

Residuated partially ordered algebras, logic and preclones: Part 1, Tonicities, inequational logic, partially ordered algebras and po-varieties

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A slightly different view of semilattices

It is well known that semilattices (and lattices) can be defined algebraically or order-theoretically (for semilattices do we get $\vee$ or $\wedge$? *lub, glb* need $\exists$)

At a **Summer School on General Algebra and Ordered Sets** we **shouldn't** have to choose, so here is a third way:

Consider the **strict Horn theory** with axioms

- $x \leq x, \quad x \leq y \ \& \ y \leq x \implies x = y, \quad x \leq y \ \& \ y \leq z \implies x \leq z,$
- $x \leq y \ \& \ z \leq w \implies x \cdot z \leq y \cdot w,$
- $x \leq x \cdot x, \quad x \cdot y \leq x, \quad x \cdot y \leq y.$

A model $\mathbf{A} = (A, \leq^{\mathbf{A}}, \cdot^{\mathbf{A}})$ of these axioms is called a **po-meet-semilattice**.

Exercise: show that $\cdot$ is associative, commutative and idempotent ($x \cdot x = x$).

In this case we can **define** $x \leq y \iff x \cdot y = x$ (a consequence of the axioms).

Partially ordered algebras

A **po-algebra** $\mathbf{A} = (A, \leq^{\mathbf{A}}, F)$ is a partially ordered set with basic operations in F that are either order-preserving **or order-reversing** in each argument.

This information is fixed by the **tonicity** $\tau : \mathcal{F} \rightarrow \bigcup_{n=0}^{\infty} \{+, -\}^n$ where $\mathcal{F}$ is a set of basic operation symbols, so $F = \{f^{\mathbf{A}} : f \in \mathcal{F}\}$ is their interpretations in $\mathbf{A}$.

$|\tau(f)| = n$ means f is an n -ary operation symbol

$\tau(f)_i = +$ means $\mathbf{A} \models x \leq y \Rightarrow f(\dots, x, \dots) \leq f(\dots, y, \dots)$

$\tau(f)_i = -$ means $\mathbf{A} \models x \leq y \Rightarrow f(\dots, y, \dots) \leq f(\dots, x, \dots)$

A **po-variety of po-algebras** is a class of similar po-algebras defined by equations **or inequations** [Pigozzi 2004]. (It's actually a strict Horn theory)

Residuated po-magmas and to-magmas

A **residuated partially ordered magma** or **rpo-magma** $\mathbf{A} = (A, \leq, \cdot, \backslash, /)$ is a partially-ordered set $(A, \leq)$ with a binary operation $\cdot$ and two **residuals** that satisfy for all $x, y, z \in A$

$$(\text{res}) \quad x \cdot y \leq z \iff x \leq z/y \iff y \leq x \backslash z$$

The operation $x \cdot y$ is usually written xy .

A **rpo-semigroup** is a rpo-magma where $\cdot$ is **associative**.

A **rpo-monoid** $\mathbf{A} = (A, \leq, \cdot, \backslash, /, 1)$ is a rpo-semigroup with an identity element 1

A **residuated to-magma** is a totally ordered (= linearly ordered) rpo-magma.

If $\cdot$ is **commutative** then $x \backslash y = y / x$ and this is usually written $x \rightarrow y$.

Examples of residuated po-magmas and to-magmas

$(A, \leq, \cdot)$ such that $(A, \leq)$ is a **complete lattice** and $\cdot$ is a **completely join-preserving** binary operation (= nonassociative quantale).

$$x \backslash z = \bigvee \{y \in A : xy \leq z\} \quad \text{and} \quad z / y = \bigvee \{x \in A : xy \leq z\}$$

E.g., any **finite to-magma** $(A, \leq, \cdot)$ with an order-preserving $\cdot$ operation.

Number of **rto-magmas**: 1, 6, 175, 24696, ... <https://oeis.org/A071095>

A rpo-magma is **integral** if $\cdot$ has an identity element that is the top of the poset.

