

Critical elements in algebras of numerical events

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Outline:

- Algebras of numerical events
- S -probabilities
- Examples
- Destructive and critical S -probabilities

Algebras of numerical events

Algebras of numerical events

Definition 1

(*M. J. Mączyński and T. Traczyk 1973*) Let S be a set. A subset P of $[0, 1]^S$ is called an *algebra of numerical events*, or more precisely, an *algebra of S -probabilities*, if the following hold for all $p, q, r \in P$:

- $0 \in P$,
- $p' := 1 - p \in P$,
- if $p \perp q \perp r \perp p$ then $p + q + r \in P$.

Here and in the following $p \perp q$ means $p + q \leq 1$. Putting $r = 0$ in the last condition shows that $p + q \in P$ whenever $p, q \in P$ and $p \perp q$.

Example 2

$\{0, 1, x, 1 - x\}$ is an algebra of $[0, 1]$ -probabilities.

Orthomodular posets

$\mathbf{P} = (P, \leq, ', 0, 1)$ is called an **orthomodular poset** if $(P, \leq, 0, 1)$ is a bounded poset, $'$ is an antitone involution on $(P, \leq)$, $x \vee y$ is defined whenever $x \perp y$ (which means $x \leq y'$) and $y = x \vee (y \wedge x')$ whenever $x \leq y$. A mapping $s: P \rightarrow [0, 1]$ is called a **state** on $\mathbf{P}$ if $s(1) = 1$ and $s(x \vee y) = s(x) + s(y)$ whenever $x \perp y$. A **set S of states** on $\mathbf{P}$ is called **full** if $s(x) \leq s(y)$ for all $s \in S$ implies $x \leq y$.

Representation Theorem

Theorem 3

(*M. J. Mączyński and T. Traczyk 1973*) *Up to isomorphism, the algebras of numerical events are exactly the orthomodular posets having a full set of states.*

Idea of proof. If $P \subseteq [0, 1]^S$ is an algebra of S -probabilities then it is an orthomodular poset having $\{p \mapsto p(s) \mid s \in S\} \subseteq [0, 1]^P$ as a full set of states. If, conversely, P is an orthomodular poset having a full set S of states then $\{s \mapsto s(p) \mid p \in P\} \subseteq [0, 1]^S$ is an algebra of S -probabilities isomorphic to P .

P corresponds to **classical mechanics** if and only if it is a **Boolean algebra**.

Direct products

Theorem 4

Let $S_i, i \in I$, be pairwise disjoint sets, for every $i \in I$ let $P_i \subseteq [0, 1]^{S_i}$ be an algebra of S_i -probabilities and put $S := \bigcup_{i \in I} S_i$ and $P := \{p \in [0, 1]^S \mid p|_{S_i} \in P_i \text{ for all } i \in I\}$. Then $P \subseteq [0, 1]^S$ is an algebra of S -probabilities, called the **direct product** of the algebras $P_i \subseteq [0, 1]^{S_i}$ of S_i -probabilities, $i \in I$.

Boolean algebras

A **filter** of a lattice $\mathbf{L} = (L, \vee, \wedge)$ is a non-empty subset F of L satisfying

- if $x, y \in F$ then $x \wedge y \in F$,
- if $x \in F$, $y \in L$ and $x \leq y$ then $y \in F$.

A **filter** F of $\mathbf{L}$ is called **proper** if $F \neq L$. An **ultrafilter** of $\mathbf{L}$ is a maximal proper filter of $\mathbf{L}$.

Theorem 5

Let $\mathbf{B} = (B, \vee, \wedge, ', 0, 1)$ be a Boolean algebra, let $\mathcal{U}$ denote the set of its ultrafilters, and for every $U \in \mathcal{U}$ define $s_U: B \rightarrow [0, 1]$ by

$$s_U(p) := \begin{cases} 1 & \text{if } p \in U, \\ 0 & \text{otherwise} \end{cases}$$

for all $p \in B$. Then $\{s_U \mid U \in \mathcal{U}\}$ is a full set of states on $\mathbf{B}$. Hence every Boolean algebra can be considered as an algebra of S -probabilities.

