

COMPLETELY JOIN-PRESERVING MAPS BETWEEN COMMUTATOR LATTICES AS ABSTRACTIONS FOR CONGRUENCE EXTENSIONS

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- 1 Commutator lattices, (minimal) prime elements, Stone/spectral topology and flat/inverse topology
- 2 Some of our main results
 - On the Stone and flat topologies on (minimal) prime spectra
 - On the abstractions of the generalizations of congruence extensions from inclusion maps to arbitrary morphisms

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The term condition commutator

Let:

- $\mathcal{V}$: variety of algebras of a similarity type τ
- $A \in \mathcal{V}$

Definition (won't be used)

The **term condition commutator** of A : $[\cdot, \cdot]_A : \text{Con}(A) \times \text{Con}(A) \rightarrow \text{Con}(A)$,
for any $\alpha, \beta \in \text{Con}(A)$,

$$[\alpha, \beta]_A = \bigcap_{(=\min)} \{ \mu \in \text{Con}(A) \mid (\forall n, k \in \mathbb{N}) (\text{for any } (n+k)\text{-ary term } t \text{ over } \tau), \\ (\forall a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_k, d_1, \dots, d_k, \in A),$$

$$[((a_1, b_1), \dots, (a_n, b_n) \in \alpha \text{ and } (c_1, d_1), \dots, (c_k, d_k) \in \beta) \Rightarrow$$

$$((t^A(a_1, \dots, a_n, c_1, \dots, c_k), t^A(a_1, \dots, a_n, d_1, \dots, d_k)) \in \mu \Leftrightarrow$$

$$(t^A(b_1, \dots, b_n, c_1, \dots, c_k), t^A(b_1, \dots, b_n, d_1, \dots, d_k)) \in \mu)] \}$$

For all $\alpha, \beta, \theta, \zeta \in \text{Con}(A)$: $[\alpha, \beta]_A \subseteq \alpha \cap \beta$ and

$$(\alpha \subseteq \theta \text{ and } \beta \subseteq \zeta) \Rightarrow [\alpha, \beta]_A \subseteq [\theta, \zeta]_A$$

The commutator in congruence-modular/distributive varieties

Let:

- $B \in \mathcal{V}$
- $f : A \rightarrow B$: morphism in $\mathcal{V}$ (if $A \hookrightarrow B$, then $i_{A,B} : A \rightarrow B$ in $\mathcal{V}$)

$$f^* := (f \times f)^{-1} : \text{Con}(B) \rightarrow \text{Con}(A).$$

If $\mathcal{V}$ is congruence-modular, then:

- for all $\alpha, \beta \in \text{Con}(A)$ and any $(\beta_i)_{i \in I} \subseteq \text{Con}(A)$: $[\alpha, \beta]_A = [\beta, \alpha]_A$ and

$$[\alpha, \bigvee_{i \in I} \beta_i]_A = \bigvee_{i \in I} [\alpha, \beta_i]_A$$

(equivalent definition for modular commutator below won't be used)

- for each $M \in \mathcal{V}$, $[\cdot, \cdot]_M : \text{Con}(M) \times \text{Con}(M) \rightarrow \text{Con}(M)$ is unique s.t., for all $\alpha, \beta \in \text{Con}(M)$, $[\alpha, \beta]_M = \min\{\mu \in \text{Con}(M) \mid \mu \subseteq \alpha \cap \beta, (\forall N \in \mathcal{V}) (\forall h : M \rightarrow N) (h : \text{surjective morphism in } \mathcal{V} \Rightarrow (\mu \vee \text{Ker}(h) = h^*([h(\alpha \vee \text{Ker}(h)), h(\beta \vee \text{Ker}(h))]_N)))\}$

- so, if f is surjective, then, for all $\alpha, \beta \in \text{Con}(A)$, $Cg_B(f([\alpha, \beta]_A)) = f([\alpha, \beta]_A \vee \text{Ker}(f)) = [f(\alpha \vee \text{Ker}(f)), f(\beta \vee \text{Ker}(f))]_B = [Cg_B(f(\alpha)), Cg_B(f(\beta))]_B$

If $\mathcal{V}$ is congruence-distributive, then $[\cdot, \cdot]_A = \cap$.

