

Direct Product of Directed Graphs of Algebraic Length 1

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Direct product and a problem

Graph: directed graph without parallel edges, but may contain loops

Let $\mathbb{A}$ and $\mathbb{B}$ be two finite, connected directed graphs of algebraic length 1. Is it true, that $\mathbb{A} \times \mathbb{B}$ also has a connected component of the same algebraic length?

Definition (Direct product).

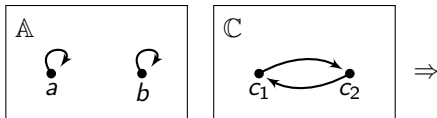
Let $\mathbb{A}$ and $\mathbb{B}$ be finite directed graphs. Then $\mathbb{A} \times \mathbb{B}$ denotes the directed graph with vertex set $A \times B$, and edges defined as follows: for $(a_1, b_1), (a_2, b_2) \in A \times B$, $(a_1, b_1) \rightarrow (a_2, b_2)$ in $\mathbb{A} \times \mathbb{B}$ iff $a_1 \rightarrow a_2$ holds in $\mathbb{A}$ and $b_1 \rightarrow b_2$ holds in $\mathbb{B}$.

Exponentiation of directed graphs

Definition (Exponentiation of directed graphs).

Let $\mathbb{A}$ and $\mathbb{C}$ be finite directed graphs. Then $\mathbb{A}^{\mathbb{C}}$ denotes the directed graph with vertex set $A^{\mathbb{C}}$ (the set of functions from C to A), and edges defined as follows: for $f, g \in A^{\mathbb{C}}$, $f \rightarrow g$ in $\mathbb{A}^{\mathbb{C}}$ iff $f(c_1) \rightarrow g(c_2)$ holds in $\mathbb{A}$ whenever $c_1 \rightarrow c_2$ holds in $\mathbb{C}$.

Example.

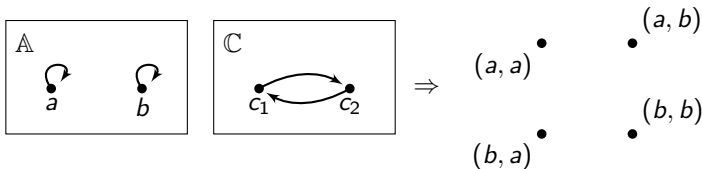


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Example.

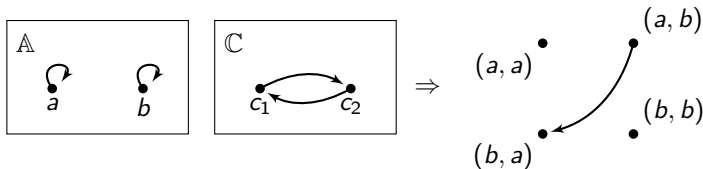


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Example.

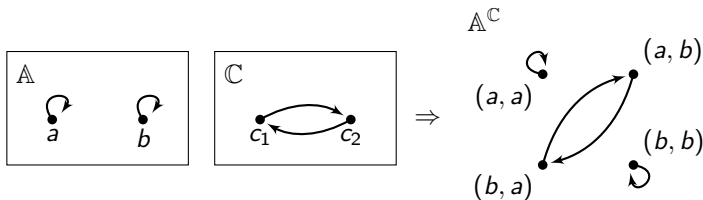


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Example.



Exponentiation of directed graphs

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Let $\mathbb{A}$ and $\mathbb{C}$ be finite directed graphs. Then $\mathbb{A}^{\mathbb{C}}$ denotes the directed graph with vertex set A^C (the set of functions from C to A), and edges defined as follows: for $f, g \in A^C$, $f \rightarrow g$ in $\mathbb{A}^{\mathbb{C}}$ iff $f(c_1) \rightarrow g(c_2)$ holds in $\mathbb{A}$ whenever $c_1 \rightarrow c_2$ holds in $\mathbb{C}$.

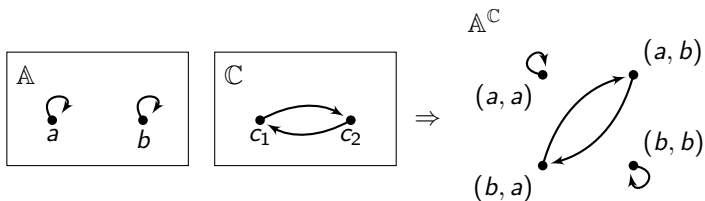
Remark. The following identities are true for this exponentiation:

$$\mathbb{A}^{\mathbb{B} \cup \mathbb{C}} \cong \mathbb{A}^{\mathbb{B}} \times \mathbb{A}^{\mathbb{C}}, \quad (\mathbb{A}^{\mathbb{B}})^{\mathbb{C}} \cong \mathbb{A}^{\mathbb{B} \times \mathbb{C}}, \quad (\mathbb{A} \times \mathbb{B})^{\mathbb{C}} \cong \mathbb{A}^{\mathbb{C}} \times \mathbb{B}^{\mathbb{C}}.$$

