

Star-Free Modules

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Motivation

Background

A module-theoretic dual

The program algebra

Main result

Conclusion

- **Dynamic algebra** expresses *weakest preconditions* of programs

$$\Box(a, p) = \text{“}p \text{ is guaranteed to hold after running } a\text{”}$$

- Workhorse of formal program verification (Hoare-style reasoning, PDL).
- What happens when the program is **quantum**?
 - Propositions no longer form a Boolean algebra.
 - They form an **orthomodular lattice** – possibly non-distributive.

Goal of this line of work

Extend dynamic-algebraic reasoning to the quantum (orthomodular) setting.

Orthomodular lattice

A lattice $(L, \leq, 0, 1)$ with an *orthocomplement* $\sim: L \rightarrow L$ such that

- $\sim$ is an involutive, order-reversing map, $x \wedge \sim x = 0$;
- (orthomodular law) $x \leq y \implies y = x \vee (\sim x \wedge y)$.

Guiding example

For a Hilbert space H , the closed subspaces $C(H)$ form a complete orthomodular lattice: $\wedge = \cap$, $\sim =$ orthogonal complement.

In general **non-distributive** – this is what makes quantum logic quantum.

Star-free dynamic orthomodular lattices (Takagi, ISMVL 2026)

- Programs built from atomic actions Π , tests $p?$, choice $\cup$, sequential composition ; – but *no* Kleene star $(\cdot)^*$ -iteration.

(the star raises genuinely new difficulties in the quantum setting)

- A necessity operator $\Box(a, p)$: weakest precondition of p w.r.t. a .

Main result of [2]

Every star-free dynamic orthomodular lattice embeds into a complete canonical extension (proper filters) – a **Stone-type representation theorem**, and one half of a discrete duality with Kripke-style frames.

Our perspective: propagate information forward

Takagi [2]

- Operator $\Box(a, p)$
- weakest precondition
- “what must hold *before*”

This talk

- Action $\ell(a, p)$
- forward image / strongest postcondition
- “what holds *after*”

Punchline

Quotienting the program terms by their induced action yields a **unital involutive m -semilattice** S_{DI} acting on L : the module-theoretic analogue of Takagi’s complex algebras.

The term algebra $QA[\Pi, L]$

Fix atomic programs Π with an involution $\pi \mapsto \pi^*$, and an orthomodular lattice L . The set of **star-free program terms** is generated by:

- atomic programs $\pi \in \Pi$;
- tests $p^?$ for $p \in L$;
- choice $a \cup b$ and composition $a ; b$.

The involution extends recursively:

$$(p^?)^* := p^?, \quad (a \cup b)^* := a^* \cup b^*, \quad (a ; b)^* := b^* ; a^*.$$

Reading the involution

$(\cdot)^*$ is the algebraic *converse* of a program: $\pi \leftrightarrow \pi^*$ plays the role of $U_\pi \leftrightarrow U_\pi^{-1}$;
 $(AB)^{-1} = B^{-1}A^{-1}$ becomes $(a ; b)^* = b^* ; a^*$.

Axioms for the module action ℓ

An interpretation $\ell: \text{QA}[\Pi, L] \times L \rightarrow L$ must satisfy, for $\pi \in \Pi$, $a, b \in \text{QA}[\Pi, L]$, $p, q \in L$:

$$(1) \quad \sim\ell(\pi, p) = \ell(\pi^*, \sim p) \quad (\text{unitarity})$$

$$(2) \quad \ell(\pi; \pi^*, p) = \ell(\pi^*; \pi, p) = p \quad (\pi, \pi^* \text{ mutually inverse})$$

$$(4) \quad \ell(a, 0) = 0, \quad \ell(a, p \vee q) = \ell(a, p) \vee \ell(a, q)$$

$$(5) \quad \ell(a; b, p) = \ell(a, \ell(b, p)) \quad (\text{composition})$$

$$(6) \quad \ell(a \cup b, p) = \ell(a, p) \vee \ell(b, p) \quad (\text{choice})$$

$$(7) \quad \ell(p^?, q) = p \wedge (\sim p \vee q) \quad (\text{Sasaki projection} = \text{test})$$

Programs act by well-behaved maps

Adjointable maps

$f: L \rightarrow L$ is **adjointable** if some f^* satisfies

$$f(x) \leq \sim y \iff x \leq \sim f^*(y) \quad (\forall x, y \in L).$$

This is exactly the orthogonality relation used to define the Hilbert-space adjoint – an “orthogonality Galois connection”.

- Adjoints, when they exist, are **unique**.
- Adjointable maps preserve all existing joins (hence are monotone, $f(0) = 0$).
- Each atomic action $f_\pi(p) := \ell(\pi, p)$ is **self-adjoint**: $f_\pi^* = f_\pi$.
- Composition and choice of adjointable maps stay adjointable: $(g \circ f)^* = f^* \circ g^*$,
 $(f_a \vee f_b)^* = f_a^* \vee f_b^*$.