Commutator lattices

Definition

$(L, \wedge, \vee, [\cdot, \cdot], 0, 1) \stackrel{\text{not.}}{=} (L, [\cdot, \cdot])$ is a **commutator lattice** iff:

- $(L, \wedge, \vee, 0, 1)$: complete (bounded) lattice
- the **commutator**: $[\cdot, \cdot] : L \times L \rightarrow L$ s.t., for all $x, y \in L$ and $(y_i)_{i \in I} \subseteq L$:

$$[x, y] = [y, x] \leq x \wedge y \quad \text{and} \quad [x, \bigvee_{i \in I} y_i] = \bigvee_{i \in I} [x, y_i]$$

Let:

- $(L, [\cdot, \cdot])$: commutator lattice

Then, for all $a, b, x, y \in L$: $(a \leq x \text{ and } b \leq y) \Rightarrow [a, b] \leq [x, y]$.

$(L, [\cdot, \cdot] = \wedge)$: commutator lattice iff L : **frame**.

- $[\cdot, \cdot]_A$ and $[\cdot, \cdot]_B$ will be commutative and distributive w.r.t. arbitrary $\vee$

Thus $(\text{Con}(A), \cap, \vee, [\cdot, \cdot]_A, \Delta_A, \nabla_A)$ and $(\text{Con}(B), \cap, \vee, [\cdot, \cdot]_B, \Delta_B, \nabla_B)$ are algebraic (i.e. compactly generated) commutator lattices.

The (minimal) prime spectrum: primes w.r.t. $[\cdot, \cdot]$

The **prime spectrum** of $(L, [\cdot, \cdot])$:

$$\text{Spec}_{(L, [\cdot, \cdot])} := \{p \in L \setminus \{1\} \mid (\forall a, b \in L)([a, b] \leq p \Rightarrow (a \leq p \text{ or } b \leq p))\}$$

$$\text{Spec}(A) := \text{Spec}_{(\text{Con}(A), [\cdot, \cdot]_A)}$$

The **minimal prime spectrum** of $(L, [\cdot, \cdot])$:

$$\text{Min}_{(L, [\cdot, \cdot])} := \text{Min}(\text{Spec}_{(L, [\cdot, \cdot])})$$

$$\text{Min}(A) := \text{Min}_{(\text{Con}(A), [\cdot, \cdot]_A)}$$

$$\begin{aligned} \text{Max}_L &:= \text{Max}(L \setminus \{1\}) \quad (\neq \emptyset \text{ if } |L| > 1 \text{ and } 1 \in \text{Cp}(L); \\ &\subseteq \text{Spec}_{(L, [\cdot, \cdot])} \text{ if } [1, 1] = 1) \end{aligned}$$

$$\text{Max}(A) := \text{Max}_{\text{Con}(A)}$$

- If $\mathcal{V}$: **semidegenerate**, i.e.: $S \hookrightarrow M$ in $\mathcal{V}$ and $|M| > 1 \Rightarrow |S| > 1$, then $\nabla_A \in \text{Cp}(\text{Con}(A))$.
- If $\mathcal{V}$: **congruence-modular and semidegenerate**, then $[\alpha, \nabla_A]_A = \alpha$ for all $\alpha \in \text{Con}(A)$, in particular $[\nabla_A, \nabla_A]_A = \nabla_A$.

Stone/spectral topology on (minimal) primes

The **Stone/spectral topology** on $\text{Spec}(L, [\cdot, \cdot])$:

$$\mathcal{S}_{\text{Spec}, (L, [\cdot, \cdot])} = \{D_{(L, [\cdot, \cdot])}(x) \mid x \in L\}$$

with closed sets : $\{V_{(L, [\cdot, \cdot])}(x) \mid x \in L\}$, where, for each $x \in L$:

- $D_{(L, [\cdot, \cdot])}(x) = \text{Spec}(L, [\cdot, \cdot]) \setminus [x]_L = \{p \in \text{Spec}(L, [\cdot, \cdot]) \mid x \not\leq p\}$
- $V_{(L, [\cdot, \cdot])}(x) = \text{Spec}(L, [\cdot, \cdot]) \cap [x]_L = \{p \in \text{Spec}(L, [\cdot, \cdot]) \mid x \leq p\}$

having the following basis if L is algebraic : $\{D_{(L, [\cdot, \cdot])}(x) \mid x \in \text{Cp}(L)\}$

$$\text{Spec}(L, [\cdot, \cdot]) := (\text{Spec}(L, [\cdot, \cdot]), \mathcal{S}_{\text{Spec}, (L, [\cdot, \cdot])})$$

