

Generalized quasiorders for maximal clones

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Summerschool on General Algebra and Ordered Sets,
Tatranská Lomnica, Sept. 6–11, 2026

Outline

Generalized quasiorders and some questions

Generalized quasiorders for the maximal clones

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A nice property: When translations are enough!?

$f : A^n \rightarrow A$ (n -ary operation), $\varrho \subseteq A^m$ (m -ary relation)

$$\forall f: \boxed{f \triangleright \varrho \iff \text{trl}(f) \triangleright \varrho} \quad (\Xi) \quad \text{▶ Def ▶}$$

$\text{trl}(f) := \{f(c_1, \dots, c_{i-1}, x, c_{i+1}, \dots, c_n) \mid i \in \{1, \dots, n\}, c_1, \dots, c_n \in A\}$
(*unary translations* of f)

(Ξ) holds for

- **equivalence relations** $\varrho \in \text{Eq}(A)$ (reflexiv, symmetric, transitive)
- **quasiorder relations** $\varrho \in \text{Quord}(A)$ (binary, reflexive, transitive)
- **generalized quasiorders** (m -ary, reflexive, transitive)

JPR 2022 (published in Algebra Universalis 2024)

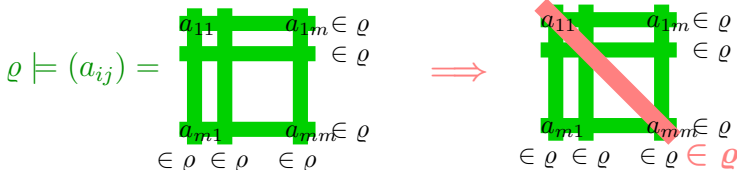
history (JPR = D. JAKUBÍKOVÁ-STUDENOVSKÁ, R.P., S. RADELECZKI):
investigation of (the lattice of) congruence and quasiorder lattices
 $\text{Con}(A, F)$, $\text{Quord}(A, F)$, (since 2007) ▶ Ref

Generalized quasiorders

Definition

Let $\varrho \subseteq A^m$ (m -ary relation)

- *reflexive* : $\iff \forall a \in A : (a, \dots, a) \in \varrho$.
- *transitive*
: $\iff \forall (a_{ij})_{i,j \in \{1, \dots, m\}} : \varrho \models (a_{ij}) \implies (a_{11}, \dots, a_{mm}) \in \varrho$



- *generalized quasiorder* : $\iff$ reflexive & transitive
- $\text{gQuord}(A) :=$ all generalized quasiorders on A ($m \in \mathbb{N}_+$)

Remark: $\text{gQuord}^{(2)}(A) = \text{Quord}(A)$

Some questions and problems

Given finite A , **how to describe all $\text{gQuord}(A)$** ?

or

Given an algebra (A, F) or a clone F , what are its invariant generalized quasiorders?

i.e., **determine $\text{gQuord}(A, F)$, $\text{gQuord } F$**

(= reflexive and transitive subalgebras of direct powers $(A, F)^m$)

in this talk we consider the **maximal clones $F = \text{Pol } \varrho$** ,

(with ϱ from one of the six relation classes from IVO

ROSENBERG's classification for maximal clones) and ask for

$$\text{gQuord } F = ?$$

Note

$$\text{gQuord } F = \text{Inv Pol } \varrho \cap \text{gQuord}(A) = [\varrho]_{(\exists, \wedge, =)} \cap \text{gQuord}(A)$$

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Generalized quasiorders for the maximal clones

$F = \text{Pol } \varrho$ for an equivalence relation ϱ

Let $\varrho \subseteq A \times A$ be a **nontrivial equivalence relation**. Then $F = \text{Pol } \varrho$ is a maximal clone.

Theorem

Each invariant relation of the maximal clone $F = \text{Pol } \varrho$ is a generalized quasiorder and can be obtained from ϱ by a pp-formula without quantifiers, i.e.,

$$\text{gQuord } F = \text{Inv } F = [\varrho]_{(\exists, \wedge, =)} = [\varrho]_{(\wedge, =)}.$$

$F = \text{Pol } \varrho$ for a lattice order relation ϱ

Let ϱ be a **partial order relation** with 0 and 1.

Then $F = \text{Pol } \varrho$ is a maximal clone.

For a **lattice order** ϱ , the generalized quasiorders gQuord F are easy to describe:

Theorem

The following are equivalent for $F = \text{Pol } \varrho$:

- (A) ϱ is a lattice order,
- (B) $\text{gQuord } F = \text{Inv } F$,
- (C) $[\varrho]_{(\wedge, =)} = [\varrho]_{(\exists, \wedge, =)} \quad (= \text{Inv } F)$.

Remark: For $A = \{0, 1\}$ we have even more:

Let $\mathcal{M} = \text{Pol } \leq$ (maximal clone of monotone Boolean functions).

Then

$$\text{Inv } \mathcal{M} = \text{gQuord}(A) \text{ (set of all generalized quasiorders)}$$

$F = \text{Pol } \varrho$ for arbitrary partial orders ϱ

Now: $F = \text{Pol } \varrho$ maximal clone for an *arbitrary* partial order relation with 0 and 1.

What is $\text{gQuord } F$?

By previous theorem we definitely have $\text{gQuord } F \subsetneq \text{Inv } F$
(if ϱ is not a lattice order)

It is easy to see that $[\varrho]_{(\wedge,=)} \subseteq \text{gQuord } F$.

Conjecture (JPR2024)

Let ϱ be a partial order relation with 0 and 1. Then

$$[\varrho]_{(\exists, \wedge, =)} \cap \text{gQuord}(A) = \text{gQuord } F = [\varrho]_{(\wedge, =)}.$$

True for lattice orders, but no final result since 2024.

