

Dealing with C^* -algebras by means of algebra and order

Thomas Vetterlein

Institute for Mathematical Methods in Medicine und Data-Based Modelling
Johannes Kepler University (Linz, Austria)

Summer School on General Algebra and Ordered Sets,
Tatranská Lomnica, 11 September 2026



Der Wissenschaftsfonds.



This work is supported by the Austrian Science Fund (FWF): project 10.55776/PIN5424624
and the Czech Science Foundation (GAČR) 25-20013L.

The mathematical framework for quantum mechanics is provided by a

complex Hilbert space H .

- The (pure) **states** of a quantum-physical system are modelled by elements of

$$P(H) = \{ \text{lin}\{u\} : u \in H \setminus \{0\} \}.$$

- The (bounded) **observables** are modelled by elements of

$$\mathcal{B}(H)_{\text{sa}} = \{a \in \mathcal{B}(H) : a^* = a\}.$$

Quantum mechanics and its basic model

Assume that $a \in \mathcal{B}(H)_{\text{sa}}$ models some physical observable.

a corresponds to a **spectral measure** s^a , which sends real intervals (α, β) to projections $s^a((\alpha, \beta))$, all pairwise compatible and such that

$$a = \int \lambda \, ds^a(\lambda).$$

If the physical system is in a state belonging to $\text{im } s^a((\alpha, \beta))$, a measure of a leads to a value in (α, β) .

Simplification, justification?

Question

Can the Hilbert space be replaced with a more elementary mathematical model?

Does the Hilbert space arise in some way from a first-order structure of a simple type?

A possible answer

For a Hilbert space H , $P(H)$ together with its orthogonality relation $\perp$ is an **orthoset**.

The orthoset $(P(H); \perp)$ determines the structure of H essentially uniquely.

Some progress has been made to characterise $(P(H); \perp)$ in an “economical” way.

Quantum mechanics and its more advanced model

In a sense, a Hilbert space is a “too large” model.

Let $\mathcal{A}$ be the smallest closed subalgebra of $\mathcal{B}(H)$ that contains all operators modelling observables.

$\mathcal{A}$ has the structure of a

C*-algebra,

a more advanced mathematical model for quantum physics.

- The (pure) **states** of the quantum-physical system are modelled by pure states $p: \mathcal{A} \rightarrow \mathbb{C}$.
- The (bounded) **observables** are modelled by self-adjoint elements of $\mathcal{A}$.

Definition

A **C*-algebra** is a Banach algebra $\mathcal{A}$ equipped with an involution such that, for any $a \in \mathcal{A}$,

$$\|a^*\| = \|a\|, \quad \|a^*a\| = \|a\|^2.$$

$\mathcal{A}$ is called **unital** if it possesses a (multiplicative) unit.

If non-unital, $\mathcal{A}$ is embeddable canonically into a unital C*-algebra $\tilde{\mathcal{A}}$, its **unitisation**.

The major examples

- Given a Hilbert space H , any norm-closed, self-adjoint subalgebra of $\mathcal{B}(H)$ is a C*-algebra.
- Given a locally compact Hausdorff space X , the algebra $C_0(X)$ of continuous function $X \rightarrow \mathbb{C}$ vanishing at infinity, equipped with the pointwise complex conjugation and with the supremum norm, is a commutative C*-algebra.

Simplification, justification?

The same question again

Can C^* -algebras be replaced with more elementary mathematical models?

Do C^* -algebras arise in some way from first-order structures of a simple type?

A first attempt

Let $P(\mathcal{A})$ be the set of pure states on the C^* -algebra $\mathcal{A}$.

For $p, q \in P(\mathcal{A})$, let $p \perp q$ if there are positive elements a and b of norm 1 such that $ab = 0$, $p(a) = q(b) = 0$, and $q(a) = p(b) = 1$.

Then $(\mathcal{A}, \perp)$ is again an **orthoset**.

To which extent does $(P(\mathcal{A}); \perp)$ determine the structure of $\mathcal{A}$?

The orthoset associated with a C^* -algebra

Theorem

Let $\mathcal{A}$ be a C^* -algebra. Then there are Hilbert spaces H_ι , $\iota \in I$, such that $(P(\mathcal{A}); \perp)$ is isomorphic to

$$(\dot{\bigcup}_\iota P(H_\iota); \perp),$$

where $\perp$ is on each $P(H_\iota)$ the usual orthogonality relation and holds between any pair of elements from distinct $P(H_\iota)$ s.

That is, $(P(\mathcal{A}); \perp)$ has a transparent structure. However:

Any commutative C^* -algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X . Then the pure states are the point evaluations and the orthogonality relation coincides with inequality.

The orthoset associated with a C^* -algebra

Conclusion

To characterise the orthoset $(P(\mathcal{A}); \perp)$ is a feasible project.

However, in general, $(P(\mathcal{A}); \perp)$ does not provide much of information about $\mathcal{A}$.



We have to search for an alternative to $(P(\mathcal{A}); \perp)$.