The **Stone/spectral topology** on $\text{Min}(L, [\cdot, \cdot])$ (induced by the one above;
similarly for Max_L):

$$\mathcal{S}_{\text{Min}, (L, [\cdot, \cdot])} = \{D_{(L, [\cdot, \cdot])}(x) \cap \text{Min}(L, [\cdot, \cdot]) \mid x \in L\}$$

with closed sets : $\{V_{(L, [\cdot, \cdot])}(x) \cap \text{Min}(L, [\cdot, \cdot]) \mid x \in L\}$

having this basis if L is algebraic : $\{D_{(L, [\cdot, \cdot])}(x) \cap \text{Min}(L, [\cdot, \cdot]) \mid x \in \text{Cp}(L)\}$

$$\text{Min}(L, [\cdot, \cdot]) := (\text{Min}(L, [\cdot, \cdot]), \mathcal{S}_{\text{Min}, (L, [\cdot, \cdot])})$$

Flat/inverse topology on minimal primes (and primes in the compact case)

The **flat/inverse topology** on $\text{Min}_{(L, [\cdot, \cdot])}$:

$$\mathcal{F}_{\text{Min}, (L, [\cdot, \cdot])}, \text{ generated by: } \{ V_{(L, [\cdot, \cdot])}(x) \cap \text{Min}_{(L, [\cdot, \cdot])} \mid x \in \text{Cp}(L) \} ,$$

which is a **basis** for $\mathcal{F}_{\text{Min}, (L, [\cdot, \cdot])}$.

$$\mathcal{M}in_{(L, [\cdot, \cdot])}^{-1} := (\text{Min}_{(L, [\cdot, \cdot])}, \mathcal{F}_{\text{Min}, (L, [\cdot, \cdot])})$$

If $L = \text{Cp}(L)$, then the **flat/inverse topology** on $\text{Spec}_{(L, [\cdot, \cdot])}$:

$$\mathcal{F}_{\text{Spec}, (L, [\cdot, \cdot])} = \{ V_{(L, [\cdot, \cdot])}(x) \mid x \in L \} ; \quad \mathcal{S}pec_{(L, [\cdot, \cdot])}^{-1} := (\text{Spec}_{(L, [\cdot, \cdot])}, \mathcal{F}_{\text{Spec}, (L, [\cdot, \cdot])}) .$$

- $\mathcal{M}in(A) := (\text{Min}(A), \mathcal{S}_{\text{Min}, (\text{Con}(A), [\cdot, \cdot]_A)})$
- $\mathcal{M}in^{-1}(A) := (\text{Min}(A), \mathcal{F}_{\text{Min}, (\text{Con}(A), [\cdot, \cdot]_A)})$
- $\mathcal{S}pec(A) := (\text{Spec}(A), \mathcal{S}_{\text{Spec}, (\text{Con}(A), [\cdot, \cdot]_A)})$
- If $\text{Con}(A)$ is compact , then $\mathcal{S}pec^{-1}(A) := (\text{Spec}(A), \mathcal{F}_{\text{Spec}, (\text{Con}(A), [\cdot, \cdot]_A)})$.

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Implication($\rightarrow$), polar($^\perp$), a commutator lattice congruence

For any $a, b \in L$: $a \rightarrow b := \bigvee \{x \in L \mid [a, x] \leq b\} = \max\{x \in L \mid [a, x] \leq b\}$
 $a^\perp := a \rightarrow 0 = \max\{x \in L \mid [a, x] = 0\}$

$$\equiv_{(L, [\cdot, \cdot])} := \{(x, y) \in L \times L \mid V_{(L, [\cdot, \cdot])}(x) = V_{(L, [\cdot, \cdot])}(y)\} \in \text{Con}(L, [\cdot, \cdot]),$$

preserves arbitrary $\bigvee$

and $(\forall x, y \in L) ([x, y] \equiv_{(L, [\cdot, \cdot])} x \wedge y)$

so:

