

Stability of Mapping Spaces in ε -Simplicial Sets

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Mapping objects often inherit the target's structure

Ab	$\text{Hom}(A, B) \in \mathbf{Ab}$
Cat	$\text{Fun}(A, B) \in \mathbf{Cat}$
Grpd	$\text{Fun}(A, G) \in \mathbf{Grpd}$
Kan complexes	$\text{Map}(K, Y) \in \mathbf{Kan}$
quasi-categories	$\text{Fun}(K, Y) \in \mathbf{qCat}$

Stability pattern:

$$Y \in \mathcal{F}, \quad (X \in \mathcal{F}) \implies \text{Fun}(X, Y) \in \mathcal{F}.$$

When does the same principle hold for ε -fibrant
simplicial sets?

Motivation



C. Heunen, I. Contreras, A. S. Cattaneo. *Relative Frobenius algebras are groupoids*. (2013).

I. Contreras, M. Keller, R. A. Mehta. *Frobenius objects in the category of spans*. (2022).

R. A. Mehta, R. Zhang. *Frobenius objects in the category of relations*. (2020).

W. H. Stern. *2-Segal objects and algebras in spans*. (2021).

I. Contreras, R. A. Mehta, W. H. Stern. *Frobenius and commutative pseudomonoids in the bicategory of spans*. (2025).

D. Lachman. *Simplicial Approach to Frobenius Algebras in the Category of Relations*. (2026).

D. Lachman. *Simplicial Perspectives on (Pseudo) Effect Algebras*. (2025).

Formal definitions

Simplicial set

A functor

$$X: \Delta^{\text{op}} \longrightarrow \mathbf{Set}.$$

Equivalently, sets X_n with face and degeneracy maps

$$d_i: X_n \rightarrow X_{n-1}, \quad s_i: X_n \rightarrow X_{n+1},$$

satisfying

$$d_i d_j = d_{j-1} d_i \quad (i < j),$$

$$s_i s_j = s_{j+1} s_i \quad (i \leq j),$$

$$d_i s_j = \begin{cases} s_{j-1} d_i, & i < j, \\ \text{id}, & i = j, j+1, \\ s_j d_{i-1}, & i > j+1. \end{cases}$$

ε -simplicial set

A simplicial set X , a set X_ε , and a map

$$X_\varepsilon \longrightarrow X_1.$$

Elements of X_ε are witnesses of the marked ε -edges.

Paracyclic set

A simplicial set X with bijections

$$t_n: X_n \xrightarrow{\sim} X_n$$

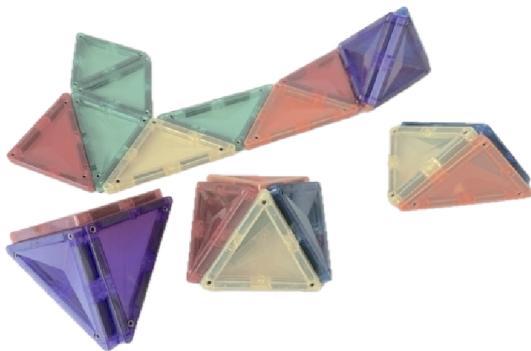
satisfying

$$d_i t_n = \begin{cases} d_n, & i = 0, \\ t_{n-1} d_{i-1}, & 1 \leq i \leq n, \end{cases}$$

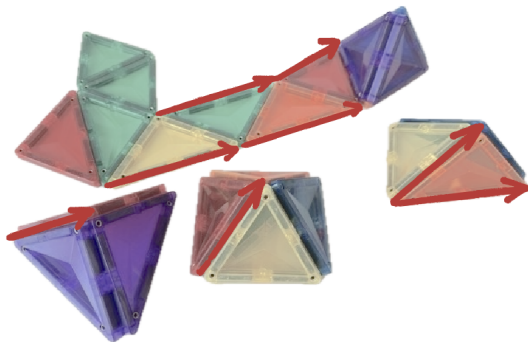
and

$$s_i t_n = \begin{cases} t_{n+1}^2 s_n, & i = 0, \\ t_{n+1} s_{i-1}, & 1 \leq i \leq n. \end{cases}$$

Good enough definition of simplicial sets



Good enough definition of ε -simplicial sets



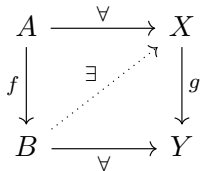
Key notion

Let $f: A \rightarrow B$ and $g: X \rightarrow Y$ be morphisms of a category.

