

Relational algebra for the wolf, goat, and cabbage problem

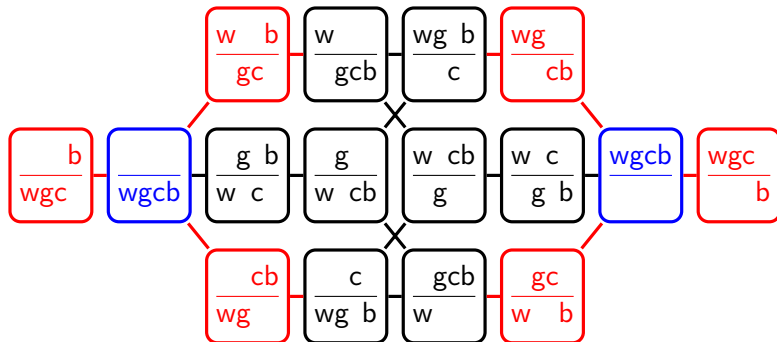
David Kruml¹

Masaryk University

SSAOS 2026, Tatranská Lomnica

¹Supported by project GAČR 25-20013L.

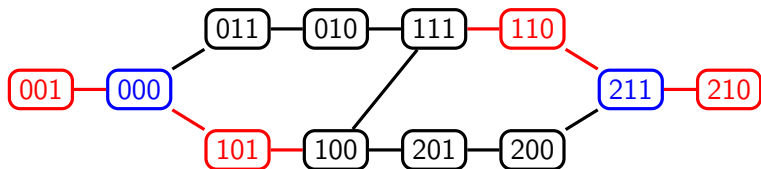
Graph of states



Two types of aggregation

- ▶ The wolf and the cabbage are equivalent (and this provides two solutions of the problem). However, they are still individuals (and travel separately).
- ▶ The farmer never leaves the boat. At start, they form a *complex resource*. It is dissolved after finishing all the transports.
- ▶ The two aggregations of resources correspond to disjoint union $\sqcup$ and cartesian product $\times$.

Reduced graph

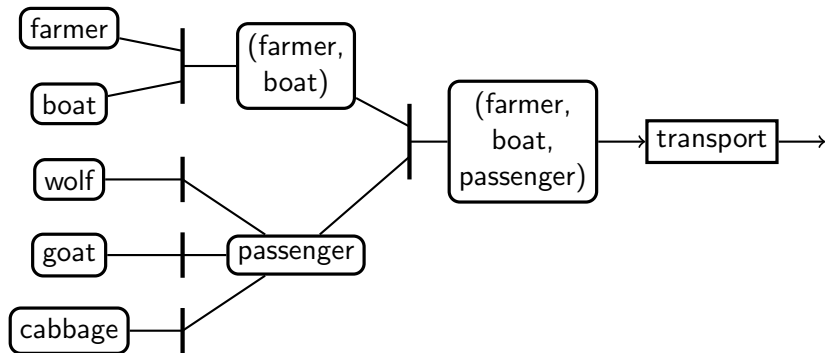


- ▶ Triples: number of wolf-cabbage, of goat, and of farmer-boat on the finishing bank.
- ▶ In practice, manufacturing is extremely standardized and aggregation is even more important.

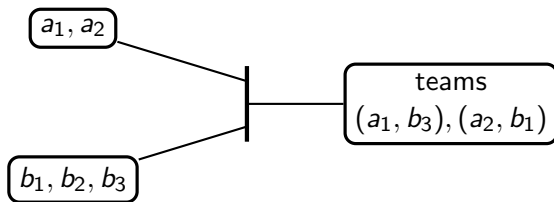
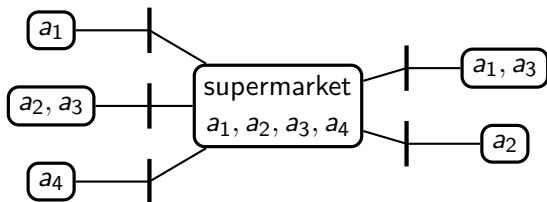
Distinction of two types of precesses

- ▶ Logical process — any number of inputs and outputs, no duration, reversible (undirected).
- ▶ Work process — one input, one output, duration, causal (directed).
- ▶ Our nets combine both types — common generalization of Petri nets or time Petri nets, as well as reversible programming.