Number of **integral** rto-magmas: 1, 2, 11, 170, 7429, 920460, 323801820, ...
<https://oeis.org/A051255>

Quasigroups: $(A, =, \cdot, \backslash, /)$ such that $xy = z \iff x = z / y \iff y = x \backslash z$

(antichain order) 1, 1, 5, 35, 1411, 1130531, ... <https://oeis.org/A057991>

Examples of rpo-semigroups and rpo-monoids

Any set $\mathcal{A}$ of subsets of a set U closed under $\cap, \rightarrow$ where $X \rightarrow Y = (U \setminus X) \cup Y$.

$\mathbf{A} = (\mathcal{A}, \cap, \rightarrow, \leftarrow, U)$ is a rpo-monoid called a **generalized Boolean algebra**.

In fact $\mathbf{A}$ is a **residuated distributive lattice-ordered** monoid (rdlo-monoid)

Also a member of the variety generated by the 2-element Brouwerian lattice.

A **residuated lattice** is a rpo-monoid such that $(A, \leq)$ is a lattice.

A rpo-magma is **divisible** if $x \leq y \implies x = (x/y)y = y(y \setminus x)$.

GBL-algebras are defined as divisible residuated lattices.

A poset $(A, \leq)$ is **self-dual** if there exist order-reversing operations $\sim, -$ such that $\sim - x = x = - \sim x$.

A rpo-magma is **involutive** if it is self-dual, $x \setminus y = \sim(-y \cdot x)$ and $x/y = -(y \cdot \sim x)$.

In that case $(A, \leq, \cdot, \sim, -)$ is called an ipo-magma.

Residuated po-magmas are a po-variety

Residuation ensures that x/y and $y \backslash x$ are **order-preserving** in the numerator

(x position) and **order-reversing** in the denominator. Tonicity: $/^{+-}$, $\backslash^{-+}$

xy is order-preserving in both arguments. Tonicity: $\cdot^{++}$

(res) is equivalent to $x \leq xy/y$, $(z/y)y \leq z$, $y \leq x \backslash xy$, $x(x \backslash z) \leq z$

hence **rpo-magmas are a po-variety of po-algebras**.

Residuated lattices are also a po-variety, but $\leq$ is definable by $x \wedge y = x$, so they are algebras.

rpo-monoids are models of the **Lambek Calculus**.

Involutive po-monoids (simpler axioms)

An **involutive po-monoid** (ipo-monoid) **A** is an algebraic structure $(A, \leq, \cdot, \sim, -, 0)$ such that $(A, \leq)$ is a poset, $\cdot$ is **associative** and

$$(\text{lin}) \quad x \leq y \iff x \cdot \sim y \leq 0 \iff -y \cdot x \leq 0$$

Even **without associativity** (lin) has some interesting consequences:

Lemma

$-, \sim$ form a **dual Galois connection**, (i.e., $-x \leq y \iff \sim y \leq x$).

Proof.

$$-x \leq y \iff -x \cdot \sim y \leq 0 \iff \sim y \leq x. \quad \square$$

Hence $-, \sim$ are order reversing, $-\sim x \leq x$, $\sim -x \leq x$,

$\sim \sim x = x$ and $-\sim -x = -x$ (Summer school homework).

The linear negations are involutive

Lemma

$\sim -x = x = -\sim x$, *i.e., $\sim, -$ are involutions.*

Proof.

$$x \leq -\sim x \iff x \cdot \sim -\sim x \leq 0 \iff x \cdot \sim x \leq 0 \iff x \leq x. \quad \square$$

For the next few results we use associativity.

Lemma

$\sim 0 = -0$ *is an identity element.*

Proof.

$$-0 \cdot x \leq y \iff -0 \cdot x \cdot \sim y \leq 0 \iff x \cdot \sim y \leq 0 \iff x \leq y.$$

Hence $-0 \cdot x = x$ and similarly $x \cdot \sim 0 = x$. Therefore $\sim 0 = -0 \cdot \sim 0 = -0$. $\square$

Naturally ~ 0 is denoted by 1.

Multiplication is residuated, hence order-preserving

Lemma

$$xy \leq z \iff x \leq -(y \cdot \sim z) \iff y \leq \sim(-z \cdot x).$$

Proof.

$$xy \leq z \iff (-\sim x)y \sim z \leq 0 \iff y \sim z \leq \sim x \iff x \leq -(y \sim z).$$

$$\text{Similarly } xy \leq z \iff -zx(\sim -y) \leq 0 \iff -zx \leq -y \iff y \leq \sim(-zx). \quad \square$$

Therefore $z/y = -(y \cdot \sim z)$ and $x \setminus z = \sim(-z \cdot x)$.

