

Solving lattice-valued relational equations and inequations

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The function E in an Ω -set is **separated** if

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If $\mu : X \rightarrow \Omega$ is an Ω -valued set on X then for $p \in \Omega$, the set

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Proposition

The collection $\{\mu_p \mid p \in \Omega\}$ of all cuts of the function $\mu : X \rightarrow \Omega$ is a closure system on X .

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For every n -ary operation $f \in F$, for all $a_1, \dots, a_n, b_1, \dots, b_n \in A$,
and for every constant (nullary operation) $c \in F$

$$\bigwedge_{i=1}^n R(a_i, b_i) \leq R(f(a_1, \dots, a_n), f(b_1, \dots, b_n));$$

and $R(c, c) = 1$.

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In addition, the collection of all cuts $\{E_p \mid p \in \Omega\}$ of E is a closure system, a subposet of the lattice of all weak equivalences on A .

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- (ii) The cut E_p of E is a congruence relation on μ_p .*

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Then, $(\mathcal{A}, E)$ **satisfies identity** $u \approx v$ (this identity **holds** on $(\mathcal{A}, E)$) if the following condition is fulfilled:

$$\bigwedge_{i=1}^n \mu(a_i) \leq E(u(a_1, \dots, a_n), v(a_1, \dots, a_n)),$$

for all $a_1, \dots, a_n \in A$ and the term-operations corresponding to terms u and v respectively.

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If an identity $u \approx v$ holds on an algebra $\mathcal{A}$, then it also holds on an Ω -algebra $(\mathcal{A}, E)$.

Theorem

Let $(\mathcal{A}, E)$ be an Ω -algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies (all identities in) $\mathcal{F}$ if and only if for every $p \in L$ the quotient algebra μ_p/E_p satisfies the same identities.

Proposition

The collection of cuts of E in an Ω -algebra $\overline{\mathcal{A}} = (\mathcal{A}, E)$ is a closure system on A^2 , a subposet of the weak congruence lattice $\text{Con}_w(\mathcal{A})$ of $\mathcal{A}$.

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Theorem (Representation)

Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\text{Con}_w(\mathcal{A})$ such that

if $a \neq b$, then $(a, b) \notin \bigcap \{R \in \mathcal{R} \mid (a, a) \in R\}$ for all $a, b \in A$.

Then there is a complete lattice Ω and an Ω -algebra $(\mathcal{A}, E)$ with the underlying algebra $\mathcal{A}$, such that $\mathcal{R}$ consists of cuts of E .

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Now we have that $E_R = R$ (the cut determined by R considered as an element of Ω , coincides with R as a weak congruence).

The structure $(\mathcal{A}, E)$ is then the required Ω -algebra, obtained by the *canonical construction*.

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Corollary

Let $\mathcal{A}$ be an algebra and $\mathcal{R}$ a closure system in $\text{Con}_w(\mathcal{A})$ fulfilling condition:

if $a \neq b$, then $(a, b) \notin \bigcap \{R \in \mathcal{R} \mid (a, a) \in R\}$ for all $a, b \in A$.

Let also $\mathcal{F}$ be a set of identities in the language of $\mathcal{A}$ and suppose that for every $R \in \mathcal{R}$, the algebra $\text{dom}R/R$ fulfills these identities. Then there is a complete lattice Ω and an Ω -algebra $(\mathcal{A}, E)$, such that $\mathcal{R}$ consists of cuts of E and $(\mathcal{A}, E)$ satisfies $\mathcal{F}$.

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$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z,$$

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In terms of Ω -algebras, these identities are equivalent with formulas:

- (i) $E(x \cdot (y \cdot z), (x \cdot y) \cdot z) \geq \mu(x) \wedge \mu(y) \wedge \mu(z),$
- (ii) $E(x \cdot e, x) \geq \mu(x)$ and $E(e \cdot x, x) \geq \mu(x),$
- (iii) $E(x \cdot x^{-1}, e) \geq \mu(x)$ and $E(x^{-1} \cdot x, e) \geq \mu(x).$

Theorem

Let $(\mathcal{G}, E)$ be an Ω -algebra. Then, $(\mathcal{G}, E)$ is an Ω -group if and only if for every $p \in \Omega$, the quotient structure μ_p/E_p is a group.

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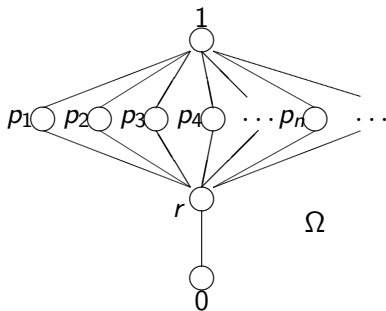
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$$\mu := \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \dots & \mathbf{n} & \dots \\ 1 & p_1 & p_2 & p_3 & \dots & p_n & \dots \end{pmatrix}.$$

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E^μ	0	1	2	3	4	5	...
0	1	0	r	0	r	0	...
1	0	p_1	0	r	0	r	...
2	r	0	p_2	0	r	0	...
3	0	r	0	p_3	0	r	...
4	r	0	r	0	p_4	0	...
5	0	r	0	r	0	p_5	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

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5	0	r	0	r	0	p_5	...
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The structure $(\mathcal{G}, E^\mu)$ is an Ω -group.