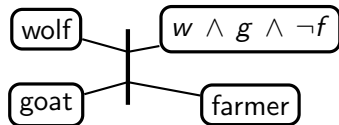
Example



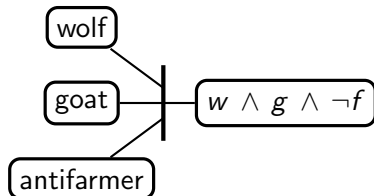
Logic



Motivating example — composed constraint



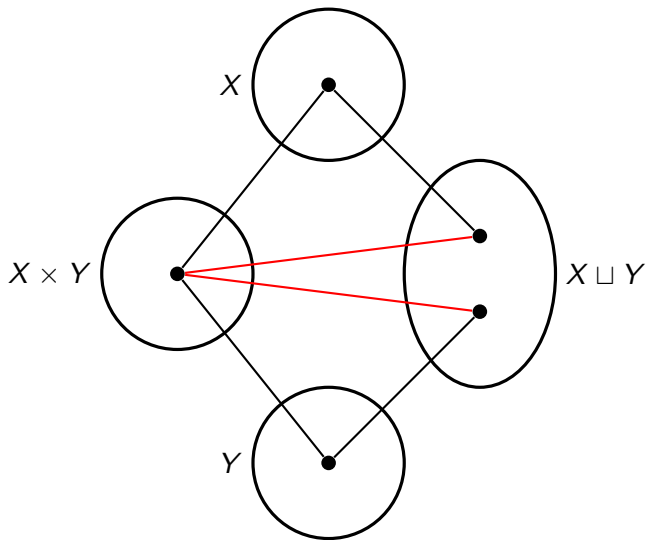
$\cong$



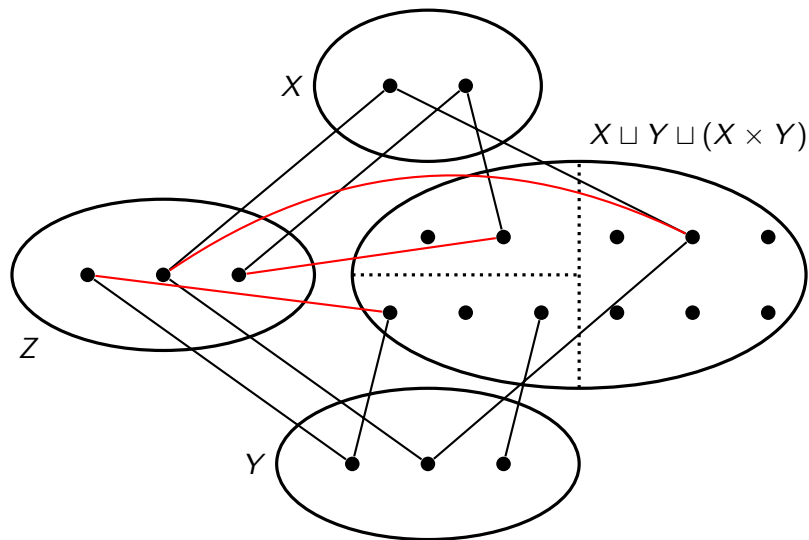
Rel and PInj

- ▶ Rel — category of *relations*, all possibilities, non-deterministic.
- ▶ PInj — category of *partial injections* (matchings), particular execution, deterministic.
- ▶ Both categories are *dagger*: $R^\dagger = R^{-1}$. PInj is *inverse*: $RR^\dagger R = R$.
- ▶ Disjoint union $\sqcup$ is both product and coproduct in Rel.
- ▶ PInj does not have products or coproducts.
- ▶ However, PInj looks more reasonable (firing in Petri nets).

Why $\mathbf{Pinj}$ does not have (co)products



Multiproduct with local projections



PInj is symmetric bimonoidal

- ▶ Let us write $\otimes$ for $\times$, and $\oplus$ for $\sqcup$.
- ▶ $(\mathsf{PInj}, \otimes, \{*\})$ is symmetric monoidal.
- ▶ $(\mathsf{PInj}, \oplus, \emptyset)$ is symmetric monoidal.
- ▶ $\otimes$ distributes over $\oplus$.
- ▶ $\emptyset$ is zero for $\otimes$.

The functor ℓ^2

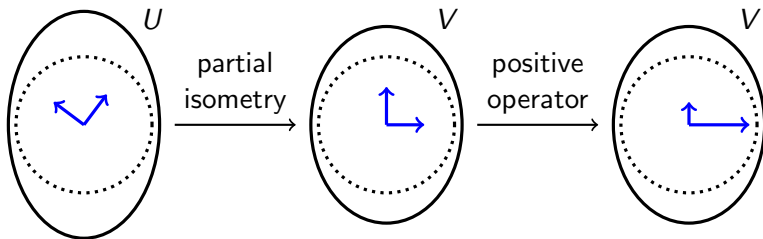
Theorem (Heunen 2013)

There is a bimonoidal functor $\ell^2 : \mathbf{PInj} \rightarrow \mathbf{Hilb}$.