- $L/\equiv_{(L, [\cdot, \cdot])}$ is a frame
- if $\text{Cp}(L)$ is closed w.r.t. $[\cdot, \cdot]$, then (since it's closed w.r.t. $\bigvee$):

$\text{Cp}(L)/\equiv_{(L, [\cdot, \cdot])}$ is a sublattice of $L/\equiv_{(L, [\cdot, \cdot])}$

If $\text{Cp}(\text{Con}(A))$ is closed w.r.t. $[\cdot, \cdot]_A$, $\nabla_A \in \text{Cp}(\text{Con}(A))$ and

$[\nabla_A, \nabla_A]_A = \nabla_A$, then:

$\text{Cp}(\text{Con}(A))/\equiv_{(\text{Con}(A), [\cdot, \cdot]_A)}$ is the **reticulation** of A ,

that is $\text{Spec}(A)$ is homeomorphic to $\text{Spec}(\text{Id}(\text{Cp}(\text{Con}(A)))/\equiv_{(\text{Con}(A), [\cdot, \cdot]_A)}, \cap)$.

$(1 \in \text{Cp}(L) \text{ and } [1, 1] = 1) \Rightarrow 1/\equiv_{(L, [\cdot, \cdot])} = \{1\} \Leftrightarrow (\forall x \in L \setminus \{1\}) (V_{(L, [\cdot, \cdot])}(x) \neq \emptyset)$

Some conditions

Conditions

Condition (CP) :

- L is algebraic
- $\text{Cp}(L)$ is closed w.r.t. $[\cdot, \cdot]$
- $1/\equiv_{(L, [\cdot, \cdot])} = \{1\}$

Condition (ID) :

$$(\forall p \in \text{Min}_{(L/\equiv_{(L, [\cdot, \cdot])}, \wedge)}) \left((p)_{L/\equiv_{(L, [\cdot, \cdot])}} \in \text{Min}_{(\text{Id}(L/\equiv_{(L, [\cdot, \cdot])}), \cap)} \right)$$

$$(L = \text{Cp}(L) \text{ and } [1, 1] = 1) \Rightarrow (L = \text{Cp}(L) \text{ and } 1/\equiv_{(L, [\cdot, \cdot])} = \{1\}) \Rightarrow (L, [\cdot, \cdot]) \models (\text{CP})$$

$$L = \text{Cp}(L) \Rightarrow (L, [\cdot, \cdot]) \models (\text{ID})$$

$$\text{If } \bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0, \text{ then: } \begin{cases} (\forall a, b \in L) ([a, b] = 0 \Leftrightarrow a \wedge b = 0) \\ 0/\equiv_{(L, [\cdot, \cdot])} = \{0\} \end{cases}$$

A is said to be **semiprime** iff $\bigcap \text{Spec}_{(\text{Con}(A), [\cdot, \cdot]_A)} = \Delta_A$.

$\vee$ -semilattice morphisms between commutator lattices abstracting the generalizations of congruence extensions from inclusion maps to arbitrary morphisms

Let:

- $(M, \langle \cdot, \cdot \rangle)$: commutator lattice
- $h : L \rightarrow M$ s.t. h preserves arbitrary $\vee$ (so $h \nearrow$)

Then h has a unique right adjoint h_* , i.e. map $h_* : M \rightarrow L$ satisfying:

$$(\forall a \in L) (\forall b \in M) (h(a) \leq b \Leftrightarrow a \leq h_*(b)) ,$$

namely: for all $b \in M$, $h_*(b) = \bigvee \{a \in L \mid h(a) \leq b\} = \max\{a \in L \mid h(a) \leq b\}$.

Recall that $f : A \rightarrow B$: morphism in $\mathcal{V}$.

$$f^\bullet : \text{Con}(A) \rightarrow \text{Con}(B) , \text{ for all } \alpha \in \text{Con}(A), \quad f^\bullet(\alpha) = Cg_B(f(\alpha))$$

Then:

- $f^\bullet$ preserves arbitrary $\vee$
- the right adjoint of $f^\bullet$ is f^* :

$$(f^\bullet)_* = f^* = (f \times f)^{-1} : \text{Con}(B) \rightarrow \text{Con}(A)$$

