

Sectionally residuated semilattices

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An integral **residuated po-monoid** (an **rpo-monoid**) is a po-monoid $(A, \leq, \cdot, 1)$, where $\cdot$ is **residuated**:

$$a \setminus b = \max\{x \in A \mid a \cdot x \leq b\} \quad \text{and} \quad b / a = \max\{x \in A \mid x \cdot a \leq b\}$$

exist for all $a, b \in A$, i.e., for all $a, b, c \in A$:

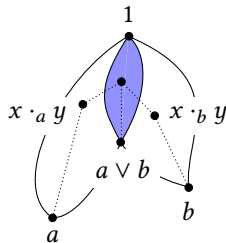
$$a \cdot b \leq c \quad \text{and} \quad b \leq a \setminus c \quad \text{and} \quad a \leq c / b.$$

A **sectionally residuated semilattice** is a $\vee$ -semilattice $(A, \leq, 1)$, where every interval $[a, 1]$ is an rpo-monoid, $([a, 1], \leq, \cdot_a, 1)$, satisfying the “compatibility condition”

$$a \vee b \leq x, y \quad \Rightarrow \quad (x \cdot_a y) \vee b = (x \cdot_b y) \vee a,$$

or equivalently,

$$a \leq b \leq x, y \quad \Rightarrow \quad (x \cdot_a y) \vee b = x \cdot_b y.$$



Residuated lattices are sectionally residuated:

$$x \cdot_a y = (x \cdot y) \vee a$$

for $x, y \in [a, 1]$.

Question: Is every SRSL representable by an RL?

$\text{SRSL } (A, \leq, (\cdot_a)_{a \in A}, 1) \longrightarrow \text{Algebra } (A, f, \backslash, /, 1) \text{ of type } (3, 2, 2, 0)$

$$\begin{aligned} f(x, y, z) &= (x \vee z) \cdot_z (y \vee z) \\ x \backslash y &= (x \vee y) \backslash_y y \quad \text{and} \quad y / x = y /_y (x \vee y) \end{aligned}$$

$\text{RL } (L, \leq, \cdot, \backslash, /, 1) \longrightarrow \text{Algebra } (L, f, \backslash, /, 1) \text{ of type } (3, 2, 2, 0)$

$$f(x, y, z) = (x \cdot y) \vee z$$

Question: Can every $\text{SRSL } (A, f, \backslash, /, 1)$ be embedded into some $\text{RL } (L, f, \backslash, /, 1)$?

Yes, in particular cases [Ch. & K. 2008]:

- Every SRLattice A embeds into $\prod_{a \in A} [a, 1]/U$ for some ultrafilter U .
- Every cone algebra embeds into the negative cone of some ℓ -group.

Yes, in general:

Every SDSL can be represented by a complete RL (quantale).

Proof: $\pm$ van Alten's embedding of bireiduation algebras into RLs [van Alten 2006].

- $(\mathcal{P}(A^*), \subseteq, \cdot, \{\varepsilon\})$, the free unital quantale over the **set** A , is too big, and is therefore replaced with its suitable **nuclear image**
- Relation $\leq$ between A^* and A :

$$a_1 \dots a_n \leq x \quad \text{iff} \quad a_n \setminus (\dots \setminus (a_1 \setminus x) \dots) = 1 \quad \text{iff} \quad (\dots (x / a_n) / \dots) / a_1 = 1$$

and $\varepsilon \leq x$ iff $x = 1$

- **Galois connection** between $\mathcal{P}(A^*)$ and $\mathcal{P}(A)$:

$$L^\uparrow = \{x \in A \mid w \leq x \text{ for all } w \in L\} \quad \text{and} \quad X^\downarrow = \{w \in A^* \mid w \leq x \text{ for all } x \in X\}$$

- $L \mapsto L^{\uparrow\downarrow}$ is a **nucleus** on $\mathcal{P}(A^*)$

- $\mathcal{D}_A = \mathcal{P}(A^*)^{\uparrow\downarrow}$ is an integral quantale, where

$$K \bullet L = (KL)^{\uparrow\downarrow}$$

$$1 = \{\varepsilon\}^{\uparrow\downarrow} = A^*$$

$$K \setminus L = \{w \in A^* \mid Kw \subseteq L\}$$

$$L / K = \{w \in A^* \mid wK \subseteq L\}$$

- $\eta_A: a \mapsto \{a\}^{\uparrow\downarrow}$ is an embedding of $(A, f, \setminus, /, 1)$ into $(\mathcal{D}_A, f, \setminus, /, 1)$
- η_A preserves all existing suprema and infima

Morphisms?

If $\varphi: A \rightarrow B$ is a homomorphism, a homomorphism $\Phi: \mathcal{D}_A \rightarrow \mathcal{D}_B$ such that

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathcal{D}_A \\ \varphi \downarrow & & \downarrow \Phi \\ B & \xrightarrow{\eta_B} & \mathcal{D}_B \end{array}$$

might not exist; in particular,

$$\Phi: L \mapsto \{\varphi^*(w) \mid w \in L\}^{\uparrow\downarrow} = \bigvee_{w \in L}^{\mathcal{D}_B} \varphi^*(w)^{\uparrow\downarrow}$$

need not be a homomorphism. Here, $\varphi^*(a_1 \dots a_n) = \varphi(a_1) \dots \varphi(a_n)$ and $\varphi^*(\varepsilon) = \varepsilon$.

“Weak” morphisms

SRSL = the category of SRSLs with **subhomomorphisms** satisfying $\varphi(1) = 1$ and

$$\left. \begin{aligned} f(\varphi(x), \varphi(y), \varphi(z)) &\leq \varphi(f(x, y, z)) \\ \varphi(x \setminus y) &\leq \varphi(x) \setminus \varphi(y) \\ \varphi(y / x) &\leq \varphi(y) / \varphi(x) \end{aligned} \right\} \text{equivalent to each other}$$

$\mathcal{V}$ = the class of maps $\varphi: A \rightarrow B$ satisfying

$$a_1 \dots a_n \leq x \text{ in } A \quad \text{iff} \quad \varphi^*(a_1 \dots a_n) = \varphi(a_1) \dots \varphi(a_n) \leq \varphi(x) \text{ in } B$$

- Quantales are the $\mathcal{V}$ -injective objects in **SRSL**.
- For every A , $\mathcal{D}_A$ is the $\mathcal{V}$ -injective hull of A .

But quantales do not form a reflective subcategory.

“Strong” morphisms

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathcal{D}_A \\ \varphi \downarrow & & \downarrow \Phi \\ B & \xrightarrow{\eta_B} & \mathcal{D}_B \end{array}$$

Φ is a quantale homomorphism iff φ satisfies

$$\varphi^*[L^{\uparrow\downarrow}] \subseteq \varphi^*[L]^{\uparrow\downarrow}$$

for every $\emptyset \neq L \subseteq A^*$.

S $\mathcal{R}\mathcal{S}\mathcal{L}^*$ = the category of $\mathcal{S}\mathcal{R}\mathcal{S}\mathcal{L}$ s with morphisms satisfying $\varphi^*[L^{\uparrow\downarrow}] \subseteq \varphi^*[L]^{\uparrow\downarrow}$

Quant = the category of quantales with quantale homomorphisms

Quant is a reflective subcategory of **S $\mathcal{R}\mathcal{S}\mathcal{L}^*$** .

THANK YOU!