Algebraic / order-theoretic approaches to C^* -algebras: common obstacles

The present situation

Certain tools are commonly employed in C^* -algebra theory from the very beginning:

- the Gelfand transform of commutative C^* -algebras
 $\rightsquigarrow$ set-theoretic topology;
- the Hilbert space representation by means of the GNS construction
 $\rightsquigarrow$ dimensions like $2^{2^{\aleph_0}}$.

We wonder whether it helps when avoiding

- the Axiom of Choice,
- Hilbert spaces of uncountable dimension.

The substructures of C^* -algebras

Definition and Theorem

The set of **self-adjoint** elements of the C^* -algebra $\mathcal{A}$,

$$\mathcal{A}_{\text{sa}} = \{a \in \mathcal{A} : a^* = a\},$$

is a real linear space. We can partially order $\mathcal{A}_{\text{sa}}$ by choosing

$$\mathcal{A}_{\text{sa}}^+ = \{a^2 : a \in \mathcal{A}_{\text{sa}}\}$$

as the set of **positive** elements.

Definition

A non-empty set $\mathcal{F} \subseteq \mathcal{A}_{\text{sa}}^+$ is called a **face** if

- (1) $\mathcal{F}$ is a cone: $a, b \in \mathcal{F}$ implies $a + b \in \mathcal{F}$,
 $a \in \mathcal{F}$ and $\alpha \in \mathbb{R}^+$ implies $\alpha a \in \mathcal{F}$;
- (2) $\mathcal{F}$ is a downset: $b \in \mathcal{F}$ and $0 \leq a \leq b$ implies $a \in \mathcal{F}$.

The lattice of closed faces

Definition

Let $\text{Faces } \mathcal{A}$ be the set of all closed faces, partially ordered by inclusion.

Theorem

$\text{Faces } \mathcal{A}$ is a complete lattice. Let $\mathcal{F}_\iota \in \text{Faces } \mathcal{A}$, $\iota \in I$. Their infimum is $\bigcap_\iota \mathcal{F}_\iota$ and their supremum is

$$\bigvee_\iota \mathcal{F}_\iota = \overline{\downarrow \sum_\iota \mathcal{F}_\iota}.$$

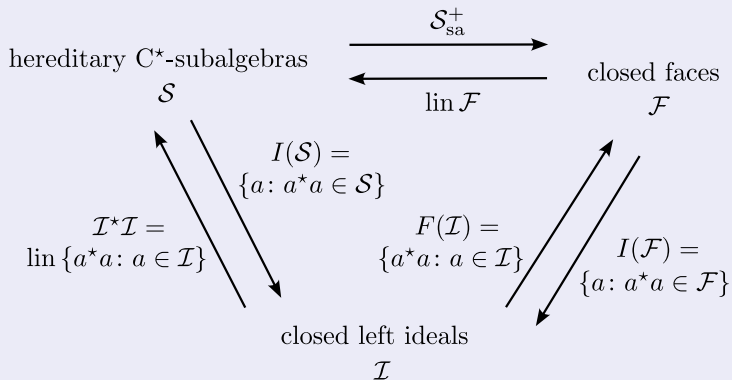
Note

There is a one-to-one correspondence between the pure states and the maximal closed faces distinct from $\mathcal{A}_{\text{sa}}^+$.

Alternative choices of substructures of C^* -algebras

Theorem

The following diagram is commutative and establishes one-to-one correspondences:



Representing elements w.r.t. Faces $\mathcal{A}$

Recall that, for a self-adjoint element a of a Hilbert space,

$$a = \int \lambda \, ds^a(\lambda),$$

where s^a is a map that sends the open real intervals to projections.

We intend to do something similar in the C^* -algebra $\mathcal{A}$.
The role of the projections is taken over by the closed faces.

B. BANASCHEWSKI, C. J. MULVEY, A globalisation of the Gelfand duality theorem, *Ann. Pure Appl. Logic*, 2006.

T. COQUAND, B. SPITTERS, Constructive Gelfand duality for C^* -algebras, *Math. Proc. Camb. Phil. Soc.* 2009.

The spectral map

Definition

- Let $b \in \mathcal{A}_{\text{sa}}^+$. We denote by $F\langle b \rangle$ the smallest closed face containing b .
- We denote by $\mathcal{O}(\mathbb{R})$ the frame of opens of $\mathbb{R}$ and by $\mathcal{OI}(\mathbb{R})$ the set of all open real intervals.

Lemma

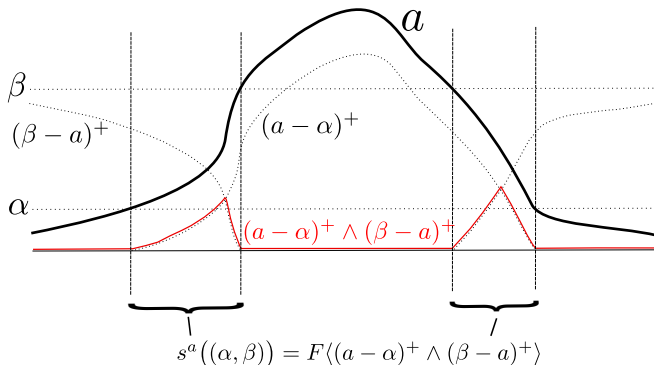
For any $b \in \mathcal{A}_{\text{sa}}^+$, $F\langle b \rangle = \overline{\downarrow \mathbb{R}^+ b}$.