Lemma

$$x \leq y \implies xz \leq yz \text{ and } zx \leq zy.$$

Proof.

Follows from residuation: Assume $x \leq y$.

Then $yz \leq yz \implies x \leq y \leq yz/z \implies xz \leq yz$, and $zx \leq zy$ is similar. $\square$

Algebraic models of multiplicative linear logic (MLL)

The previous results show that ipo-monoids are po-algebraic models of **non-commutative, non-cyclic multiplicative linear logic**

$(x + y = -(\sim y \cdot \sim x) = \sim(-y \cdot -x)$ and $x^\perp = \sim x$, ${}^\perp x = -x$).

With **cyclicity** ($xy \leq 0 \Leftrightarrow yx \leq 0$) added, we obtain models of **cyclic multiplicative linear logic** (CyMLL), where $\sim x = -x$.

With commutativity ($xy \leq yx$) added, we obtain models of the **multiplicative linear logic**. MLL entailment has FMP and is decidable (NP-complete!).

Some of these algebras (reducts of distributive commutative involutive residuated lattices) are models of the **multiplicative fragment of contraction-free relevance logic** (RW_m^t).

We will see that ipo-monoids come in a form of **group**-indexed unions of directed (bounded) connected components $\bigcup_{g \in G} A_g$. Depending on additional axioms or assumptions on the unital component A_e , we can say a bit more.

Examples: binary relations, pointed groups, po-groups

Any set of binary relations $A \subseteq \mathcal{P}(X^2)$ including id_X , closed under composition and **complement-converse** $\sim R = -R = X^2 \setminus R^{-1}$ is a **cyclic ipo-monoid**.

Weakening relations on a poset with composition and complement-converse.

$wk(P, \leq) = \{R \in P^2 : \leq \circ R \circ \leq = R\}$, the identity element is $\leq$.

Closed under $\cap, \cup$, hence **distributive involutive residuated lattices**.

A **pointed group** $(A, \cdot, ^{-1}, 1, 0)$ is a group with an (arbitrary) constant 0.

$\Rightarrow$ obtain an **ipo-monoid** with $\leq$ as equality, $\sim x = x^{-1} \cdot 0$ and $-x = 0 \cdot x^{-1}$.

Partially ordered groups

A **partially ordered group** is a structure $\mathbf{A} = (A, \leq, \cdot, ^{-1}, 1)$ such that

- 1 $(A, \leq)$ is a poset
- 2 $(A, \cdot, ^{-1}, 1)$ is a group and
- 3 multiplication is order preserving (hence $^{-1}$ is order reversing).

$\Rightarrow$ obtain a cyclic ipo-monoid with $0 = 1^{-1} = 1$ and $\sim x = -x = x^{-1}$.

Examples: $(\mathbb{Z}, \leq, +, -, 0)$, $(\mathbb{R}, \leq, +, -, 0)$, $(\mathbb{Q}^+, \leq, \cdot, ^{-1}, 1)$,

all groups (with $\leq$ as $=$, e.g. $(\mathbb{Z}, =, +, -, 0)$)

Excellent textbook by **Andrew Glass** (1999) <https://doi.org/10.1142/3811>

Why is $\leq$ not algebraically definable in ipo-monoids?

$(\mathbb{Z}, \leq, +, -, 0)$ and $(\mathbb{Z}, =, +, -, 0)$ are non-isomorphic po-groups that have the same algebra reduct.

Can we find non-isomorphic finite ipo-monoids with the same $\cdot, \sim, -$ operations?

Consider $\mathbb{Z}_3 = (\{0, 1, 2\}, +_3, -, 0)$, $\mathcal{S} = \{\emptyset, \{0\}, \{1, 2\}, \{0, 1, 2\}\}$.

Then the **relation algebra reduct** $\mathbf{A}_3 = (\mathcal{S}, \subseteq, \cdot^c, \{0\}^c)$ with complex addition and complement is an ipo-monoid.

However it is also an ito-monoid with the ordering $\emptyset < \{0\} < \{1, 2\} < \{0, 1, 2\}$.