However, some progress was made last month:

A larger class satisfying the conjecture

The class of lattice orders can be enlarged:

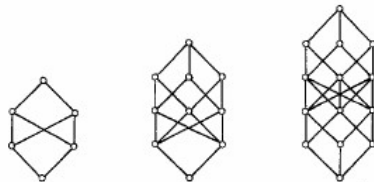
Theorem (JPR, Aug. 2026)

Let (A, ϱ) be a finite partially ordered set (with 0 and 1) satisfying the strong selection property. Then the conjecture is true, i.e., we have:

$$[\varrho]_{(\exists, \wedge, =)} \cap \text{gQuord}(A) = \text{gQuord Pol } \varrho = [\varrho]_{\wedge, =}.$$

Remark: The *strong selection property* was introduced by P. NEVERMANN AND R. WILLE, *The strong selection property and ordered sets of finite length*. Algebra Universalis 18(1), (1984), 18–28.

Lattices do have this property,
but there are much more posets
satisfying it, e.g.:



The remaining maximal clones

The maximal clones (on a finite set) are of the form $F = \text{Pol } \varrho$, where ϱ is of one of the following types (IVO ROSENBERG):

- (1) ϱ is a partial order with least and greatest element,
- (2) ϱ is the graph of a (fixed point free) permutation where all cycles have equal prime length,
- (3) ϱ is an affine relation ($\varrho = \{(x, y, u, v) \in A^4 \mid x + y = u + v\}$),
- (4) ϱ is a nontrivial equivalence relation,
- (5) ϱ is a central relation,
- (6) ϱ is a m -regular relation ($m \geq 3$).

Theorem (JPR2026)

Let $F = \text{Pol } \varrho$ with ϱ of type (2), (3), (5), (6). Then there is no nontrivial generalized quasiorder, i.e., gQuord F contains only (so-called) diagonal relations.

How to prove this Theorem?

Proof by contradiction. $F = \text{Pol } \varrho$ maximal clone.

Assume that there exists a nontrivial generalized quasiorder $\sigma \in \text{gQuord } F$ (this must lead to a contradiction).

By maximality of F , we must have $F = \text{Pol } \sigma$
(since $\text{Pol } \varrho \subseteq \text{Pol } \sigma \subset \text{Op}(A)$).

Let $M := \text{End } \varrho = F^{(1)}$ (unary operations in F) and
 $M^* = \{f \in \text{Op}(A) \mid \text{trl}(f) \subseteq M\}$. Then:

$\sigma \in \text{gQuord}(A)$ implies $F = M^*$ (“nice” property (Ξ))

Moreover we have [JakPR2024, 3.10(iii)] (here $(\Delta f)(x) := f(x, x)$).

($^\circ$) $\text{For every binary } f \in \text{Op}^{(2)}(A): f \in M^* \implies \Delta f \in M$

We can assume $M \subsetneq A^A$ (otherwise $F = (\text{Op}^{(1)}(A))^* = \text{Op}(A)$).

For each maximal clone $F = \text{Pol } \varrho$ of type (2),(3),(5),(6) one can find a contradiction to the property ($^\circ$).

Proof for central (5) and m -regular (6) relations

We do **not** give a definition of relations of type (5),(6).

Because: all what we need to know about these maximal clones $\text{Pol } \varrho$ is the following property:

there exists an element $c \in A$ such that for each $a \in A$ and each unary function g with image $\{c, a\}$ we have $g \in \text{Pol } \varrho$, i.e., $g \in M$.

for central relations one must choose c to be a central element; for m -regular relations even any element $c \in A$ does the job, see, e.g., the survey article of Á. SZENDREI [Sze2025, pp. 546,547].

Now, take a unary function h which does not belong to M .

Define $f(x, y) := h(x)$ if $x = y$ and $f(x, y) = c$ otherwise.

Then each translation of f has image $\{h(a), c\}$ for some $a \in A$ and therefore belongs to M . Thus $f \in M^*$.

But $h = \Delta f$ does not belong to M by assumption, a contradiction to condition

(°) For every binary $f \in \text{Op}^{(2)}(A)$: $f \in M^* \implies \Delta f \in M$

□

References

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Function f preserves relation ϱ function f (n -ary) **preserves** relation ϱ (m -ary):

$$f \triangleright \varrho$$

$$\begin{array}{ccccccc}
 f(& \text{green bar} & \text{green bar} & \dots & \text{green bar} &) = & \text{red bar with 3 dots} \\
 f(& a_{11} & a_{12} & \dots & a_{1n} &) = & \bullet \\
 f(& a_{21} & a_{22} & \dots & a_{2n} &) = & \bullet \\
 & & & & & & \\
 f(& \text{green bar} & \text{green bar} & \dots & \text{green bar} &) = & \text{red bar with 1 dot} \\
 f(& a_{m1} & a_{m2} & \dots & a_{mn} &) = & \bullet \\
 & \underbrace{\hspace{1cm}}_{\in \varrho} & \underbrace{\hspace{1cm}}_{\in \varrho} & \dots & \underbrace{\hspace{1cm}}_{\in \varrho} & \Rightarrow & \underbrace{\hspace{1cm}}_{\in \varrho}
 \end{array}$$

 $F \subseteq \text{Op}(A)$ (set of all finitary operations $f : A^n \rightarrow A$) $Q \subseteq \text{Rel}(A)$ (set of all finitary relations $\varrho \subseteq A^m$)

$$\text{Inv } F := \{\varrho \in R_A \mid \forall f \in F : f \triangleright \varrho\}$$

invariant relations

$$\text{Pol } Q := \{f \in \text{Op}(A) \mid \forall \varrho \in Q : f \triangleright \varrho\}$$

polymorphisms(Galois connection $\text{Pol} - \text{Inv}$) [◀ back1](#)