

# Old but NU: Finding Large Monoidal Intervals

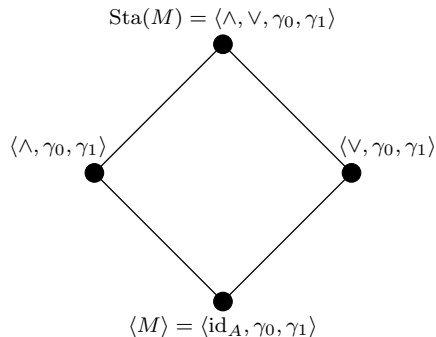
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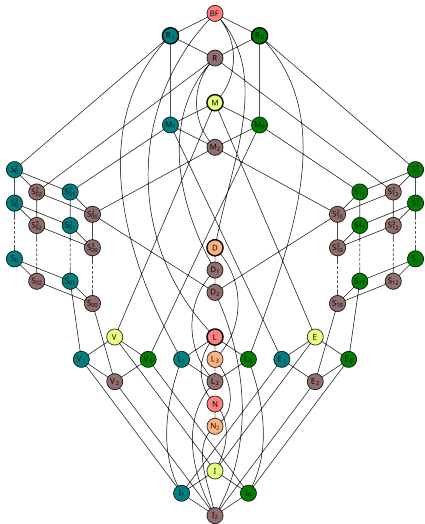
10 September 2026

Joint work with Miklós Dormán

- Let  $A$  be a finite set. We can say, that  $A = \{0, 1, 2, \dots, k - 1\}$ .
- A set of finitary operations on  $A$  that is closed under composition and contains all the projections is called a *clone*.
- Let  $\mathcal{O}_A$  be the clone of all finitary operations on  $A$ . For the set of  $n$ -ary operations on  $A$ , we use the  $\mathcal{O}_A^{(n)}$  notation. The clone lattice on  $A$  will be denoted by  $\text{Lat}_A$ .
- $A_i$  will denote the set  $A \setminus \{i\}$ .



- All the clones in  $\text{Lat}_A$  with unary part  $M \leq T(A)$  form an interval in the lattice, this is the monoidal interval corresponding to  $M$ , denoted by  $\text{Int}(M)$ .
- The least element is  $\langle M \rangle$  and the largest is  $\text{Sta}(M)$ .



- Blue  $\rightarrow$  Infinite
- Brown  $\rightarrow$  Infinite
- Green  $\rightarrow$  Infinite
- Red  $\rightarrow$  Finite
- Orange  $\rightarrow$  Finite
- Yellow  $\rightarrow$  Finite

# Infinite intervals

In the  $|A| = 3$  case there are 699 transformation monoids forming 160 conjugacy classes. Two transformation monoids are conjugate if one is obtained from the other by conjugation with a permutation of  $A$ .

There are at least 26 finite intervals and 108 intervals are known to have cardinality  $|\mathcal{P}(\mathbb{N})|$ , while 11 intervals are known to be infinite. Furthermore there are monoidal intervals for which the cardinality is completely unknown.

Isomorphic transformation monoids can have intervals with different cardinality.

- $M_a = \{\gamma_0, \gamma_1, \text{id}_A\}$ ;  $|\text{Int}_{M_a}| = 4$
- $M_b = \{\tau_{002}, \tau_{112}, \text{id}_A\}$ ;  $|\text{Int}_{M_b}| = \infty$

## Where are the countably infinite intervals?

The smallest base set of at least 3 elements with known transformation monoid with countably infinite interval has 9 elements and the monoid is  $\{ax + b : a, b \in \mathbb{Z}_9\}$  (Krokhin 1997).

## Large intervals ( $|\mathcal{P}(\mathbb{N})|$ )

From now on for simplicity's sake we will refer to monoidal intervals of cardinality  $|\mathcal{P}(\mathbb{N})|$  as **large intervals**.

Based on a method due to J. Demetrovics and L. Hannák the mostly used method for finding a large interval is to search for an infinite set of independent operations. The clones generated by these operations form an antichain in  $\text{Lat}_A$ .

The interval corresponding to monoid  $\{\gamma_0, \tau_{002}, \text{id}_A\}$  on  $A = \{0, 1, 2\}$  is a large interval.

Define  $\alpha_n = \{(2, 1, 0, \dots, 0, 0, 0), \dots, (0, 0, 0, \dots, 0, 2, 1), (1, 0, 0, \dots, 0, 0, 2)\} \subseteq A^n$ .

The family of independent operations is  $f_n: A^n \rightarrow A$ ,  $f_n(\mathbf{a}) = \begin{cases} 2, & \text{if } \mathbf{a} \in \alpha_n \\ 0, & \text{otherwise.} \end{cases}$

Based on the well-known near-unanimity operations, we define a relations that allow us to separate operations.

$$\text{NU}_m(a, b) = \{\mathbf{a} \in \{a, b\}^m : |\{i : \mathbf{a}_{[i]} = b\}| = 1\} \quad (m \in \mathbb{N}, m \geq 3)$$

## Proposition

*Let  $f$  be an  $n$ -ary operation on  $A$  ( $n \in \mathbb{N}, n \geq 3$ ) such that  $f(\mathbf{a}) = 1$  for every  $\mathbf{a} \in \text{NU}_n(1, 2)$ . Let  $\varrho$  be an  $n$ -ary relation on  $A$  such that  $\hat{1} \notin \varrho$  and  $\text{NU}_n(1, 2) \subseteq \varrho$ . Then  $f$  does not preserve  $\varrho$ .*

$$f \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

For a nonempty subset  $B$  of  $A$  such that  $|B| \leq |A| - 2$ , let  $B^A = \{\tau \in T(A) \mid \tau(A) \subseteq B\}$ . Then  $B^A \cup \{\text{id}_A\}$  is a submonoid of  $T(A)$ .