We say g has **the right lifting property with respect to f**

$$f \sqsupseteq g$$

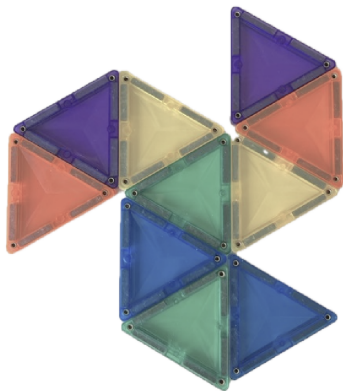
if



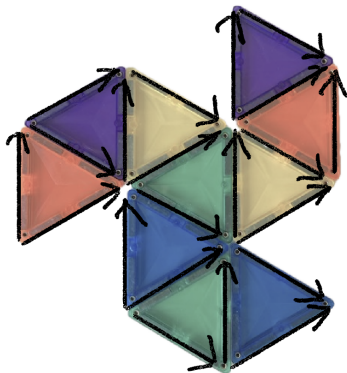
In simplicial sets, a morphism g is a **trivial Kan fibration** if:

$$\{\text{monomorphisms}\} \sqsupseteq g$$

Capturing algebraic data in a simplicial set



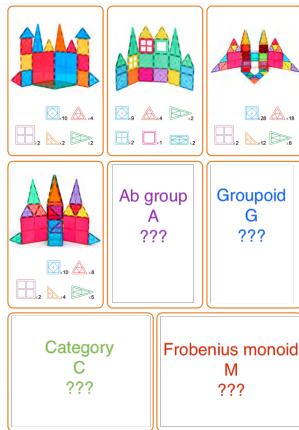
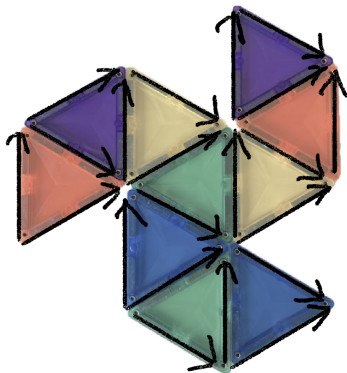
Capturing algebraic data in a simplicial set



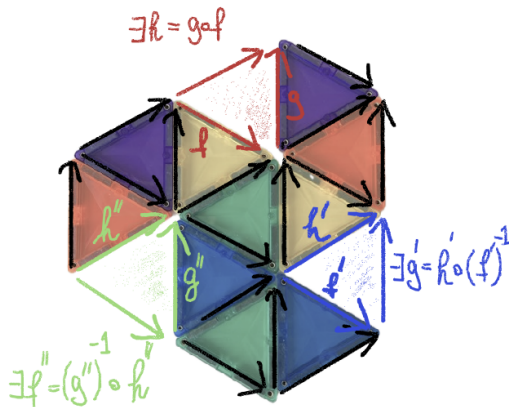
Edges are oriented arrows.

Tiles are perfectly flexible:
they can be freely stretched and
compressed.

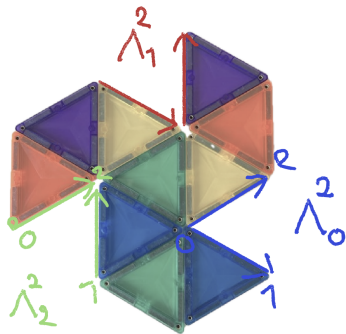
Capturing algebraic data in a simplicial set



Capturing algebraic data in a simplicial set



Capturing algebraic data in a simplicial set



Category $\mathcal{C}$

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & N(\mathcal{C}) \\ \downarrow & \nearrow \exists! & \\ \Delta^n & & \end{array} \quad \begin{array}{l} 1 \leq i \leq n-1, \\ 2 \leq n \end{array}$$

Groupoid $\mathcal{G}$

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & N(\mathcal{G}) \\ \downarrow & \nearrow \exists! & \\ \Delta^n & & \end{array} \quad \begin{array}{l} 0 \leq i \leq n, \\ 2 \leq n \end{array}$$

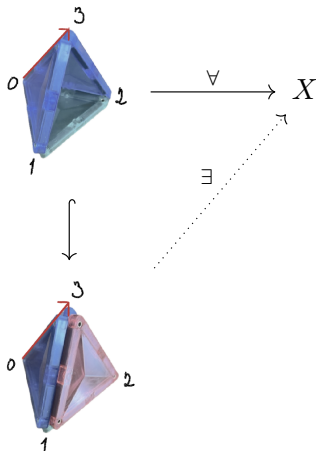
$$\exists! \rightsquigarrow \exists : \quad \left\{ \begin{array}{l} \text{categories} \rightsquigarrow \infty\text{-categories,} \\ \text{groupoids} \rightsquigarrow \infty\text{-groupoids.} \end{array} \right.$$

