P is *finitely q-based* if $Q(P)$ is axiomatizable by a finite number of quasi-identities. We study the following question.

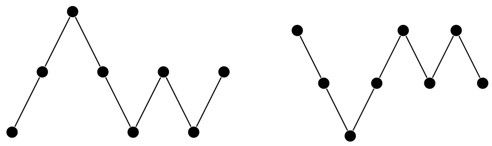
Problem 1 *Which finite relational structures are finitely q-based?*

Related questions for finite algebras have stimulated research in universal algebra for decades. Perhaps the most prominent result in this direction is McKenzie's solution to Tarski's problem.[3]. Deep results were also obtained for quasivarieties, see e.g. [2, 4]. However the results for relational structures are rare. As an example we recall Pultr's and Nešetřil's theorem stating that the only finitely q-based finite simple graphs are disjoint unions of complete bipartite graphs [1].

In the hope of discovering a general pattern, we decided to look closer at a class of digraphs that, on one side, behave essentially different than simple graphs and, on the other, are not too complicated. We focus on *directed paths*, that is, digraphs obtained from a path in a simple graph by choosing a direction for each of its edges. One may visualize directed paths by Hasse-like diagrams where arrows are edges whose direction is always upwards.



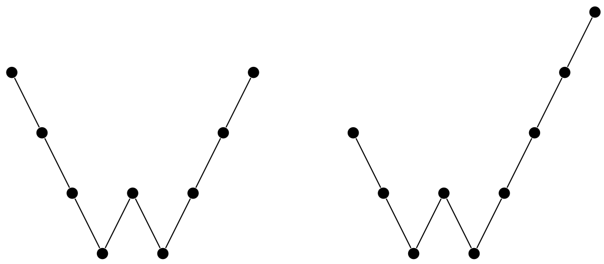
w m wIm



Λw $\qquad$ Vm

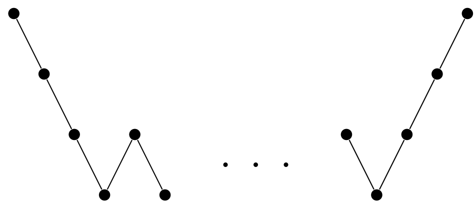
Our most general resulthas a rather technical formulation. But, during the talk, we will be mainly focused on proof technique that allows answering question in Problem 1, at least in some cases. Namely the lack of a finite q-base may be witnessed by the existence of an infinite family of so called critical digraphs.

Then is not difficult to show that if an directed path P contains one of wIm, Λw or Vm (see the picture above), then P is not finitely q-based. Let us consider directed paths that contain w or m but $\{wIm, \Lambda w, Vm\} \cap Q(P) = \emptyset$. We call them valleys and hills. Critical digraphs relative to valleys and hills may be very complex. As a not complicated examples, consider the following valleys



W_{33} $\qquad$ W_{24}

Then W_{33} is finitely q-based while W_{24} is not. The latter fact is witnessed by the depicted infinite family of critical digraphs relative to W_{24}:



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Adjointable maps for graph theory

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Given graphs $(V, E), (V', E')$, a function $f : V \to V'$ is said to be *adjointable* if there exists a function $g : V' \to V$ such that for all vertices $v \in V, v' \in V'$, we have:

$$f(v)v' \in E' \quad \iff \quad vg(v') \in E.$$

This notion plays a significant role in the setting of Hilbert spaces and their associated projective geometries [3], [1, Chapter 14], and a special case of a bijective f that is equal to g plays a role in the *cancellation problem* for graphs, as described in [2, Chapter 9].

Despite the notion being elementary, the general case has not received much attention from the graph theory community. In this talk we give examples of adjointable maps between graphs, compare them to graph homomorphisms, mention their connection to perfect codes, and give results about the number of adjointable maps between graphs that are cycles and paths.

Finally, we list some open problems that we find interesting.