The spectral map

Definition

Let $a \in \mathcal{A}_{\text{sa}}$. The **spectral map** of a is

$$s^a: \mathcal{OG}(\mathbb{R}) \rightarrow \text{Faces } \tilde{\mathcal{A}}, (\alpha, \beta) \mapsto F\langle (a - \alpha)^+ \wedge (\beta - a)^+ \rangle$$



Commutative C^* -(sub)algebras

Theorem (no (AC))

Let $\mathcal{C}$ be a commutative C^* -algebra.

Then $\text{Faces } \mathcal{C}$ is a continuous regular frame.

If $\mathcal{C}$ is unital, $\text{Faces } \mathcal{C}$ is a compact regular frame.

Theorem

Let $\mathcal{C}$ be a commutative C^* -subalgebra of the C^* -algebra $\mathcal{A}$.

For any closed face $\mathcal{F}$ of $\mathcal{C}$, $\downarrow \mathcal{F}$ is a closed face of $\mathcal{A}$.

Moreover, the map

$$\downarrow : \text{Faces } \mathcal{C} \rightarrow \text{Faces } \mathcal{A}, \mathcal{F} \mapsto \downarrow \mathcal{F}$$

is an embedding of lattices preserving arbitrary suprema and non-empty finite infima.

The spectral map as a frame homomorphism

Lemma

Let $a \in \mathcal{A}_{\text{sa}}$ and consider its spectral map

$$s^a: \mathcal{O}\mathcal{I}(\mathbb{R}) \rightarrow \text{Faces } \tilde{\mathcal{A}}, (\alpha, \beta) \mapsto F\langle (a - \alpha)^+ \wedge (\beta - a)^+ \rangle.$$

Let $\mathcal{C}$ be any commutative C^* -subalgebra containing a . Then there is a uniquely determined frame homomorphism $h^a: \mathcal{O}(\mathbb{R}) \rightarrow \text{Faces } \tilde{\mathcal{C}}$ such that $\downarrow \circ h^a|_{\mathcal{O}\mathcal{I}(\mathbb{R})} = s^a$:

$$\begin{array}{ccccc} \mathcal{O}\mathcal{I}(\mathbb{R}) & \hookrightarrow & \mathcal{O}(\mathbb{R}) & \xrightarrow{h^a} & \text{Faces } \tilde{\mathcal{C}} \\ & & & \searrow s^a & \downarrow \downarrow \\ & & & & \text{Faces } \tilde{\mathcal{A}} \end{array}$$

Integration over a spectral map

Definition

Let $\tilde{\mathcal{C}}$ be a commutative C^* -subalgebra of $\tilde{\mathcal{A}}$.

Let $s: \mathcal{O}(\mathbb{R}) \rightarrow \text{Faces } \tilde{\mathcal{C}}$ be a frame homomorphism.

Let $\varrho = \sup\{|r|: r \notin \ker s\}$ and assume $0 < \varrho < \infty$.

Assume that there is an element $a \in \mathcal{A}_{\text{sa}}$ such that:

For any cover $\{A_1, \dots, A_k\} \subseteq \mathcal{O}\mathcal{I}(\mathbb{R})$ of $[-\varrho, \varrho]$ with mesh $\varepsilon > 0$,
for any $e_1 \in s(A_1), \dots, e_k \in s(A_k)$ such that $e_1 + \dots + e_k = 1$,
and for any $\nu_1 \in A_1, \dots, \nu_k \in A_k$,

$$\|a - \sum_i \nu_i e_i\| < \varepsilon.$$

In this case, we write

$$a = \int \lambda \, ds(\lambda).$$

The spectral decomposition of a self-adjoint element

Theorem

For any $a \in \mathcal{A}_{\text{sa}}$,

$$a = \int \lambda \, \mathrm{d}s^a(\lambda).$$

Conclusion

With a C^* -algebra $\mathcal{A}$, we associate

Faces $\mathcal{A}$,

the complete lattice of closed faces.

Each self-adjoint element a can be represented by a homomorphism from the frame $\mathcal{O}(\mathbb{R})$ to a subframe of Faces $\mathcal{A}$.

- For $a, b \in \mathcal{A}_{\text{sa}}$, their linear combinations $\alpha a + \beta b$ as well as their product ab are expressible in terms of s^a and s^b , provided that a and b commute.
- For $a \in \mathcal{A}_{\text{sa}}$, we have $\|a\| = \max \{|r| : r \notin \ker s^a\}$.

Open issues

- How can we characterise the subframes of Faces $\mathcal{A}$ that are images of spectral maps?
- Can we say in the present framework anything about operations between non-commuting elements?