Capturing rotational data in an ε -simplicial set

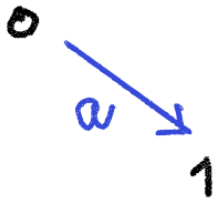
An ε -simplicial set “has enough structure to model rotations” if

$$(\mathbf{\Delta}_i^n \hookrightarrow \mathbf{\Delta}^n) \sqsupseteq X$$

for all $n \geq 1$, $i \in \{0, n\}$.

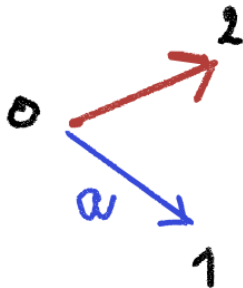


Capturing rotational data in an ε -simplicial set



a is an edge of an ε -simplicial set X which we want to **rotate**.

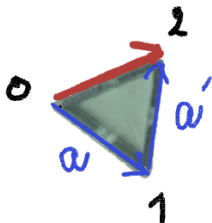
Capturing rotational data in an ε -simplicial set



Attach an ε -edge with source 0 .

$$\Lambda_0^1 \hookrightarrow \Delta^1$$

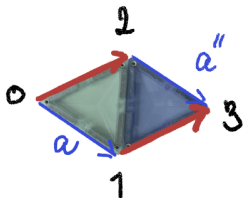
Capturing rotational data in an ε -simplicial set



Fill the ε -horn.

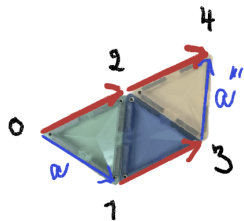
$$\Lambda_0^2 \hookrightarrow \Delta^2$$

Capturing rotational data in an ε -simplicial set



Repeat the process.

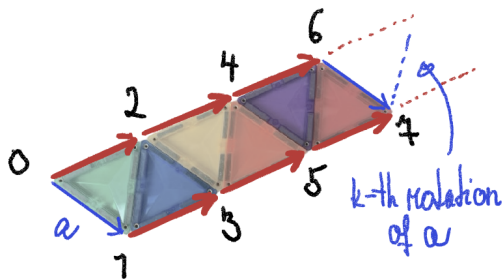
Capturing rotational data in an ε -simplicial set



Repeat the process...
repeat again.

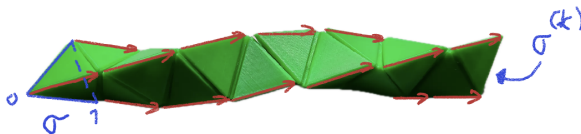
Capturing rotational data in an ε -simplicial set

ε -horns

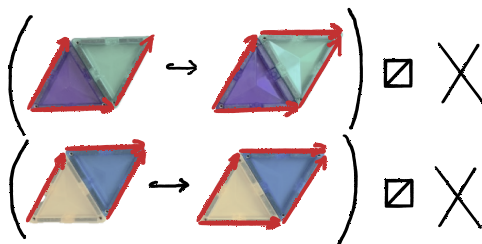


$$\begin{array}{ccc} \Lambda_i^n & \rightarrow & X \\ \downarrow & \nearrow \exists & \\ \Delta^n & & \end{array}$$

$$i \in \{0, n\}$$



ε -Coherence conditions



Definition

We call an ε -simplicial set X ε -fibrant if it satisfies the following properties:

$$(\Lambda_i^n \hookrightarrow \Delta^n) \square X, \quad i = 0, n, \quad n \geq 1, \quad (1)$$

$$X \text{ satisfies the two } \varepsilon\text{-coherence conditions.} \quad (2)$$

Joyal's stability theorem

Let Y be a simplicial set. We say that Y is a *quasi-category* if it satisfies the inner horn-filling condition:

$$(\Lambda_i^n \hookrightarrow \Delta^n) \boxtimes Y, \quad 1 \leq i \leq n-1$$

A theorem of Joyal states:

Theorem

Let X and Y be simplicial sets. If Y is a quasi-category, then $\mathrm{Fun}(X, Y)$ is also a quasi-category.

