

Universal algebraic geometry of normal bands

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A semigroup $\mathbb{A} = (A; \cdot)$ is a **band** if all elements of A are idempotent.

A band is a **normal band** if it satisfies the identity $xyz u = xzy u$.

Semigroup equations are of the form $V_0 = V_1$, where $V_0, V_1 \in X^+$ and X is a finite set of variables.

A system of equations is a set of equations over the same set of variables.

A set of tuples $T \subseteq A^n$ is an **algebraic set** if there is a system of equations $\mathcal{E}$ such that T is the set of solutions to $\mathcal{E}$.

The set of all n -ary algebraic sets of $\mathbb{A}$ forms a lattice, this lattice is denoted by **$\text{Alg}_n(\mathbb{A})$** .

Characterization of the algebraic sets is an open problem for most algebraic structures.

We determined $\text{Alg}_n(\mathbb{A})$ up to isomorphism if $\mathbb{A}$ is a normal band.

Quasiidentities are statements of the form $(\forall x_1, x_2, \dots, x_n \in A)(\mathcal{E} \rightarrow e)$ where $\mathcal{E}$ is a finite system of equations over X and e is an equation.

A class $\mathcal{K}$ of semigroups forms a **quasivariety** if and only if it can be defined by quasiidentities.

The quasivariety generated by a semigroup $\mathbb{A}$ will be denoted by $\mathcal{Q}(\mathbb{A})$.

Quasivarieties and algebraic sets

Over any given algebra, there is a Galois connection between the implicationally closed systems of equations over $X_n := \{x_1, \dots, x_n\}$ and $\text{Alg}_n(\mathbb{A})$.

This closure system over the equations on $\mathbb{A}$ is completely determined by $\mathcal{Q}(\mathbb{A})$.

Quasivarieties of normal bands

$\mathcal{NB}$: normal bands

$li\mathcal{NB}$: left-semiinjective normal bands

$\mathcal{LNB}$: left normal bands

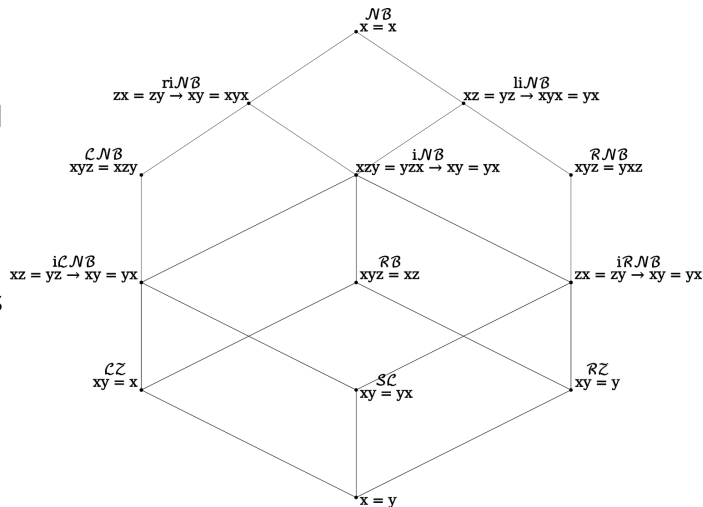
$i\mathcal{NB}$: injective normal bands

$i\mathcal{LNB}$: injective left normal bands

$\mathcal{RB}$: rectangular bands

$\mathcal{LZ}$: left zero semigroups

$\mathcal{SL}$: semilattices



Algebraic sets of injective normal bands

Let $\mathcal{Q}(\mathbb{A}) = i\mathcal{NB} = \langle xzy = yzx \rightarrow xy = yx \rangle_{\mathcal{NB}}$.

Every system of equations on $\mathbb{A}$ over X_n is equivalent to one of the following form:

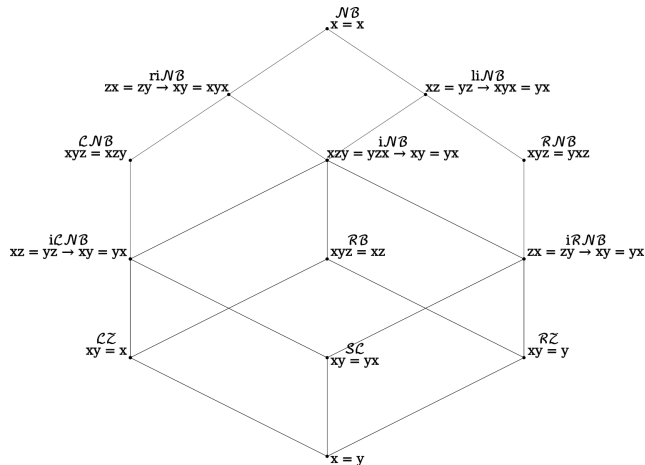
$$\{x_I = x_I x_{\text{cl}(I)} x_I : I \subseteq [n]\} \cup \{x_i x_j = x_j x_i x_j : (i, j) \in \lambda\} \cup \{x_k x_\ell = x_k x_\ell x_k : (k, \ell) \in \rho\},$$