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$$\begin{array}{c|cc} \oplus & \mathbf{0} & \mathbf{n} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{n} \\ \mathbf{n} & \mathbf{n} & \mathbf{0} \end{array} ; \quad \begin{array}{c|cc} E_{p_n}^\mu & \mathbf{0} & \mathbf{n} \\ \hline \mathbf{0} & 1 & 0 \\ \mathbf{n} & 0 & 1 \end{array} .$$

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For every $p_n \in \emptyset$, the quotient structure $\mu_{p_n}/E_{p_n}^\mu$ is a two-element group.

2. Consider the symmetric group S_3 (given by the table) as a basic (underlying) algebra.

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f	f	e	h	g	k	j
g	g	j	e	k	f	h
h	h	k	f	j	e	g
j	j	g	k	e	h	f
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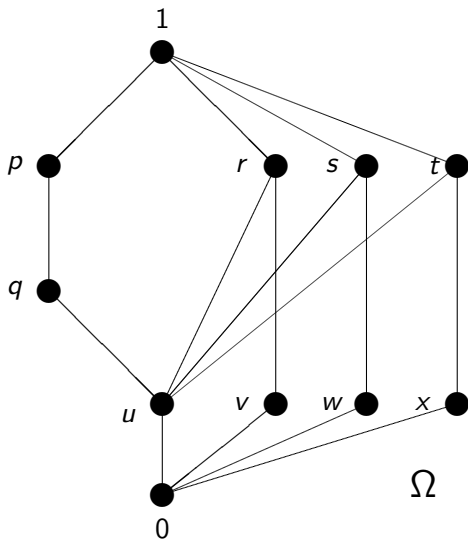
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k	k	h	j	f	g	e

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The lattice Ω is presented by the diagram.



E^μ	e	f	g	h	j	k
e	1	x	w	q	q	v
f	x	t	u	0	0	u
g	w	u	s	0	0	u
h	q	0	0	p	q	0
j	q	0	0	q	p	0
k	v	u	u	0	0	r .

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$$\mu = \begin{pmatrix} e & f & g & h & j & k \\ 1 & t & s & p & p & r \end{pmatrix}.$$

All the structures μ_z/E_z^μ , $z \in \Omega$ are groups of order 3, 2 or 1, hence Abelian.

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Therefore, this structure is an Abelian Ω -group, identity $x \cdot y \approx y \cdot x$ holds as the formula

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$$\mu(x) \wedge \mu(y) \leq E^\mu(x \cdot y, y \cdot x).$$

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j	0	0	0	0	1	0
k	0	0	0	0	0	0.

Hence, E_p^μ is a weak congruence on S_3 , a diagonal of $\mu_p = \{e, h, j\}$ and μ_p/E_p^μ is a group of order 3.

$$\mu_u = \begin{pmatrix} e & f & g & h & j & k \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \text{hence } \mu_u \text{ is the underlying group } S_3.$$

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$\mu_u/E_u^\mu = \{\{e, h, j\}, \{f, g, k\}\}$ i.e., it is a two-element group, similarly for other cuts.

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An **equasigroup** is an algebra $(Q, \cdot, \backslash, /)$ which satisfies the following identities:

$$Q1 : y = x \cdot (x \backslash y);$$

$$Q2 : y = x \backslash (x \cdot y);$$

$$Q3 : y = (y / x) \cdot x;$$

$$Q4 : y = (y \cdot x) / x.$$

Theorem

If $(Q, \cdot)$ is a quasigroup, then $(Q, \cdot, \backslash, /)$ is an equasigroup, where the additional binary operations $\backslash$ and $/$ are defined by:

$$a \backslash b = c \quad \text{iff} \quad b = a \cdot c \quad \text{and} \quad a / b = c \quad \text{iff} \quad a = c \cdot b.$$

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Here we consider groups in the language with a binary operation $\cdot$, unary operation $^{-1}$ and a constant e .

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We say that an equation $a \cdot x = b$ is **solvable over (Q, E)** if there is $c \in Q$ such that

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Analogously, an equation $y \cdot a = b$ is **solvable over (Q, E)** if there is $d \in Q$ such that

$$\mu(a) \wedge \mu(b) \leq \mu(d) \wedge E(d \cdot a, b).$$

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Analogously, if d is a solution of the equation $y \cdot a = b$ over (Q, E) and $d_1 \in Q$ fulfills $E(d_1 \cdot a, b) \geq p$ for some $p \leq \mu(a) \wedge \mu(b)$, then

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Theorem (Krapež, Šešelja, Tepavčević, Information Sciences, 2019)

Let (Q, E) be an Ω -groupoid. If equations $a \cdot x = b$ and $y \cdot a = b$, are E -uniquely solvable over (Q, E) for all $a, b \in Q$, then for every $p \in L$ the quotient groupoid μ_p/E_p is a quasigroup.