- $(\forall a, b \in A) (f^\bullet(Cg_A(a, b)) = Cg_B(f(a), f(b)))$
- hence $f^\bullet(\text{Cp}(\text{Con}(A))) \subseteq \text{Cp}(\text{Con}(B))$

Conditions

For any $S \subseteq L$:

- condition $[,]_{\leq}(S)$: $(\forall a, b \in S) (h([a, b]) \leq \langle h(a), h(b) \rangle)$
- condition $[,]_{=}(S)$: $(\forall a, b \in S) (h([a, b]) = \langle h(a), h(b) \rangle)$

If $S \subseteq T \subseteq L$ s.t. S is $\vee$ -dense in T , then:

$$h \models [,]_{\leq}(S) \Leftrightarrow h \models [,]_{\leq}(T) \text{ and } h \models [,]_{=}(S) \Leftrightarrow h \models [,]_{=}(T) .$$

In particular, if L is algebraic , then:

$$h \models [,]_{\leq}(\text{Cp}(L)) \Leftrightarrow h \models [,]_{\leq}(L) \text{ and } h \models [,]_{=}(\text{Cp}(L)) \Leftrightarrow h \models [,]_{=}(L) .$$

If $\mathcal{V}$ is congruence-modular , then $f^{\bullet} \models [,]_{\leq}(\text{Con}(A))$.

If $\mathcal{V}$ is congruence-modular and f is surjective , then $f^{\bullet} \models [,]_{=}(\text{Con}(A))$.

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Characterization(s) for minimal primes

Proposition

Let $p \in \text{Spec}(L, [\cdot, \cdot])$. Assume that $\bigwedge \text{Spec}(L, [\cdot, \cdot]) = 0$ and let:

$$\textcircled{1} \ p \in \text{Min}(L, [\cdot, \cdot]) ;$$

$$\textcircled{2} \ (\forall a \in \text{Cp}(L)) (a \leq p \Rightarrow a^\perp \not\leq p) ;$$

$$\textcircled{3} \ (\forall a \in \text{Cp}(L)) (a \leq p \Leftrightarrow a^\perp \not\leq p) ;$$

$$\textcircled{4} \ (\forall a \in L) (a \leq p \Rightarrow a^\perp \not\leq p) ;$$

$$\textcircled{5} \ (\forall a \in L) (a \leq p \Leftrightarrow a^\perp \not\leq p) .$$

- If $(L, [\cdot, \cdot]) \models (\text{CP})$, then $\textcircled{1} \Leftrightarrow \textcircled{2} \Leftrightarrow \textcircled{3}$.
- If $(L, [\cdot, \cdot]) \models (\text{ID})$, then $\textcircled{1} \Leftrightarrow \textcircled{4} \Leftrightarrow \textcircled{5}$.
- Thus, if $(L, [\cdot, \cdot]) \models (\text{CP}) \& (\text{ID})$, in particular if $L = \text{Cp}(L)$ and $[1, 1] = 1$, then $\textcircled{1} \Leftrightarrow \textcircled{2} \Leftrightarrow \textcircled{3} \Leftrightarrow \textcircled{4} \Leftrightarrow \textcircled{5}$.

Lattice structures of the Stone/flat topologies on (minimal) primes as bounded sublattices of the power sets of the (minimal) prime spectra

Proposition

- $L/\equiv_{(L,[\cdot,\cdot])} \cong \mathcal{S}_{\text{Spec},(L,[\cdot,\cdot])} \hookrightarrow \mathcal{P}(\text{Spec}_{(L,[\cdot,\cdot])})$, hence:
- $\mathcal{S}_{\text{Min},(L,[\cdot,\cdot])} \hookrightarrow \mathcal{P}(\text{Min}_{(L,[\cdot,\cdot])})$:
bounded sublattices closed w.r.t. arbitrary $\bigcup$.

Thus $\mathcal{S}_{\text{Spec},(L,[\cdot,\cdot])}$ and $\mathcal{S}_{\text{Min},(L,[\cdot,\cdot])}$ are frames .