where cl is a closure operator on $[n]$ with $\text{cl}(\emptyset) = \emptyset$ and $\lambda, \rho \in \text{Eq}([n])$.

$$\text{Alg}_n(\mathbb{A}) \cong \text{Cl}(n) \times \text{Eq}^d([n]) \times \text{Eq}^d([n]).$$

The lattice of algebraic sets up to isomorphism

$\mathcal{Q}(\mathbb{A})$	$\text{Alg}_n(\mathbb{A})$
$\mathcal{NB}$	$\text{ClEqEq}(n)$
$\ell i \mathcal{NB}$	$\text{ClEq}(n) \times \text{Eq}^d([n])$
$\mathcal{LNB}$	$\text{ClEq}(n)$
$i \mathcal{NB}$	$\text{Cl}(n) \times \text{Eq}^d([n]) \times \text{Eq}^d([n])$
$i \mathcal{LNB}$	$\text{Cl}(n) \times \text{Eq}^d([n])$
$\mathcal{RB}$	$\text{Eq}^d([n]) \times \text{Eq}^d([n])$
$\mathcal{LZ}$	$\text{Eq}^d([n])$
$\mathcal{SL}$	$\text{Cl}(n)$



Let $\mathcal{C} \in \text{Cl}(n)$ and for each $M \in \mathcal{C}$ let σ_M be an equivalence relation on M such that if $M \subseteq M'$ then $\sigma_M \subseteq \sigma'_{M'}$.

Let $\Sigma: \mathcal{C} \rightarrow \mathcal{P}([n] \times [n])$, $\Sigma(M) = \sigma_M$.

One can obtain the lattice $\text{ClEq}(n)$ of the pairs $(\mathcal{C}, \Sigma)$ using the partial order $\leq$ defined by the following:

$(\mathcal{C}, \Sigma) \leq (\mathcal{C}', \Sigma')$ if and only if
 $\mathcal{C} \subseteq \mathcal{C}'$, and for all $M \in \mathcal{C}$, we have $\Sigma(M) \subseteq \Sigma'(M)$.

Clones and algebraic sets of clones

The set of all operations over A is denoted by $\mathcal{O}_A$.

$F \subseteq \mathcal{O}_A$ is a **clone** over A if F is closed under composition and contains all projections.

Equations on a clone $\mathcal{C}$ over a set of variables X are of the form $f(\mathbf{x}) = g(\mathbf{x})$ where f and g are k -ary operations in the clone and $\mathbf{x} \in X^k$.

Systems of equations, algebraic sets, and the lattice $\text{Alg}_n(\mathcal{C})$ are defined the same way as before.

Two clones $\mathcal{C}_1$ and $\mathcal{C}_2$ are **algebraically equivalent** if for all $n \in \mathbb{N}$, we have $\text{Alg}_n(\mathcal{C}_1) = \text{Alg}_n(\mathcal{C}_2)$.

Algebraic equivalence to the clone of a normal band

The clone (of term operations) of a semigroup is the smallest clone containing the semigroup operation.

Let $\mathbb{A}$ be a normal band that does not have the same clone of term operations as

- $\mathbb{L} \times \mathbb{S}$, where $\mathbb{L}$ is a left zero semigroup and $\mathbb{S}$ is a semilattice,
- $\mathbb{L} \times \mathbb{B}$, where $\mathbb{L}$ is a left zero semigroup and $\mathbb{B}$ is a right normal band, or
- $\mathbb{L} \times \mathbb{S} \times \mathbb{R}$, where $\mathbb{L}$ is a left zero semigroup, $\mathbb{R}$ is a right zero semigroup and $\mathbb{S}$ is a semilattice.

Then a clone $\mathcal{C}$ is algebraically equivalent to the clone of $\mathbb{A}$ if and only if $\mathcal{C}$ is the clone of $\mathbb{A}$.

