

Representation of double Boolean algebras

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- 1 Preliminaries
- 2 Example of dBas
- 3 Representation

Revisiting some work of R. Wille: 2000-2006

- 1 Boolean Concept Logic [Wil00]
- 2 Algebras of semiconcepts and double Boolean algebras [HLSW01]
- 3 Preconcept Algebras and Generalized Double Boolean Algebras [Wil04]
- 4 Semiconcept and Protoconcept Algebras: The Basic Theorems [VW05]
- 5 The Basic Theorem on Preconcept Lattices [BW06]

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Double Boolean algebras [Wil00]

A *double Boolean algebra* (dBa) is an algebra $\mathbf{D} := (D; \sqcup, \sqcap, \neg, \lrcorner, \top, \perp)$ of type (2,2,1,1,0,0) which satisfies the equations (1a.) – (11a.), (1b.) – (11b.) and (12.).

Idempotence:

$$1a. (x \sqcap x) \sqcap y = x \sqcap y$$

$$8a. \neg(x \sqcap x) = \neg x$$

$$1b. (x \sqcup x) \sqcup y = x \sqcup y$$

$$8b. \lrcorner(x \sqcup x) = \lrcorner x$$

Commutative and associative:

$$2a. x \sqcap y = y \sqcap x$$

$$3a. x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$$

$$2b. x \sqcup y = y \sqcup x$$

$$3b. x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$$

Absorption:

$$4a. x \sqcap (x \sqcup y) = x \sqcap x$$

$$5a. x \sqcap (x \vee y) = x \sqcap x$$

$$4b. x \sqcup (x \sqcap y) = x \sqcup x$$

$$5b. x \sqcup (x \wedge y) = x \sqcup x$$

Distributive:

$$6a. x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$$

$$6b. x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$$

Axioms for negations:

$$7a. \neg\neg(x \sqcap y) = x \sqcap y$$

$$9a. x \sqcap \neg x = \perp$$

$$7b. \lrcorner\lrcorner(x \sqcup y) = x \sqcup y$$

$$9b. x \sqcup \lrcorner x = \top$$

where $x \vee y := \neg(\neg x \sqcap \neg y)$ and $x \wedge y := \lrcorner(\lrcorner x \sqcup \lrcorner y)$.

Axioms for constants:

$$10a. \neg \perp = \top \sqcap \top$$

$$10b. \lceil \top = \perp \sqcup \perp$$

$$11a. \lceil \perp = \top$$

$$11b. \neg \top = \perp$$

Proto's axiom:

$$12. (x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x).$$

Axiom for semiconcepts:

$$13. x \sqcap x = x \text{ or } x \sqcup x = x.$$

Axioms for precepts:

$$14a. x_{\sqcap \sqcup \sqcap} = x_{\sqcap \sqcup}$$

$$14b. x_{\sqcup \sqcap \sqcup} = x_{\sqcup \sqcap}$$

where $t_{\sqcap} := t \sqcap t$ and $t_{\sqcup} := t \sqcup t$, for any t .

- A weak dBa satisfies equations (1a.) – (11a.) and (1b.) – (11b.)
- A pure dBa satisfies equations (1a.) – (11a.), (1b.) – (11b.), 12. and 13.
- A generalized dBa satisfies (1a.) – (11a.), (1b.) – (11b.), 14a. and 14b.

Axioms for constants:

$$10a. \neg \perp = \top \sqcap \top$$

$$10b. \lceil \top = \perp \sqcup \perp$$

$$11a. \lceil \perp = \top$$

$$11b. \neg \top = \perp$$

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A small subset of Wille's dBas axioms [HKLB26]

Howlader, Kwuida, Behrisch, Liao (2026)

An algebra $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \perp)$ is a weak dBa if and only if it satisfies

Commutative:

$$2a. x \sqcap y = y \sqcap x$$

$$2b. x \sqcup y = y \sqcup x$$

Idempotence:

$$8a. \neg(x \sqcap x) = \neg x$$

$$8b. \lrcorner(x \sqcup x) = \lrcorner x$$

Axioms for negations:

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$$9a. x \sqcap \neg x = \perp$$

$$9b. x \sqcup \lrcorner x = \top$$

Absorption:

$$4a. x \sqcap (x \sqcup y) = x \sqcap x$$

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Distributive:

$$6a. x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$$

$$6b. x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$$

A quasi-order on weak dBas[HLSW01]

Let $\mathbf{D}$ be a weak dBa. Define $\sqsubseteq$ by

$$x \sqsubseteq y : \Longleftrightarrow x \sqcap y = x \sqcap x \text{ and } x \sqcup y = y \sqcup y.$$

Set $D_{\sqcap} := \{x \in D \mid x \sqcap x = x\}$ and $D_{\sqcup} := \{x \in D \mid x \sqcup x = x\}$.