Proposition

- $|L| < \infty \Rightarrow |\text{Min}_{(L,[\cdot,\cdot])}| < \infty \Rightarrow \mathcal{S}_{\text{Min},(L,[\cdot,\cdot])} = \mathcal{P}(\text{Min}_{(L,[\cdot,\cdot])})$.
- $|L| < \infty \Rightarrow (|\text{Min}_{(L,[\cdot,\cdot])}| < \infty \text{ and } L = \text{Cp}(L)) \Rightarrow$
 $(|\text{Min}_{(L,[\cdot,\cdot])}| < \infty \text{ and } (\forall S \subseteq \text{Min}_{(L,[\cdot,\cdot])}) (\bigwedge S \in \text{Cp}(L))) \Rightarrow$
 $\mathcal{F}_{\text{Min},(L,[\cdot,\cdot])} = \mathcal{P}(\text{Min}_{(L,[\cdot,\cdot])})$.

Proposition (flat versus Stone topologies on minimal primes)

If $\bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0$ and $(L, [\cdot, \cdot]) \models ((\text{CP}) \text{ or } (\text{ID}))$, then:

- 1 $\mathcal{F}_{\text{Min}, (L, [\cdot, \cdot])} \subseteq \mathcal{S}_{\text{Min}, (L, [\cdot, \cdot])}$;
- 2 if $\text{Cp}(L)$ is closed w.r.t. $\cdot^\perp$, in particular if $L = \text{Cp}(L)$, then $\mathcal{F}_{\text{Min}, (L, [\cdot, \cdot])} = \mathcal{S}_{\text{Min}, (L, [\cdot, \cdot])}$, that is:

$$\text{Min}_{(L, [\cdot, \cdot])}^{-1} = \mathcal{M}_{\text{in}}(L, [\cdot, \cdot]) .$$

Lemma (abstract version of the reticulation property)

If $(L, [\cdot, \cdot]) \models (\text{CP})$, then:

- 1 $\text{Spec}_{(L, [\cdot, \cdot])}$ and $\text{Spec}_{(\text{Id}(\text{Cp}(L)/\equiv_{(L, [\cdot, \cdot])}), \cap)}$ are homeomorphic ;
- 2 $\text{Min}_{(L, [\cdot, \cdot])}$ and $\text{Min}_{(\text{Id}(\text{Cp}(L)/\equiv_{(L, [\cdot, \cdot])}), \cap)}$ are homeomorphic ;
- 3 $\text{Min}_{(L, [\cdot, \cdot])}^{-1}$ and $\text{Min}_{(\text{Id}(\text{Cp}(L)/\equiv_{(L, [\cdot, \cdot])}), \cap)}^{-1}$ are homeomorphic .

Compactness and separability of the Stone/flat topologies

Proposition

If $(L, [\cdot, \cdot]) \models (\text{CP})$, then:

$\text{Min}_{(L, [\cdot, \cdot])}^{-1}$ is compact T_1 .

Theorem

If $\bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0$ and $(L, [\cdot, \cdot]) \models (\text{CP})$, then the following are equivalent:

- ① $\text{Min}_{(L, [\cdot, \cdot])} = \text{Min}_{(L, [\cdot, \cdot])}^{-1}$;
- ② $\text{Min}_{(L, [\cdot, \cdot])}$ is compact;
- ③ $\text{Min}_{(L, [\cdot, \cdot])}$ is compact T_1 ;
- ④ $(\forall a \in \text{Cp}(L)) (\exists b \in \text{Cp}(L)), (b \leq a^\perp \text{ and } (a \vee b)^\perp = 0)$.

$\Uparrow$

$\text{Cp}(L)$ is closed w.r.t. $\cdot^\perp$

Proposition

If $1 \in \text{Cp}(L)$ and $\text{Spec}_{(L, [\cdot, \cdot])}$ is an antichain, then

$\text{Min}_{(L, [\cdot, \cdot])} (= \text{Spec}_{(L, [\cdot, \cdot])})$ is compact.

Theorem

If:

- $\bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0$,
- $\text{Cp}(L)$ is closed w.r.t. $\cdot^\perp$ and
- $(L, [\cdot, \cdot]) \models ((\text{CP}) \text{ or } (\text{ID}))$,

in particular if $\bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0$ and $L = \text{Cp}(L)$,

then $\text{Min}_{(L, [\cdot, \cdot])}$ is Hausdorff and consists solely of clopens, thus

$\mathcal{S}_{\text{Min}, (L, [\cdot, \cdot])}$ is a complete Boolean sublattice of $\mathcal{P}(\text{Min}_{(L, [\cdot, \cdot])})$.