Integral rpo-monoids are **algebras**: $x \leq y \iff 1 \leq x \setminus y \iff 1 = x \setminus y$

Square increasing ($x \leq x^2$) ipo-monoids are **algebras**: $x \leq y \iff x \setminus y = (x \setminus y) \setminus (x \setminus y)$

Lattice-free reducts of involutive residuated lattices

An **involutive residuated lattice** $(A, \wedge, \vee, \cdot, \sim, -, 0)$ is a lattice $(A, \wedge, \vee)$ such that $(A, \leq, \cdot, \sim, -, 0)$ is an ipo-monoid.

Hence **lattice-free** reducts of involutive residuated lattices are ipo-monoids.

Recall that **IMTL-algebras** are commutative ($xy = yx$) integral ($x \leq 1$) involutive prelinear ($1 \leq x/y \vee y/x$) residuated lattices, and

MV-algebras are divisible ($x \wedge y = (x/y) \cdot y$) IMTL-algebras

$\Rightarrow$ **MV-algebras are ipo-monoids** (using only $\cdot, -, 0$ since $x \vee y = -(-x \wedge -y)$).

Sugihara ipo-monoids (reducts of algebraic models of RM_m)

$\mathbf{S}_{2n}$: a $2n$ -element chain $a_n < \dots < a_2 < \mathbf{0} < \mathbf{1} < b_2 < \dots < b_n$

$\mathbf{S}_{2n-1}$: a $2n-1$ -element chain $a_n < \dots < a_2 < \mathbf{0} = \mathbf{1} < b_2 < \dots < b_n$.

Define $a_1 = 0, \quad b_1 = 1, \quad \sim a_i = -a_i = b_i, \quad \sim b_i = -b_i = a_i,$

$$a_i \cdot a_j = a_{\max(i,j)}, \quad b_i \cdot b_j = b_{\max(i,j)}, \quad a_i \cdot b_j = b_j \cdot a_i = \begin{cases} b_j & \text{if } i < j \\ a_i & \text{otherwise} \end{cases}$$

$(\mathbf{S}_m, \leq, \cdot, \sim, -, 0)$ is the m -element Sugihara ipo-monoid (the m -element lattice-free reduct of the Sugihara monoid)

$\mathbf{S}_2 = \mathbf{2}$ is the two-element Boolean algebra $\mathbf{2} = (\{0, 1\}, \wedge, \neg, \neg, 0)$

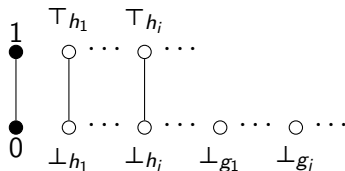
$\mathbf{S}_3$ is also known as the Gaifman-Pratt ipo-monoid

Models from products of groups and Boolean algebras

Let $\mathbf{G}$ be a group with subgroup $\mathbf{H}$.

Define $\mathbf{2}_{\mathbf{G},\mathbf{H}} = \mathbf{H} \times \mathbf{2} \cup (\mathbf{G} \setminus \mathbf{H}) \times \mathbf{1}$ with

$$(g, a) \cdot (g', a') = (gg', aa') \text{ and } \sim(g, a) = (g^{-1}, \sim a) = -(g, a).$$



Then $\mathbf{A}$ is an ipo-monoid. $\mathbf{A}$ is a **conuclear image** of $\mathbf{G} \times \mathbf{2}$ (the conucleus is the map $(g, x) \mapsto (g, \top_g \cdot \top_{g^{-1}} \cdot x)$).

Conuclei on rpo-monoids and ipo-monoids

A **conucleus** δ on an rpo-monoid $\mathbf{A}$ is a **conuclear interior** operator:

- $\delta(x) \cdot \delta(y) \leq \delta(xy)$, and $\delta(x)\delta(1) = \delta(x) = \delta(1)\delta(x)$ (conuclear)
- $x \leq y \implies \delta(x) \leq \delta(y)$, and $\delta(\delta(x)) = \delta(x) \leq x$ (interior)

The **conuclear image** of $\mathbf{A}$ is $\mathbf{A}_\delta = (\delta[A], \leq, \cdot, \backslash_\delta, /_\delta, \delta(1))$,

where $x \backslash_\delta y = \delta(x \backslash y)$ and $x /_\delta y = \delta(x / y)$.