Building the program algebra: a first attempt

Idea: identify two programs when they induce the same action.

$$a \sim_{\text{DI}} b :\iff (\forall p \in L) \ \ell(a, p) = \ell(b, p).$$

We would like to define, on $\text{QA}[\Pi, L]/\sim_{\text{DI}}$,

$$[a] \sqcup [b] := [a \cup b], \quad [a] \cdot [b] := [a; b], \quad [a]^* := [a^*].$$

Question

Is $[a]^* := [a^*]$ well defined?

i.e. does $a \sim_{\text{DI}} b$ imply $a^* \sim_{\text{DI}} b^*$?

No! A concrete counterexample

Let $L =$ subspaces of $\mathbb{R}^2$, $p := \text{span}(e_1)$. Take symmetric invertible matrices

$$T = \begin{pmatrix} -4 & -4 \\ -4 & -3 \end{pmatrix}, \quad S = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix},$$

interpreting atomic programs by $\ell(\pi, X) = TX$, $\ell(\rho, X) = SX$ (and π^*, ρ^* by T^{-1}, S^{-1}).

- Let $a := \pi ; p^?$ and $b := \rho ; p^?$.
- $Te_1 = (-4, -4) = T(p)$ and $Se_1 = (2, 2)$ span the **same line** as $T(p)$.

Consequence

$\ell(a, X) = \ell(b, X)$ for every X : $a \sim_{\text{DI}} b$.

...but the duals disagree

Recall $a^* = p^\perp$; π^* and $b^* = p^\perp$; ρ^* . Test the witness $X := T(p^\perp) = \text{span}(-4, -3)$.

$T^{-1}X = p^\perp$, so

$S^{-1}X = \text{span}(1, 1) \not\subseteq p^\perp$, so

$$\ell(a^*, X) = \ell(p^\perp, p^\perp) = 0.$$

$$\ell(b^*, X) = \ell(p^\perp, S^{-1}X) = p \neq 0.$$

Conclusion

$a \sim_{\text{DI}} b$ but $a^* \not\sim_{\text{DI}} b^*$.

The involution **does not descend** to $\text{QA}[\Pi, L]/\sim_{\text{DI}}$.

The fix: track the dual as well

Strengthened equivalence

$$a \approx_{\text{DI}} b :\iff (\forall p) \ell(a, p) = \ell(b, p) \text{ and } (\forall p) \ell(a^*, p) = \ell(b^*, p).$$

- Genuine congruence relation; set $S_{\text{DI}} := \text{QA}[\Pi, L] / \approx_{\text{DI}}$.
- Operations $\sqcup$, $\cdot$, $*$ and constants $0 = [0^?]$, $e = [1^?]$ are now **well defined** – the extra clause is exactly what is needed in the compatibility proof for $(\cdot)^*$.
- Mirrors the frame-level *self-adjointness axiom* for test relations in [2].

Main theorem

S_{DI} is a unital involutive m -semilattice, and L is a module

1. $(S_{DI}, \sqcup, 0, \cdot, e, *)$ is a unital involutive m -semilattice.
2. The induced action $\bullet: S_{DI} \times L \rightarrow L$, $[a] \bullet p := \ell(a, p)$, is a module action:

$$\begin{aligned}x \bullet (p \vee q) &= (x \bullet p) \vee (x \bullet q), & (x \sqcup y) \bullet p &= (x \bullet p) \vee (y \bullet p), \\(x \cdot y) \bullet p &= x \bullet (y \bullet p), & e \bullet p &= p, \quad 0 \bullet p = 0.\end{aligned}$$

- S_{DI} = the **program algebra**; L = the **state algebra** it acts on.
- Module-theoretic counterpart of Theorems 24–25 of [2] (complex algebra of a frame / canonical frame).

Two dual pictures of the same phenomenon

	Takagi [2]	This work
Operator	$\Box(a, p)$	$\ell(a, p)$
Reading	weakest precondition	forward image
Test axiom	$\Box(p^?, q) = \sim p \vee (p \wedge q)$	$\ell(p^?, q) = p \wedge (\sim p \vee q)$
Structure	canonical extension	program algebra S_{DI}
Subtlety	self-adjointness of $R_{p^?}$	involution on $\approx_{DI}$ -quotient

Both subtleties are the *same* phenomenon, seen from the algebra side and the frame side.

What we did

- Dualized Takagi's necessity operator $\Box$ to a forward module action ℓ .
- Showed atomic program actions are self-adjoint / adjointable.
- Built the induced program algebra S_{DI} via a **strengthened** equivalence $\approx_{DI}$, and proved it is a unital involutive m -semilattice acting as a module on L .

Ongoing / future work

- Full module-theoretic duality matching Takagi's Stone-type representation.
- Adding the Kleene star to the module-theoretic picture.

-  M. Botur, J. Paseka, M. Lekár, *Foulis m -semilattices and their modules*. In: Proc. 55th ISMVL 2025, pp. 196–201, IEEE (2025).
-  T. Takagi, *A Stone-type representation theorem for star-free dynamic orthomodular lattice*. In: Proc. 56th ISMVL 2026, pp. 129–134, IEEE (2026).

Thank you!

Questions?

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