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Lemma ("surjectivity" of h_* on minimal primes)

If L and M are algebraic, $1 \in \text{Cp}(M)$, $h(\text{Cp}(L)) \subseteq \text{Cp}(M)$,

$h \models [\cdot]_{\leq}(\text{Cp}(L))$, then:

- if $h_*(\text{Spec}_{(M, \langle \cdot, \cdot \rangle)}) \subseteq \text{Spec}_{(L, [\cdot, \cdot])}$, then $h_*(\text{Min}_{(M, \langle \cdot, \cdot \rangle)}) \supseteq \text{Min}_{(L, [\cdot, \cdot])}$
- if $h_*(\text{Min}_{(M, \langle \cdot, \cdot \rangle)}) \subseteq \text{Min}_{(L, [\cdot, \cdot])}$, then $h_*(\text{Min}_{(M, \langle \cdot, \cdot \rangle)}) = \text{Min}_{(L, [\cdot, \cdot])}$

We call $f : A \rightarrow B$:

- **admissible** iff $f^*(\text{Spec}_{(\text{Con}(B), [\cdot, \cdot]_B)}) \subseteq \text{Spec}_{(\text{Con}(A), [\cdot, \cdot]_A)}$
- **Min-admissible** iff $f^*(\text{Min}_{(\text{Con}(B), [\cdot, \cdot]_B)}) \subseteq \text{Min}_{(\text{Con}(A), [\cdot, \cdot]_A)}$

If $A \hookrightarrow B$, then:

- $A \hookrightarrow B$ is said to be **admissible** iff $i_{A,B}$ is admissible
- $A \hookrightarrow B$ is said to be **Min-admissible** / an **m -extension** iff $i_{A,B}$ is Min-admissible
- $i_{A,B}^*(\beta) = \beta \cap \nabla_A$ for all $\beta \in \text{Con}(B)$, thus, by the previous lemma, if $\nabla_B \in \text{Cp}(\text{Con}(B))$ and $f^\bullet \models [\cdot]_{\leq}(\text{Cp}(\text{Con}(A)))$, in particular if $\mathcal{V}$ is congruence-modular and semidegenerate, then:
if $A \hookrightarrow B$ is admissible or Min-admissible, then $A \hookrightarrow B$ satisfies property Lying Over for minimal prime congruences

Theorem

If L and M are algebraic , $1 \in \text{Cp}(M)$, $h(\text{Cp}(L)) \subseteq \text{Cp}(M)$, $h \models [\cdot, \cdot]_{\leq}(\text{Cp}(L))$ and $h_*(\text{Min}_{(M, \langle \cdot, \cdot \rangle)}) \subseteq \text{Min}_{(L, [\cdot, \cdot])}$, then:

① $h_* : \text{Min}_{(M, \langle \cdot, \cdot \rangle)} \rightarrow \text{Min}_{(L, [\cdot, \cdot])}$ is continuous ;

② $h_* : \text{Min}_{(M, \langle \cdot, \cdot \rangle)}^{-1} \rightarrow \text{Min}_{(L, [\cdot, \cdot])}^{-1}$ is continuous ;

③ if one of the following holds:

- $\bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0$ and $(L, [\cdot, \cdot]) \models ((\text{CP}) \text{ or } (\text{ID}))$,
- $\bigwedge \text{Spec}_{(M, \langle \cdot, \cdot \rangle)} = 0$ and $(M, \langle \cdot, \cdot \rangle) \models ((\text{CP}) \text{ or } (\text{ID}))$,

then $h_* : \text{Min}_{(M, \langle \cdot, \cdot \rangle)} \rightarrow \text{Min}_{(L, [\cdot, \cdot])}^{-1}$ is continuous ;

④ if one of the following holds:

- $\bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0$, $(L, [\cdot, \cdot]) \models ((\text{CP}) \text{ or } (\text{ID}))$ and $(\forall a \in \text{Cp}(L)) (h(a^\perp) \in \text{Cp}(M))$ ($\Leftarrow \text{Cp}(L)$ is closed w.r.t. $^\perp$) ,
- $\bigwedge \text{Spec}_{(M, \langle \cdot, \cdot \rangle)} = 0$, $(M, \langle \cdot, \cdot \rangle) \models ((\text{CP}) \text{ or } (\text{ID}))$ and $(\forall a \in \text{Cp}(L)) (h(a)^\perp \in \text{Cp}(M))$ ($\Leftarrow \text{Cp}(M)$ is closed w.r.t. $^\perp$) ,

then $h_* : \text{Min}_{(M, \langle \cdot, \cdot \rangle)}^{-1} \rightarrow \text{Min}_{(L, [\cdot, \cdot])}$ is continuous .