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Solving lattice-valued relational equations and inequations

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We will present our investigation on the existence of optimal solutions to lattice-valued relational equations and inequations, extending the classical notions of greatest and least solutions to maximal and minimal ones [2, 3, 4, 5]. A key result establishes that infinite distributivity of infimum relative to supremum in the codomain lattice is both necessary and sufficient for typical inequations to admit a greatest solution. We show that minimal solutions may fail to exist even under strong lattice assumptions, while meet-continuity, though sufficient for maximal solutions, is not always necessary.

A distinctive aspect of our work is the connection between approximate solutions and factor algebras, where weak congruences provide the structural framework for transferring solvability properties [1]. This perspective links the study of relational systems with algebraic structures, clarifying how approximate solutions can be systematically understood through weak-congruence theory.

Connected to this are quotient structures of lattice-valued matrix spaces, constructive algorithms for least solutions that terminate after countably many steps, and refined lattice-theoretic conditions for solution existence. Altogether, the results advance a connection of the theory of lattice-valued equations and inequations with the theory of weak congruences (solutions of equations on factor subalgebras).

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Universal algebraic geometry of normal bands

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Given an algebra $\mathbb{A} = (A; F)$, we say that a set of n-tuples $S \subseteq A^n$ is an (n-ary) *algebraic set* if S is the set of solutions of a system of equations over $\mathbb{A}$. The collection of all algebraic sets is called the *algebraic geometry* of $\mathbb{A}$ [1]. The algebraic geometry of an algebra only depends on the clone of its term operations. If two clones defined on the same underlying set have the same algebraic geometry, then we say that the two clones are *algebraically equivalent.* For a fixed algebra $\mathbb{A}$, two natural problems arise:

- characterize the algebraic sets over $\mathbb{A}$;
- characterize the clones on the set A that are algebraically equivalent to the clone of term operations of $\mathbb{A}$.

The set of n-ary algebraic sets of $\mathbb{A}$ forms a lattice with respect to inclusion; furthermore, the structure of this lattice is completely determined by the quasivariety generated by $\mathbb{A}$ if it is a normal band. Using this connection, we managed to characterize the lattice of n-ary algebraic sets of every normal band up to isomorphism.

We obtained a similarly complete answer for the second problem as well: except for some special cases, a clone is algebraically equivalent to the clone of a normal band $\mathbb{A}$ if and only if it is the clone of $\mathbb{A}$.

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Dealing with C*-algebras by means of algebra and order

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Considerable effort has been devoted to reconstructing complex Hilbert spaces – the foundational models of quantum physics – from simpler structures such as ortholattices or orthosets. It was shown, for instance, that in a sense, a Hilbert space can be reduced to its orthogonality relation. A similar challenge arises for a class of mathematical models actually much better tailored to the needs of quantum mechanics: C*-algebras. While a C*-algebra likewise gives rise to a notion of orthogonality, this relation alone does not provide much of information, unlike in the Hilbert space case. It remains unclear how to fully characterize a C*-algebra through purely algebraic or order-theoretic means.

The unsatisfactory situation has motivated our attempt to develop the basic theory of C*-algebras with a primary focus on algebraic and order-theoretic aspects. While standard approaches employ methods of analysis and set-theoretic topology from the outset, a considerable portion of the theory can actually be developed without spectral theory. The order unit space of self-adjoint elements, the unitary group, approximate units, faces, and ideals can all be treated using methods limited to algebraic reasoning and the convergence of power series.

On this basis, we may then develop Gelfand theory following the philosophy of pointless topology: rather than a locally compact Hausdorff space, the space associated with a commutative C*-algebra is identified as a continuous regular frame. Ultimately, this approach allows us to represent elements of a C*-algebra as spectral maps, in analogy to the case of operators on a Hilbert space.

Constructive Gelfand duality for C*-algebras was developed by Banaschewski and Mulvey [1] as well as by Coquand and Spitters [2], with the non-unital case subsequently studied by Henry [3]. While our approach is possibly not constructive, it similarly avoids the (full) Axiom of Choice. Moreover, the procedure applies not only to an isolated commutative C*-algebra but also to a C*-subalgebra embedded within a larger C*-algebra. Consequently, the structural relationships between the constituents of commutative C*-subalgebras and those of the ambient algebra become to some extent transparent.

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