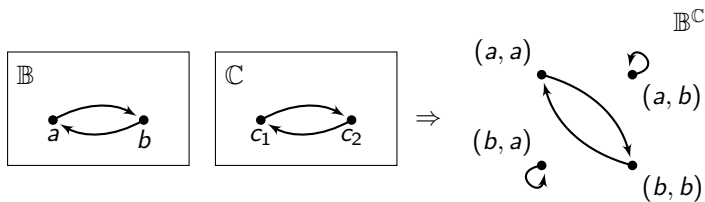
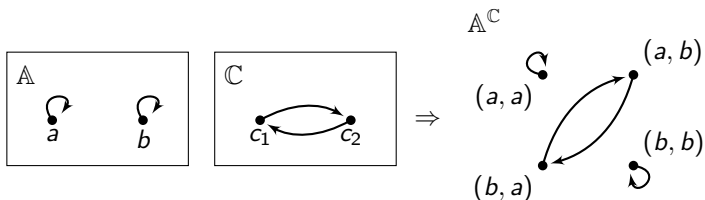
Definition (Injective exponents).

A directed graph $\mathbb{C}$ is injective if $\mathbb{A}^{\mathbb{C}} \cong \mathbb{B}^{\mathbb{C}}$ implies $\mathbb{A} \cong \mathbb{B}$ for arbitrary finite directed graphs $\mathbb{A}, \mathbb{B}$.

Exponentiation of directed graphs



Exponentiation of directed graphs



Similar problems for posets, and direct powers of graphs

Theorem (McKenzie, 2000).

If $\mathbb{A}^{\mathbb{P}} \cong \mathbb{B}^{\mathbb{P}}$, where $\mathbb{A}$, $\mathbb{B}$, $\mathbb{P}$ are finite partially ordered sets, then $\mathbb{A} \cong \mathbb{B}$.

Theorem (Duffus, Rival, 1978).

Let $\mathbb{A}$ be a finite partially ordered set that is not an antichain and let $\mathbb{B}$ and $\mathbb{C}$ be finite partially ordered sets. If $\mathbb{A}^{\mathbb{B}} \cong \mathbb{A}^{\mathbb{C}}$ then $\mathbb{B} \cong \mathbb{C}$.

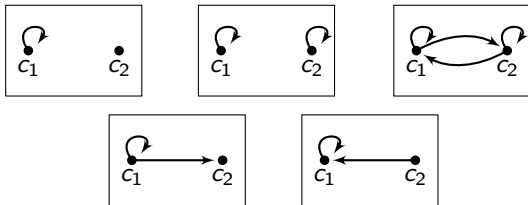
Theorem (László Lovász, 1967).

If $\mathbb{A}$ and $\mathbb{B}$ are finite directed graphs and $\mathbb{A}^n \cong \mathbb{B}^n$, then $\mathbb{A} \cong \mathbb{B}$.

Injective exponents

Examples for injective exponents:

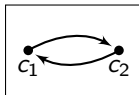
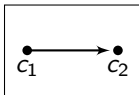
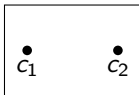
- Injective exponents with 2 vertices:



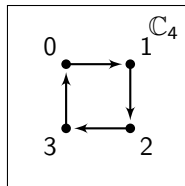
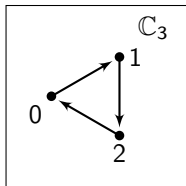
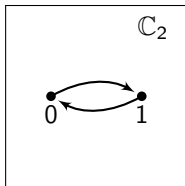
- Directed graph on n vertices which contains only the loops (Lovász)
- Complete directed graph on n vertices

Examples for non-injective exponents

- Non-injective exponents with 2 vertices:



- Directed cycles of length greater than 1:



...

- Disjoint union of directed cycles
- Every directed graph in which every connected component has algebraic length greater than 1.

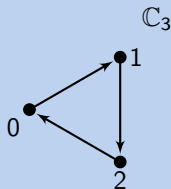
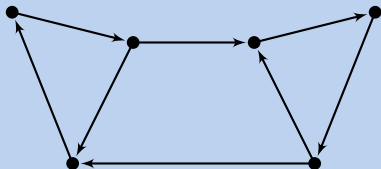
Are there any other non-injective graphs?

Algebraic length and vertex coloring

Definition (Algebraic length).

The algebraic length of a connected directed graphs is the smallest $k > 1$ such that the vertices of the graph can be colored by $\mathbb{C}_k$. If such a $k > 1$ does not exist, then the algebraic length of the graph is 1.