Lemma (Exercise 1)

(a) $\mathbf{A}_\delta$ is an rpo-monoid.

(b) If $\mathbf{A}$ is an ipo-monoid and δ satisfies

- $\delta(-(\delta(\sim\delta(x)))) \leq \delta(x)$ and $\delta(\sim(\delta(-\delta(x)))) \leq \delta(x)$

then $\mathbf{A}_\delta = (\delta[A], \leq, \cdot, \delta(0), \sim_\delta, -_\delta)$ is an ipo-monoid,

where $\sim_\delta x = \delta(\sim x)$ and $-_\delta x = \delta(-x)$.

$2_{\mathbb{Z}_2, \{e\}}$ and $\mathbb{Z}_2 \times 2$

One of the simplest examples of ipo-monoids $2_{\mathbb{Z}_2, \{e\}}$ arises as a conuclear image of the product algebra $\mathbb{Z}_2 \times 2$:



(black dots denote idempotent elements)

$a = \sim a$ acts as an infectious value, but $a^2 = 0$, and $a^3 = a$.

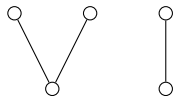
We will later discuss inequational bases for the (po-)varieties generated by $2_{\mathbb{Z}_2, \{e\}}$ and by $\mathbb{Z}_2 \times 2$.

The posets of residuated po-magmas

Up to isomorphism, there are **many** rpo-magmas:

cardinality $n =$	1	2	3	4
number of rpo-magmas	1	3	28	1200

Although rpo-magmas are very general, (res) imposes restrictions on the posets that can occur.



For example, could be the poset of a rpo-magma? (Hint: no)

Lemma (1)

For rpo-magmas, if $a, b \leq c$ then $(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a, b$.

Proof.

Assume $a \leq c$.

Then

$$a/(a \setminus c) \leq c/(a \setminus c)$$



Lemma (1)

For rpo-magmas, if $a, b \leq c$ then $(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a, b$.

Proof.

Assume $a \leq c$.

$\Rightarrow$

Then

$$\begin{aligned} a/(a \setminus c) &\leq c/(a \setminus c) \\ (c/(a \setminus c)) \setminus b &\leq (a/(a \setminus c)) \setminus b \end{aligned}$$



Lemma (1)

For rpo-magmas, if $a, b \leq c$ then $(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a, b$.

Proof.

Assume $a \leq c$.

Then $a/(a \setminus c) \leq c/(a \setminus c)$

$\Rightarrow$

$(c/(a \setminus c)) \setminus b \leq (a/(a \setminus c)) \setminus b$

$\Leftrightarrow$

$(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq b$



Lemma (1)

For rpo-magmas, if $a, b \leq c$ then $(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a, b$.

Proof.

Assume $a \leq c$.

Then $a/(a \setminus c) \leq c/(a \setminus c)$

$\Rightarrow$

$(c/(a \setminus c)) \setminus b \leq (a/(a \setminus c)) \setminus b$

$\Leftrightarrow$

$(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq b$

Assume $b \leq c$.

Then $a \setminus c \leq a \setminus b$



Lemma (1)

For rpo-magmas, if $a, b \leq c$ then $(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a, b$.

Proof.

Assume $a \leq c$.

Then

$$a/(a \setminus c) \leq c/(a \setminus c)$$

$\Rightarrow$

$$(c/(a \setminus c)) \setminus b \leq (a/(a \setminus c)) \setminus b$$

$\Leftrightarrow$

$$(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq b$$

Assume $b \leq c$.

Then

$$a \setminus c \leq a \setminus b$$

$\Leftrightarrow$

$$a(a \setminus c) \leq c$$



Lemma (1)

For *rpo*-magmas, if $a, b \leq c$ then $(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a, b$.

Proof.

Assume $a \leq c$.

$\Rightarrow$

$\Leftrightarrow$

Assume $b \leq c$.