If, furthermore, $h_* \upharpoonright_{\text{Min}(M, \langle \cdot, \cdot \rangle)} : \text{Min}(M, \langle \cdot, \cdot \rangle) \rightarrow \text{Min}(L, [\cdot, \cdot])$ is injective, then:

- if $h(L) \supseteq \text{Cp}(M)$, then $h_* : \text{Min}(M, \langle \cdot, \cdot \rangle) \rightarrow \text{Min}(L, [\cdot, \cdot])$ is a homeomorphism ;
 - if $h(\text{Cp}(L)) = \text{Cp}(M)$, then $h_* : \text{Min}^{-1}_{(M, \langle \cdot, \cdot \rangle)} \rightarrow \text{Min}^{-1}_{(L, [\cdot, \cdot])}$ is a homeomorphism ;
 - if one of the following holds:
 - $\bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0$, $(L, [\cdot, \cdot]) \models ((\text{CP}) \text{ or } (\text{ID}))$, $h(\text{Cp}(L)) = \text{Cp}(M)$ and $\text{Cp}(L)$ is closed w.r.t. $^\perp$,
 - $\bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0$, $(L, [\cdot, \cdot]) \models ((\text{CP}) \text{ or } (\text{ID}))$ and $\{h(a^\perp) \mid a \in \text{Cp}(L)\} \supseteq \text{Cp}(M)$,
 - $\bigwedge \text{Spec}_{(M, \langle \cdot, \cdot \rangle)} = 0$, $(M, \langle \cdot, \cdot \rangle) \models ((\text{CP}) \text{ or } (\text{ID}))$ and $\{h(a)^\perp \mid a \in \text{Cp}(L)\} \supseteq \text{Cp}(M)$,
- then $h_* : \text{Min}(M, \langle \cdot, \cdot \rangle) \rightarrow \text{Min}^{-1}_{(L, [\cdot, \cdot])}$ is a homeomorphism ;
- if one of the following holds:
 - $\bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0$, $(L, [\cdot, \cdot]) \models ((\text{CP}) \text{ or } (\text{ID}))$ and $h(\text{Cp}(L)) = \text{Cp}(M)$,
 - $\bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0$, $(L, [\cdot, \cdot]) \models ((\text{CP}) \text{ or } (\text{ID}))$ and $\{h(a^\perp) \mid a \in \text{Cp}(L)\} = \text{Cp}(M)$,
 - $\bigwedge \text{Spec}_{(L, [\cdot, \cdot])} = 0$, $(L, [\cdot, \cdot]) \models (\text{ID})$ and $\{h(a^\perp) \mid a \in L\} \supseteq \text{Cp}(M) \supseteq \{h(a^\perp) \mid a \in \text{Cp}(L)\}$,

- $\bigwedge \text{Spec}_{(M, \langle \cdot, \cdot \rangle)} = 0$, $(M, \langle \cdot, \cdot \rangle) \models ((\text{CP}) \text{ or } (\text{ID}))$ and $\{h(a)^\perp \mid a \in \text{Cp}(L)\} = \text{Cp}(M)$,
 - $\bigwedge \text{Spec}_{(M, \langle \cdot, \cdot \rangle)} = 0$, $(M, \langle \cdot, \cdot \rangle) \models (\text{ID})$ and
- $$\{h(a)^\perp \mid a \in L\} \supseteq \text{Cp}(M) \supseteq \{h(a)^\perp \mid a \in \text{Cp}(L)\} ,$$
- then $h_* : \text{Min}_{(M, \langle \cdot, \cdot \rangle)}^{-1} \rightarrow \text{Min}_{(L, [\cdot, \cdot])}$ is a homeomorphism .



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THANK YOU FOR YOUR ATTENTION!