S -probabilities

Properties of S -probabilities

Definition 6

Let P be an algebra of S -probabilities and q an S -probability not belonging to P . Then q is called

- **destructive** if $P \cup \{q\}$ cannot be embedded into an algebra of S -probabilities,
- **critical** if either P is a Boolean algebra, but $P \cup \{q\}$ cannot be embedded into a Boolean algebra, or if P is not a Boolean algebra, but $P \cup \{q\}$ can be embedded into a Boolean algebra.

Definition 7

An S -probability p is called

- **trivial** if $p \in \{0, 1\}$,
- **varying** if there exist $s_1, s_2 \in S$ with $p(s_1) < 1/2 < p(s_2)$,
- **proper** if p is non-trivial and varying.

Properties of S -probabilities (continued)

Definition 8

Proper S -probabilities p and q are called

- *partially reciprocal below $1/2$ if $\min\{p(s), q(s)\} \leq 1/2$ for all $s \in S$,*
- *partially reciprocal above $1/2$ if $\max\{p(s), q(s)\} \geq 1/2$ for all $s \in S$,*
- *reciprocal if they are both partially reciprocal below and above $1/2$.*

Examples

Examples

Example 9

Let $|S| = 2$, $P := \{(0,0), (1/8, 5/8), (7/8, 3/8), (1,1)\}$ and $q := (3/8, 7/8)$. Then $P \cup \{q\}$ cannot be embedded in an algebra of S -probabilities since $(1/8, 5/8) + q' = (6/8, 6/8)$ is non-trivial and not varying. Hence q is destructive (see page 15).

Example 10

Let $|S| = 4$,
 $P := \{(0,0,0,0), (0,1,0,1), (0,1,1,0), (1,0,0,1), (1,0,1,0), (1,1,1,1)\}$
and q an S -probability not belonging to P . Then P is not a Boolean algebra, but $P \cup \{q\}$ can be embedded into the Boolean algebra $\{0,1\}^4$. Hence q is critical.

Destructive and critical S -probabilities

Destructive S -probabilities

Lemma 11

Every non-trivial element p of an algebra of S -probabilities is varying.

Proof. If $p \leq 1/2$ then $p \perp p \perp p' \perp p$ and hence $p + 1 = p + p + p' \leq 1$ whence $p = 0$, a contradiction. If $p \geq 1/2$ then $p \perp p' \perp p' \perp p$ and hence $1 + p' = p + p' + p' \leq 1$ whence $p' = 0$, i.e. $p = 1$, a contradiction.

Corollary 12

If P is an algebra of S -probabilities then every not varying S -probability not belonging to P is destructive.

Critical S-probabilities

Theorem 13

Let P be an algebra of S-probabilities consisting of the six elements $0, p_1, p'_1, p_2, p'_2, 1$ and q a varying S-probability not belonging to P and assume $p_1 - q, p_2 - q$ and $1 + q - p_1 - p_2$ to be varying. Then q is critical.

Theorem 14

Let P be a Boolean algebra of S-probabilities and q a varying S-probability not belonging to P . Then q is critical if there exists some $p \in P$ such that one of the following conditions hold:

- p and q are partially reciprocal below $1/2$ and there exists some $s \in S$ with $p(s) + q(s) > 1$,*
- p and q are partially reciprocal above $1/2$ and there exists some $s \in S$ with $p(s) + q(s) < 1$,*
- p and q are reciprocal.*

Critical S-probabilities (continued)

Theorem 15

Let P be an algebra of S-probabilities that is not a Boolean algebra and q an S-probability not belonging to P . Then q is not critical if one of the following conditions holds:

- there exists some non-trivial $p \in P$ such that p, q, p', q' are pairwise different and pairwise reciprocal,*
- there exist non-trivial $p_1, p_2 \in P$ such that p_1, p_2, p'_1, p'_2 are pairwise different and pairwise reciprocal,*
- there exist $p_1, p_2 \in P$ such that p_1, q and p_2, q are partially reciprocal below $1/2$, $p_1 \perp p_2$ and $0 \neq q \leq p_1 + p_2$.*

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Thank you for your attention!