- ① $x \sqcap y \sqsubseteq x \sqsubseteq x \sqcup y$
- ② $x \mapsto x \sqcap y$ preserves $\sqsubseteq$ and $\sqcap$. Dually, $x \mapsto x \sqcup y$ preserves $\sqsubseteq$ and $\sqcup$.
- ③ The relation $\sqsubseteq$ is a quasi-order on D .
- ④ $y_{\sqcap} \sqsubseteq x_{\sqcap} \Longleftrightarrow y_{\sqcap} \sqsubseteq x$ and $y_{\sqcup} \sqsupseteq x_{\sqcup} \Longleftrightarrow y_{\sqcup} \sqsupseteq x$
- ⑤ $\mathbf{D}_{\sqcap} := (D_{\sqcap}; \sqcap, \vee, \neg, \perp, \neg\perp)$ and $\mathbf{D}_{\sqcup} := (D_{\sqcup}; \wedge, \sqcup, \lrcorner, \lrcorner\top, \top)$ are Boolean algebras, whose order relation are the restrictions of $\sqsubseteq$.
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- ⑦ $x \sqcap y = x_{\sqcap} \sqcap y_{\sqcap}$, $x \sqcup y = x_{\sqcup} \sqcup y_{\sqcup}$, $\neg x = \neg x_{\sqcap}$, $\lrcorner x = \lrcorner x_{\sqcup}$
- ⑧ $x \sqsubseteq y \Longleftrightarrow x_{\sqcap} \sqsubseteq y_{\sqcap}$ and $x_{\sqcup} \sqsubseteq y_{\sqcup}$

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Weak dBas as quasi-ordered structures[HLSW01]

Let $\mathbf{P} := (P, \leq, P_\wedge, P_\vee, 0, 1)$ with $P_\wedge \cup P_\vee \subseteq P$ be a structure such that

- ❶ $(P, \leq, 0, 1)$ is a bounded quasi-ordered set,
- ❷ $(P_\wedge, \leq)$ is a Boolean algebra with bottom element 0,
- ❸ $(P_\vee, \leq)$ is a Boolean algebra with top element 1,
- ❹ for each $x \in P$ there are $x_\wedge \in P_\wedge$ and $x_\vee \in P_\vee$ with

$$y \leq x_\wedge \iff y \leq x \text{ for all } y \in P_\wedge$$

$$y \geq x_\vee \iff y \geq x \text{ for all } y \in P_\vee.$$

Then $\mathbf{D(P)} := (P, \sqcap, \sqcup, \neg, \lrcorner, \perp, \top)$ is a weak dBa, with operations given by

$$x \sqcap y := x_\wedge \wedge y_\wedge, x \sqcup y := x_\vee \vee y_\vee, \neg x := \neg x_\wedge, \lrcorner x := \lrcorner x_\vee, \perp := 0, \top := 1.$$

- ❶ The $\mathbf{D(P)}$ -quasi-order coincides with $\leq$ iff

$$x \leq y \iff x_\wedge \leq y_\wedge \text{ and } x_\vee \leq y_\vee.$$

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Contexts, concepts, semi-, proto- and preconcepts [Wil00]

- $\mathbb{K} := (G, M, R)$ with $R \subseteq G \times M$ is called a formal context (or polarity).
- Concept forming operator. For $A \subseteq G, B \subseteq M$, define
$$A' := \{m \in M \mid gRm \ \forall g \in A\} \quad \text{and} \quad B' := \{g \in G \mid gRm \ \forall m \in B\}$$
- Concepts and friends: A pair (A, B) is a
 - concept if $A' = B$ and $B' = A$
 - semiconcept if $A' = B$ or $B' = A$.
 - protoconcept if $A'' = B'$.
 - preconcept if $A \subseteq B'$
- $\mathfrak{V}(\mathbb{K}), \mathfrak{P}(\mathbb{K}), \mathfrak{H}(\mathbb{K}), \mathfrak{B}(\mathbb{K})$ denotes the set of pre-, proto-, semi- and concepts of $\mathbb{K}$, respectively.
- $\mathfrak{B}(\mathbb{K}) \subseteq \mathfrak{H}(\mathbb{K}) \subseteq \mathfrak{P}(\mathbb{K}) \subseteq \mathfrak{V}(\mathbb{K})$.