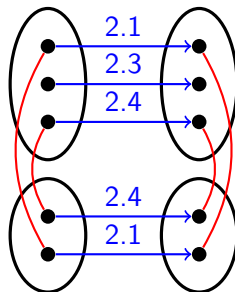
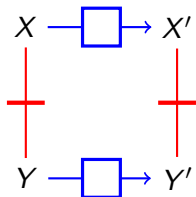
- ▶ Idea: $\text{set} \mapsto \text{Hilbert basis}$.
- ▶ The functor *chooses* a basis for each object, it is only essentially full.
- ▶ The bimonoidal structure of $\mathbf{Hilb}$ has standard meaning.
- ▶ ℓ^2 does not have adjoints.
- ▶ Since the basis is fixed, every object $\ell^2(X)$ in the image is a Frobenius algebra (Coecke, Pavlović, and Vicary 2008).

More operator theory

- ▶ After extraction of all logic, a work process is reduced to one or more numbers: duration (process time), cost, carbon footprint, etc. — *spectrum* of the process.
- ▶ An iterated process could be variable. This form a collection of process times/random sample.
- ▶ The sample provides a self-adjoint operator on Hilbert space associated with the input (or output) resource.
- ▶ In this context, decomposition to logical and work components of a process corresponds to the *polar decomposition* of an operator.

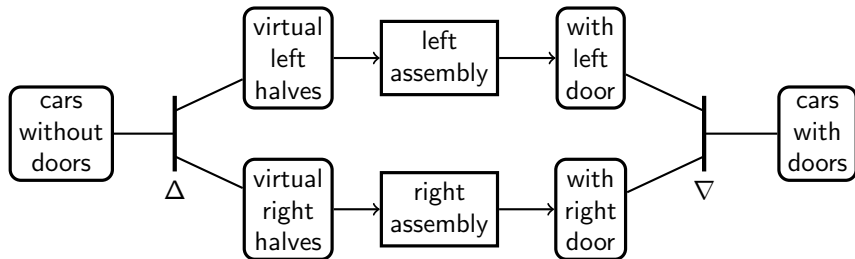


Double category of classical processes



- ▶ Vertical morphisms (red): logic, partial injections, reversible, *qualitative processes*.
- ▶ Horizontal morphisms (blue): work, spectra = valued identities (real functions), *quantitative processes*.

Double-slit in automotive



- ▶ Codagonal gate $\nabla (= \Delta^\dagger) : X \otimes X \rightarrow X$ checks that “both halves are of the same car”.
- ▶ The two virtual halves of a car are “entangled” or coherent, i. e. the car do not proceed until both assemblies are finished.
- ▶ The same holds for each item in a batch process.

Pop (push) for LIFO/FIFO stacks

- ▶ Queue: (a, b, c, d) .
- ▶ Head: $\{a\}$, body: $\{(a, b), (b, c), (c, d)\}$, tail: $\{d\}$.
- ▶ If there is one element in the stack, body is empty and head = tail. The elements are compared and popped by ∇ .
- ▶ If there are more elements, body is non-empty and the head element a is matched with (a, b) , popped, and replaced by b . Formally, $(a, (a, b)) \mapsto ((a, a), b) \mapsto (b, a)$.
- ▶ There is no conflict of the processes: if body is non-empty, head $\neq$ tail and the former one can not be performed.
- ▶ There is also no confusion of elements of head, body, and tail and they can be mixed by $\oplus$ to one supermarket.
- ▶ Push is defined dually.

Conclusion

- ▶ The problem of wolf, goat, and cabbage provides a concise example of composed constraints.
- ▶ Logical processing of resources is described by symmetric bimonoidal dagger category $(\mathbf{PInj}, \otimes, \oplus, \dagger)$.
- ▶ There is a functor to $\mathbf{Hilb}$ including the topic to operator theory, quantum protocols, linear logic, etc.
- ▶ $\mathbf{PInj}$ is a logical component of process modeling, the other component is work. Together they form a double category.
- ▶ The topic provides a low-level language for production/logistic process modeling and can be easily turned to relational databases.
- ▶ Open questions: continuous processes (spaces of functions, probability measures), quantum river crossing problem, more quantum interpretation of manufacturing.
- ▶ Thank you for your attention!