## Theorem (Krokhin)

*Let  $M$  be a non one-element submonoid of  $B^A \cup \{\text{id}_A\}$ . Then  $|\text{Int}(M)| = 2^{\aleph_0}$ .*

This result from Krokhin can be easily proven using NU relations.  
But the usefulness of the theorem is quite limited on  $A = \{0, 1, 2\}$ .

# The new NU

Define

$$\sigma_m = (A^m \setminus A_0^m) \cup \text{NU}_m(1, 2) \cup \{\hat{a} : a \in A_1\}.$$

For the transformations in  $\text{Pol}(\sigma_m)^{(1)}$  there are 2 cases:

(a)

$$\tau(0) = 0, \quad \tau(A_1) \subseteq A_1,$$

and

$$\tau(1) = 0 \quad \text{or} \quad \tau(2) = 0 \quad \text{or} \quad \tau(1) = \tau(2) \quad \text{or} \quad (\tau(1), \tau(2)) = (1, 2).$$

(b) There exists  $c \in A \setminus \{0, 1\}$  such that

$$\tau(0) = c \quad \text{and} \quad \tau(A) \subseteq \{0, c\}.$$

$$g_n: A^n \rightarrow A, g_n(\mathbf{a}) = \begin{cases} 1, & \text{if } \mathbf{a} \in \text{NU}_n(1, 2), \\ 0, & \text{otherwise.} \end{cases}$$

and

$$h_n: A^n \rightarrow A, h_n(\mathbf{a}) = \begin{cases} 2, & \text{if } \mathbf{a} = \hat{2} \\ 1, & \text{if } \mathbf{a} \in \text{NU}_n(1, 2), \\ 0, & \text{otherwise.} \end{cases}$$

### Proposition

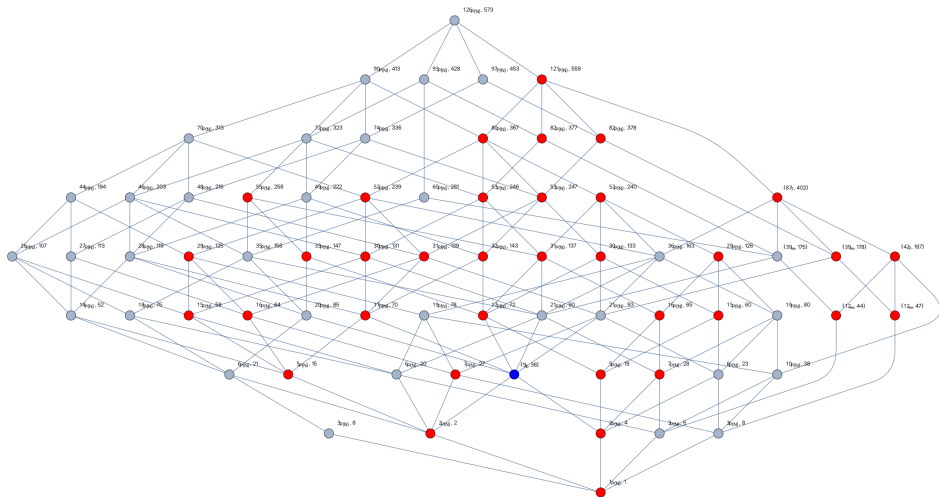
*The operations  $g_n$  and  $h_n$  preserve  $\sigma_m$  if and only if  $n \neq m$ .*

### Proposition

*Let  $N$  be a submonoid of  $\text{Pol}(\sigma_m)^{(1)}$ . The operation  $g_n$  is in  $\text{Sta}(N)$  if and only if  $\gamma_0, \tau_{010\dots 0} \in N$  or  $\gamma_0 \in N$  and for every  $\tau \in N$   $\tau(1) \neq 2$ .*

# The submonoids of $\text{Pol}(\sigma_m)^{(1)}$ and the corresponding intervals on $\{0, 1, 2\}$

$$\text{Pol}(\sigma_m)^{(1)} = \{\gamma_0, \gamma_2, \text{id}_A, \tau_{010}, \tau_{002}, \tau_{020}, \tau_{200}, \tau_{022}, \tau_{202}, \tau_{220}\} \bowtie M_{126}.$$



## New results

Using the  $g_n$  operations we get that the following intervals are large:

$$\begin{aligned}\{\text{id}_A, \gamma_0, \tau_{010}, \tau_{002}\} &= M_{18}, \\ \{\text{id}_A, \gamma_0, \gamma_2, \tau_{010}\} &\bowtie M_{20}, \\ \{\text{id}_A, \gamma_0, \gamma_2, \tau_{010}, \tau_{002}, \tau_{220}\} &\bowtie M_{60}.\end{aligned}$$

With the help of  $h_n$  we can prove that

$$\begin{aligned}\{\text{id}_A, \gamma_0, \tau_{002}, \tau_{022}\} &\bowtie M_{19}, \\ \{\text{id}_A, \gamma_0, \gamma_2, \tau_{002}\} &= M_{21}, \\ \{\text{id}_A, \gamma_0, \gamma_2, \tau_{002}, \tau_{022}\} &= M_{36}\end{aligned}$$

are large intervals.

## Open problems

Motivated by Krokhin's work on the cardinalities of monoidal intervals and by the Demetrovics–Hannák construction of continuum-sized intervals using independent families, one may ask whether every continuum-sized monoidal interval necessarily contains an infinite independent family of operations.

Are there countably infinite intervals on base sets with a size from 3 to 9?

Thank you for your attention!