Contexts, concepts, semi-, proto- and preconcepts [Wil00]

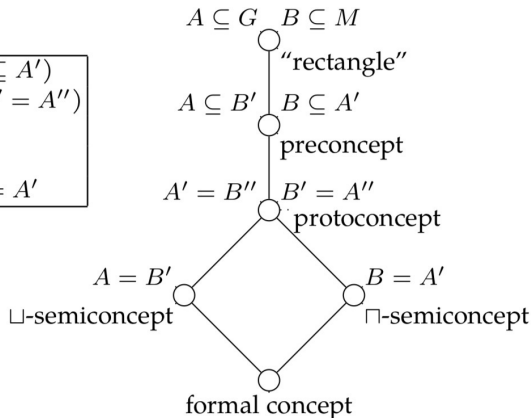
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Concepts and friends [GW24]

preconcept	$A \subseteq B' \ (\Leftrightarrow B \subseteq A')$
protoconcept	$A' = B'' \ (\Leftrightarrow B' = A'')$
$\sqcup$ -semiconcept	$A = B'$
$\sqcap$ -semiconcept	$B = A'$
concept	$A = B' \text{ and } B = A'$



- Logical operator on concepts: (meet and join)

$$(A, B) \wedge (C, D) = (A \cap C, (B \cup D)'') = (A \cap C, (A \cap C)')$$

$$(A, B) \vee (C, D) = ((A \cup C)'', B \cap D) = ((B \cap D)', B \cap D)$$

- Logical operators on preconcepts: For (A, B) and (C, D) in $\mathfrak{P}(\mathbb{K})$:

$$(A, B) \sqcap (C, D) := (A \cap C, (A \cap C)')$$

$$(A, B) \sqcup (C, D) := ((B \cap D)', B \cap D)$$

$$\neg(A, B) := (G \setminus A, (G \setminus A)')$$

$$\lrcorner(A, B) := ((M \setminus B)', M \setminus B)$$

$$\top := (G, G') \quad \text{and} \quad \perp := (M', M).$$

- $\underline{\mathfrak{P}}(\mathbb{K}) := (\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \perp)$ is a dBa.
- $\underline{\mathfrak{H}}(\mathbb{K}) := (\mathfrak{H}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \perp)$ is a pure dBa.
- $\underline{\mathfrak{V}}(\mathbb{K}) := (\mathfrak{V}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \perp)$ is a generalized dBa.

- Logical operator on concepts: (meet and join)

$$(A, B) \wedge (C, D) = (A \cap C, (B \cup D)'') = (A \cap C, (A \cap C)')$$

$$(A, B) \vee (C, D) = ((A \cup C)'', B \cap D) = ((B \cap D)', B \cap D)$$

- Logical operators on preconcepts: For (A, B) and (C, D) in $\mathfrak{P}(\mathbb{K})$:

$$(A, B) \sqcap (C, D) := (A \cap C, (A \cap C)')$$

$$(A, B) \sqcup (C, D) := ((B \cap D)', B \cap D)$$

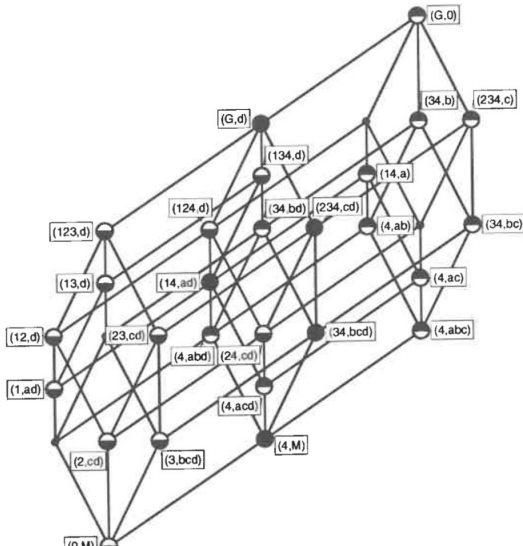
$$\neg(A, B) := (G \setminus A, (G \setminus A)')$$

$$\lrcorner(A, B) := ((M \setminus B)', M \setminus B)$$

$$\top := (G, G') \quad \text{and} \quad \perp := (M', M).$$

- $\underline{\mathfrak{P}}(\mathbb{K}) := (\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \perp)$ is a dBa.
- $\underline{\mathfrak{H}}(\mathbb{K}) := (\mathfrak{H}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \perp)$ is a pure dBa.
- $\underline{\mathfrak{V}}(\mathbb{K}) := (\mathfrak{V}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \perp)$ is a generalized dBa.