Let X, Y be ε -simplicial sets.

$$X, Y \text{ } \varepsilon\text{-fibrant} \xRightarrow{???} \mathrm{Fun}(X, Y) \text{ is } \varepsilon\text{-fibrant}$$

A. Joyal, *The Theory of Quasi-Categories and Its Applications*, lecture notes, 2008.

Joyal's proof provides the template

1. Abstract machinery.

Weakly saturated classes, cellular decompositions, and pushout-products reduce the problem to

$$(\partial\Delta^k \hookrightarrow \Delta^k) \square (\Lambda_i^n \hookrightarrow \Delta^n) \in \overline{\text{InnHorn}}, \quad 0 < i < n, \quad k \geq 0.$$

2. A clever reduction.

A retract argument reduces a key exceptional case to $n = 2$.

3. Explicit combinatorics.

Some simplicial combinatorics simply has to be done.

$$\begin{array}{ccccc}
\Lambda_i^n & \longrightarrow & \mathrm{Fun}(X, Y) & & \\
\downarrow & \nearrow & & & \\
\Delta^n & & & &
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccccc}
\Lambda_i^n \times X & \longrightarrow & Y & & \\
\downarrow & \nearrow & & & \\
\Delta^n \times X & & & &
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccccc}
\emptyset & \longrightarrow & \mathrm{Fun}(\Delta^n, Y) & & \\
\downarrow & \nearrow & \downarrow_{\mathrm{res}} & & \\
X & \longrightarrow & \mathrm{Fun}(\Lambda_i^n, Y) & &
\end{array}$$

$$\begin{array}{ccccc}
\Lambda_i^n & \longrightarrow & \text{Fun}(X, Y) & & \emptyset \longrightarrow \text{Fun}(\Delta^n, Y) \\
\downarrow & \nearrow \text{dashed} & & \Leftrightarrow & \downarrow \quad \nearrow \text{dashed} \quad \downarrow \text{res} \\
\Delta^n & & & & X \longrightarrow \text{Fun}(\Lambda_i^n, Y).
\end{array}$$

Every simplicial set X has a **cellular decomposition**. Hence it suffices to consider one elementary step

$$\partial\Delta^k \hookrightarrow \Delta^k,$$

corresponding to attaching a simplex along its boundary.

$$\begin{array}{ccccc}
\Lambda_i^n & \longrightarrow & \text{Fun}(X, Y) & & \Lambda_i^n \times X \longrightarrow Y \\
\downarrow & \nearrow & & \Leftrightarrow & \downarrow \nearrow \\
\Delta^n & & & & \Delta^n \times X
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccccc}
\emptyset & \longrightarrow & \text{Fun}(\Delta^n, Y) & & \\
\downarrow & \nearrow & \downarrow_{\text{res}} & & \\
X & \longrightarrow & \text{Fun}(\Lambda_i^n, Y) & &
\end{array}$$

Every simplicial set X has a **cellular decomposition**. Hence it suffices to consider one elementary step

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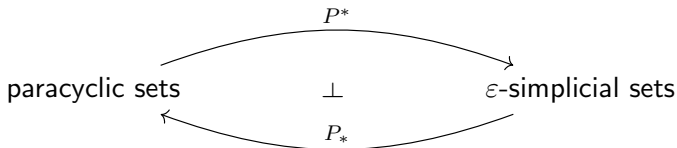
$$\begin{array}{ccc}
\partial\Delta^k \longrightarrow \text{Fun}(\Delta^n, Y) & & \partial\Delta^k \times \Delta^n \coprod_{\partial\Delta^k \times \Lambda_i^n} \Delta^k \times \Lambda_i^n \longrightarrow Y \\
\downarrow \nearrow \downarrow_{\text{res}} & \Longleftrightarrow & \downarrow \nearrow \\
\Delta^k \longrightarrow \text{Fun}(\Lambda_i^n, Y) & & \Delta^k \times \Delta^n
\end{array}$$

The left vertical map in the latter lifting problem is the box product

$$(\partial\Delta^k \hookrightarrow \Delta^k) \square (\Lambda_i^n \hookrightarrow \Delta^n).$$

By Joyal's lemma, this map is inner anodyne, so the lift exists.