We say that an Ω -groupoid (Q, E) is an **Ω -quasigroup**, if every equation of the form $a \cdot x = b$ or $y \cdot a = b$ is E -uniquely solvable over (Q, E) .

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Theorem (Krapež, Šešelja, Tepavčević, Information Sciences, 2019)

Let (Q, E) be an Ω -groupoid. If for all $a, b \in Q$ and for every $p \leq \mu(a) \wedge \mu(b)$ the quotient groupoid μ_p/E_p is a quasigroup, then (Q, E) is an Ω -quasigroup.

Theorem (Krapež, Šešelja, Tepavčević, Information Sciences 2019)

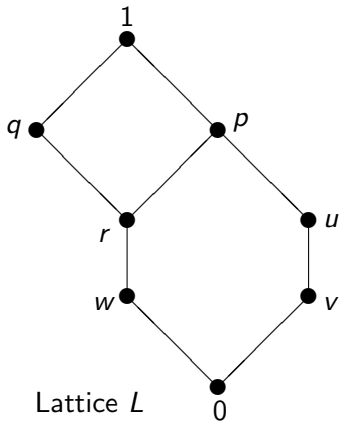
Let $\mathcal{Q} = (Q, \cdot)$ be an arbitrary groupoid, let a, b be particular elements in Q , and let $E : Q^2 \rightarrow L$ be an Ω -valued equality over $\mathcal{Q}$. Then the equation $a \cdot x = b$ has a unique solution w.r.t. E , if the equation $[a]_{E_p} \cdot X = [b]_{E_p}$, for $p = \mu(a) \wedge \mu(b)$, has a (classical) unique solution in the quotient groupoid $(\mu_p / E_p, \cdot)$.

Example 3

Let $(Q, \cdot)$ be a groupoid given in the table. Obviously, this groupoid is not a quasigroup, e.g., equation $a \cdot x = d$ does not have a solution in Q .

$\cdot$	a	b	c	d	e
a	b	c	a	a	e
b	a	b	c	d	e
c	c	a	b	b	e
d	d	a	b	b	e
e	e	e	e	e	a

The lattice L is given by the diagram, and with an L -valued equality in the next page, $((Q, \cdot), E)$ is an L -groupoid.



E	a	b	c	d	e
a	1	p	p	r	v
b	p	1	p	r	v
c	p	p	1	q	v
d	r	r	q	q	0
e	v	v	v	0	u

The function $\mu : Q \rightarrow L$ ($\mu(x) = E(x, x)$ for all $x \in Q$):

$$\mu = \begin{pmatrix} a & b & c & d & e \\ 1 & 1 & 1 & q & u \end{pmatrix}.$$

The subgroupoids of $((Q, \cdot), E)$, which are cuts of μ :

$$\mu_1 = \mu_p = \{a, b, c\},$$

$$\mu_q = \mu_r = \mu_w = \{a, b, c, d\},$$

$$\mu_u = \mu_v = \{a, b, c, e\},$$

$$\mu_0 = \{a, b, c, d, e\}.$$

The quotient groupoids over the corresponding cuts of E are the following:

$$\mu_1/E_1 = \{\{a\}, \{b\}, \{c\}\},$$

$$\mu_p/E_p = \{\{a, b, c\}\},$$

$$\mu_q/E_q = \{\{a\}, \{b\}, \{c, d\}\},$$

$$\mu_r/E_r = \mu_w/E_w = \{\{a, b, c, d\}\},$$

$$\mu_u/E_u = \{\{a, b, c\}, \{e\}\},$$

$$\mu_v/E_v = \{\{a, b, c, e\}\},$$

$$\mu_0/E_0 = \{\{a, b, c, d, e\}\}.$$

All these quotient structures are quasigroups, hence the starting L -groupoid is an L -quasigroup, and every linear equation is E -uniquely solvable over it.

E.g., the mentioned equation $a \cdot x = d$ which does not have a classical solution in Q , possesses a solution with respect to fuzzy equality E . Indeed, due to $\mu(a) \wedge \mu(d) = q$, this solution is element b , since the class $x = \{b\}$ is the unique solution of the equation $[a]_{E_q} \cdot x = [d]_{E_q}$ over the quasigroup μ_q/E_q (observe that $[d]_{E_q} = \{c, d\}$). Further, we have

$$\mu(a) \wedge \mu(d) = q \leq \mu(b) \wedge E(a \cdot b, d) = \mu(b) \wedge E(c, d) = 1 \wedge q = q.$$

Hence, $a \cdot b$ and d are E -equal with grade q .

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