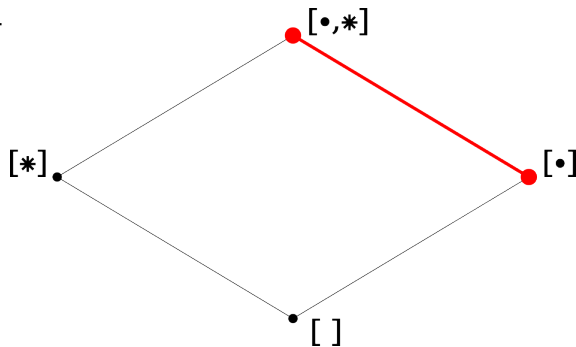
Left zero $\times$ semilattice

Let $\mathbb{A} = \mathbb{L} \times \mathbb{S}$, where $\mathbb{L}$ is a left zero semigroup, $\mathbb{S}$ is a semilattice and $x_i = (\ell_i, s_i)$.

- $x_1 * x_2 = (\ell_1, s_2)$,
- $x_1 \cdot x_2 = (\ell_1, s_1 s_2)$, the original operation of $\mathbb{A}$.

A clone $\mathcal{C}$ is algebraically equivalent to $[\cdot]$ if and only if

$$\mathcal{C} \in \{[\cdot], [\cdot, *]\}.$$



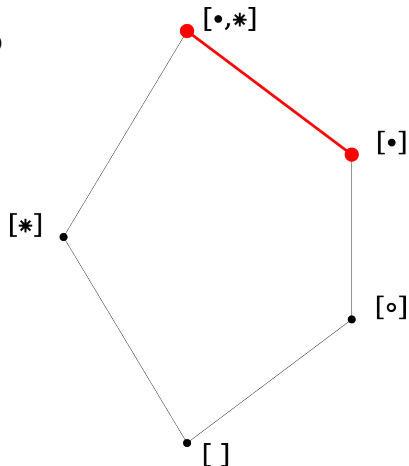
Left zero $\times$ right normal band

Let $\mathbb{A} = \mathbb{L} \times \mathbb{B}$, where $\mathbb{L}$ is a left zero semigroup, $\mathbb{B}$ is a right normal band and $x_i = (\ell_i, s_i)$.

- $x_1 * x_2 = (\ell_1, s_2)$,
- $x_1 \cdot x_2 = (\ell_1, s_1 s_2)$, the original operation of $\mathbb{A}$,
- $x_1 \circ x_2 = (\ell_1, s_2 s_1)$.

A clone $\mathcal{C}$ is algebraically equivalent to $[\cdot]$ if and only if

$$\mathcal{C} \in \{[\cdot], [\cdot, *]\}.$$



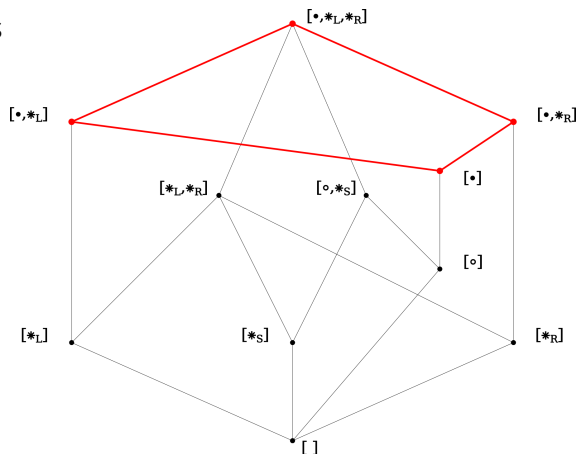
Left zero $\times$ right zero $\times$ semilattice

Let $\mathbb{A} = \mathbb{L} \times \mathbb{S} \times \mathbb{R}$, where $\mathbb{L}$ is a left zero semigroup, $\mathbb{R}$ is right zero semigroup, $\mathbb{S}$ is a semilattice and $x_i = (\ell_i, s_i, r_i)$.

- $x_1 *_L x_2 = (\ell_1, s_2, r_2)$,
- $x_1 *_R x_2 = (\ell_1, s_1, r_2)$,
- $x_1 *_S x_2 = (\ell_1, s_2, r_1)$,
- $x_1 \cdot x_2 = (\ell_1, s_1 s_2, r_2)$, the original operation of $\mathbb{A}$,
- $x_1 \circ x_2 = (\ell_1, s_1 s_2, r_1)$.

A clone $\mathcal{C}$ is algebraically equivalent to $[\cdot]$ if and only if

$$\mathcal{C} \in \{[\cdot], [\cdot, *_L], [\cdot, *_R], [\cdot, *_L, *_R]\}.$$



References

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