My proof



Horn-filling characterization

X admits fillers of all ε -horns

$\iff \epsilon_X: P^*P_*X \longrightarrow X$ is a trivial Kan fibration.

$$\begin{array}{ccc}
 \text{Rot}(\sigma) & \xrightarrow{\quad} & \text{Map}(\Delta^n, P^*P_*X) \\
 \downarrow & \lrcorner & \downarrow (\epsilon_X)_* \\
 \Delta^0 & \xrightarrow{\quad \sigma \quad} & \text{Map}(\Delta^n, X)
 \end{array}
 \qquad \text{Rot}(\sigma) \simeq \Delta^0.$$

My trick

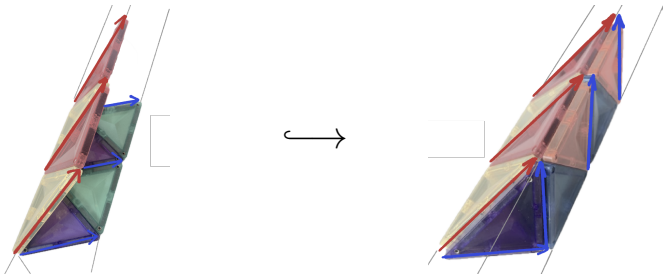
Moreover, there is an extension $\hat{\epsilon}_X: \hat{X} \rightarrow X$ such that

$$X \text{ is } \varepsilon\text{-fibrant} \iff \{\text{monomorphisms in } \text{Set}_{\Delta}^{\varepsilon}\} \boxtimes \hat{\epsilon}_X$$

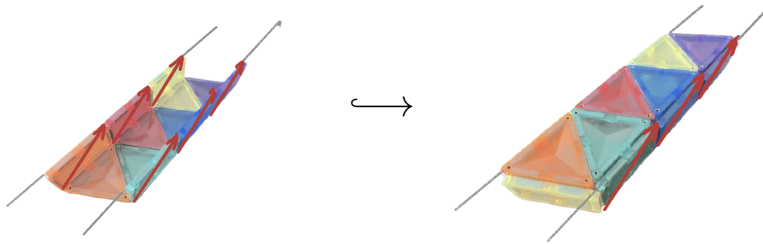
In particular, ε -fibrant X is a retract of $\hat{X}$, so $\text{Fun}(X, Y)$ is a retract of $\text{Fun}(\hat{X}, Y)$.

$$\begin{array}{ccc}
 \emptyset & \hookrightarrow & \hat{X} \\
 \downarrow & \nearrow s & \downarrow \hat{\epsilon}_X \\
 X & \xlongequal{\quad} & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & \text{Fun}(\hat{X}, Y) \\
 & \nwarrow s^* & \uparrow (\hat{\epsilon}_X)^* \\
 \text{Fun}(X, Y) & \xlongequal{\quad} & \text{Fun}(X, Y)
 \end{array}$$

Moreover, $\hat{X}$ has a cellular decomposition with cells of the form $\partial C_k[n] \hookrightarrow C_k[n]$ and $C_k[1] \hookrightarrow C_k[\varepsilon]$.



$$(\Lambda_0^2 \times C_\infty[0]) \coprod_{\Lambda_0^2 \times \partial C_\infty[0]} (\Delta^2 \times \partial C_\infty[0]) \hookrightarrow \Delta^2 \times C_\infty[0]$$



$$(\Lambda_0^1 \times C_\infty[1]) \coprod_{\Lambda_0^1 \times \partial C_\infty[1]} (\Delta^1 \times \partial C_\infty[1]) \hookrightarrow \Delta^1 \times C_\infty[1]$$

Stability results

Stability theorem

Let $X, Y \in \text{Set}_\Delta^\epsilon$ be ϵ -fibrant. Then $\text{Fun}(X, Y)$

1. admits fillers for $\Lambda_0^n \hookrightarrow \Delta^n$ and $\Lambda_n^n \hookrightarrow \Delta^n$, for every $n \geq 2$;
2. satisfies the ϵ -coherence conditions.

Only the one-dimensional ϵ -horns remain uncontrolled.

An $\eta\epsilon$ -simplicial set records both ϵ - and η -edges.

Full stability theorem

If $X, Y \in \text{Set}_\Delta^{\eta\epsilon}$ are $\eta\epsilon$ -fibrant, then $\text{Fun}(X, Y)$ is $\eta\epsilon$ -fibrant.

Consequence for ϵ -simplicial sets

For ϵ -fibrant X, Y , the subobject $\text{Fun}^\eta(X, Y) \subseteq \text{Fun}(X, Y)$ spanned by the η -edge-preserving maps is ϵ -fibrant.

Thank you for your attention!