Example. The vertices of the following graph can be colored by $\mathbb{C}_3$.

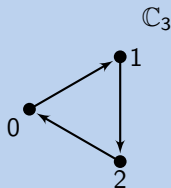
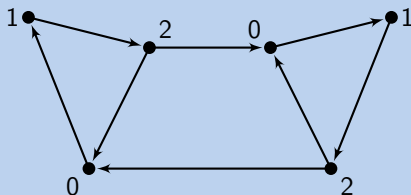


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Direct product of injective exponents

Conjecture.

Directed graphs which have a connected component of algebraic length 1 are exactly the injective exponents.

Remark.

If $\mathbb{C}$ and $\mathbb{D}$ are injective exponents, then $\mathbb{C} \times \mathbb{D}$ is also an injective exponent.

Let $\mathbb{A}$ and $\mathbb{B}$ be two finite, connected directed graphs of algebraic length 1. Is it true that $\mathbb{A} \times \mathbb{B}$ also has a connected component of algebraic length 1?

A related problem for direct product

Which finite directed graphs $\mathbb{C}$ has the property that
 $\mathbb{A} \times \mathbb{C} \cong \mathbb{B} \times \mathbb{C} \Rightarrow \mathbb{A} \cong \mathbb{B}$ for any finite directed graphs $\mathbb{A}, \mathbb{B}$?

Theorem (László Lovász, 1971).

There exist $\mathbb{A}$ and $\mathbb{B}$ non-isomorphic directed graphs such that
 $\mathbb{A} \times \mathbb{C} \cong \mathbb{B} \times \mathbb{C}$ if and only if all the connected components in $\mathbb{C}$ have algebraic length greater than 1.

Directed graphs of algebraic length 1

Definition (Algebraic length).

The algebraic length of a path is

$$|\#\{\text{Forward edges}\} - \#\{\text{Backward edges}\}|.$$

The algebraic length of a connected directed graph is the smallest algebraic length of its closed paths.

Remark.

A finite, connected graph has algebraic length 1 if and only if it has a closed path of algebraic length 1.

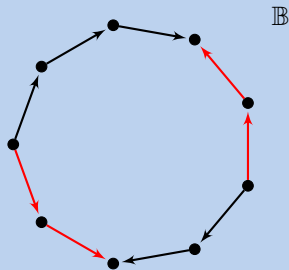
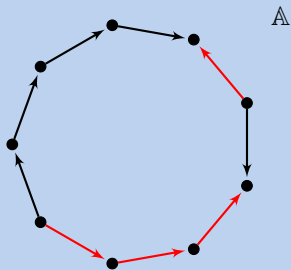
Let $\mathbb{A}$ and $\mathbb{B}$ be two finite, connected directed graphs of algebraic length 1. Then $\mathbb{A} \times \mathbb{B}$ also has a connected component of algebraic length 1.

- There exists a closed path of algebraic length 1 in both $\mathbb{A}$ and $\mathbb{B}$.
- It is sufficient to prove that there exists a closed path in $\mathbb{A} \times \mathbb{B}$ of algebraic length 1.

It can be proved by induction, that the direct product of two directed circle of algebraic length 1 will contain a closed path of algebraic length 1.

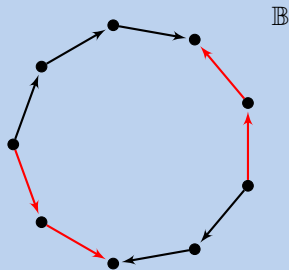
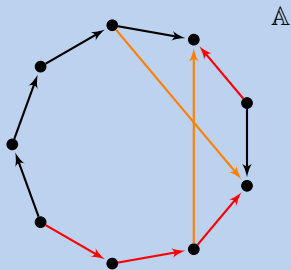
About the proof – The first type of inductive step

Example. Replacing a „forward-backward-forward” walk by one directed edge to obtain a shorter circle of algebraic length 1.



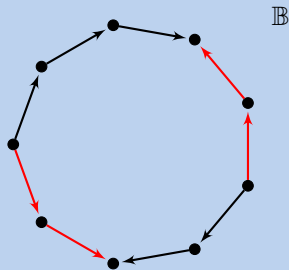
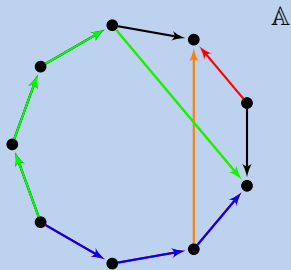
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Ralph McKenzie, *Arithmetic of Finite Ordered Sets: Cancellation of Exponents, II.*, Order **17** (2000), 309-332.

Thank you for your attention!