An algebra of semiconcepts with $\sqsubseteq$ not lattice order [HLSW01]



	a	b	c	d
1	X			X
2			X	X
3		X	X	X
4	V	V	V	V

An algebra of protoconcepts [Wil00]

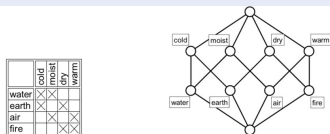


Fig. 1. A context and its concept lattice representing Empedocles' conception of the four elements

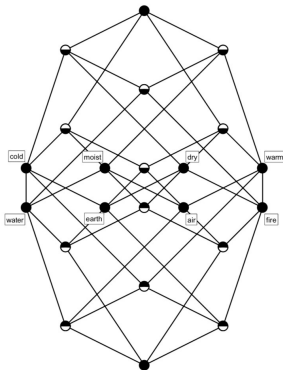
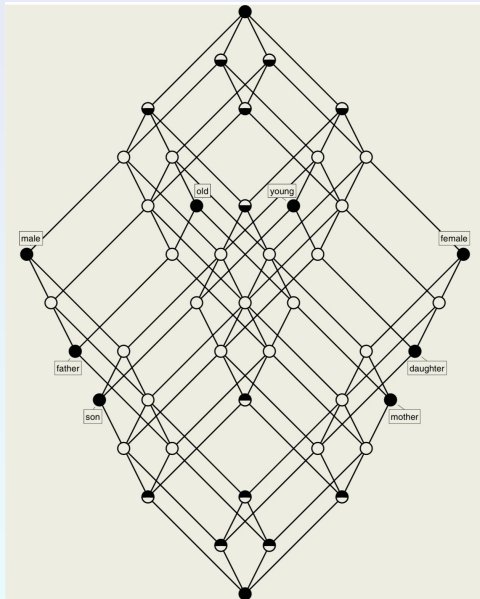


Fig. 2. The algebra of protoconcepts of the formal context in Figure 1

A small context: family members

	male	female	old	young
father	×		×	
mother		×	×	
son	×			×
daughter		×		×

Algebra of precepts of family members[Wil04]



Filters and ideals of weak dBas

- $F \subseteq D$ is a filter of $\mathbf{D}$ if
$$\begin{cases} x, y \in F \implies x \sqcap y \in F \\ x \in F, x \sqsubseteq y \implies y \in F. \end{cases}$$
- $I \subseteq D$ is an ideal of $\mathbf{D}$ if
$$\begin{cases} x, y \in I \implies x \sqcup y \in I \\ x \in I, x \sqsupseteq y \implies y \in I. \end{cases}$$
- A filter F is proper if $F \neq D$, and primary if it is proper and satisfies $x \in F$ or $\neg x \in F$ for all $x \in D$. Primary ideals are defined dually.

Howlader (2020)

Let D be a (weak) dBa and F a filter of $\mathbf{D}$. Then F is a primary filter iff $F \cap D_{\sqcap}$ is a prime filter of the Boolean algebra $\mathbf{D}_{\sqcap}$.

$\mathfrak{F}_p(D)$ (resp. $\mathfrak{I}_p(D)$) denotes the set of all primary filters (resp. ideals) of D .

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Standard context of a (weak) dBa

- 1 Define $x \sqsubseteq y : \iff x \sqsubseteq y$ and $y \sqsubseteq x$.
- 2 A map $\alpha : D \rightarrow D$ is quasi-injective if $x \sqsubseteq y \iff \alpha(x) \sqsubseteq \alpha(y), \forall x, y$
- 3 $\sqsubseteq$ is a congruence on $\mathbf{D}$, whose restriction on D_p is the identity.

Standard context

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Quasi-embedding: Wille (2000)

- 1 $x \mapsto (\mathfrak{F}_x, \mathfrak{I}_x)$ is a quasi-injective homomorphism $i : D \rightarrow \mathfrak{P}(\mathbb{K}(D))$ having $\sqsubseteq$ as kernel.

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- 1 Each double Boolean algebra quasi-embeds into the algebra of protoconcepts of a certain context.
- 2 The dBa axioms generate the equational theory of the algebras of protoconcepts [Wil00]
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Thank You