$\Leftrightarrow$

$\Leftrightarrow$

Then

$$\begin{aligned} a/(a \setminus c) &\leq c/(a \setminus c) \\ (c/(a \setminus c)) \setminus b &\leq (a/(a \setminus c)) \setminus b \\ (a/(a \setminus c))((c/(a \setminus c)) \setminus b) &\leq b \end{aligned}$$

Then

$$\begin{aligned} a \setminus c &\leq a \setminus c \\ a(a \setminus c) &\leq c \\ a &\leq c/(a \setminus c) \end{aligned}$$



Lemma (1)

For rpo-magmas, if $a, b \leq c$ then $(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a, b$.

Proof.

Assume $a \leq c$.

Then

$$a/(a \setminus c) \leq c/(a \setminus c)$$

$$\Rightarrow$$

$$(c/(a \setminus c)) \setminus b \leq (a/(a \setminus c)) \setminus b$$

$$\Leftrightarrow$$

$$(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq b$$

Assume $b \leq c$.

Then

$$a \setminus c \leq a \setminus b$$

$$\Leftrightarrow$$

$$a(a \setminus c) \leq c$$

$$\Leftrightarrow$$

$$a \leq c/(a \setminus c)$$

$$\Rightarrow$$

$$(c/(a \setminus c)) \setminus b \leq a \setminus b \leq a \setminus c$$



Lemma (1)

For rpo-magmas, if $a, b \leq c$ then $(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a, b$.

Proof.

Assume $a \leq c$.

$\Rightarrow$

$\Leftarrow$

Then

$$a/(a \setminus c) \leq c/(a \setminus c)$$

$$(c/(a \setminus c)) \setminus b \leq (a/(a \setminus c)) \setminus b$$

$$(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq b$$

Assume $b \leq c$.

$\Leftarrow$

$\Leftarrow$

$\Rightarrow$

$\Rightarrow$

Then

$$a \setminus c \leq a \setminus c$$

$$a(a \setminus c) \leq c$$

$$a \leq c/(a \setminus c)$$

$$(c/(a \setminus c)) \setminus b \leq a \setminus b \leq a \setminus c$$

$$a/(a \setminus c) \leq a/((c/(a \setminus c)) \setminus b)$$

□

Lemma (1)

For rpo-magmas, if $a, b \leq c$ then $(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a, b$.

Proof.

Assume $a \leq c$.

Then

$$a/(a \setminus c) \leq c/(a \setminus c)$$

$\Rightarrow$

$$(c/(a \setminus c)) \setminus b \leq (a/(a \setminus c)) \setminus b$$

$\Longleftrightarrow$

$$(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq b$$

Assume $b \leq c$.

Then

$$a \setminus c \leq a \setminus b$$

$\Longleftrightarrow$

$$a(a \setminus c) \leq c$$

$\Longleftrightarrow$

$$a \leq c/(a \setminus c)$$

$\Rightarrow$

$$(c/(a \setminus c)) \setminus b \leq a \setminus b \leq a \setminus c$$

$\Rightarrow$

$$a/(a \setminus c) \leq a/((c/(a \setminus c)) \setminus b)$$

$\Longleftrightarrow$

$$(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a$$

□

Finite rpo-magmas have bounded components

Lemma (Exercise 2; proof is similar to previous slide)

In any rpo-magma, if $d \leq a, b$ then

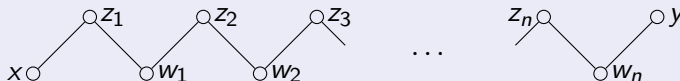
$$a, b \leq d / (((a \setminus d) / (a \setminus (d \setminus d)))((d \setminus d) / (a \setminus (d \setminus d))) \setminus (b \setminus d)))$$

Theorem

*In an rpo-magma every connected component of $\leq$ is up-directed and down-directed, hence for **finite** rpo-magmas every connected component is **bounded**.*

Proof.

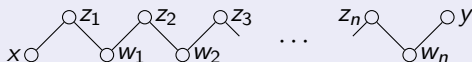
Two elements x, y in a poset are connected iff there exists a zigzag



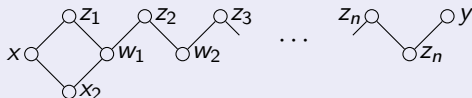
We need to find an upper and a lower bound of x, y .



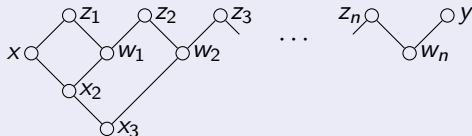
Proof (continued).



Note that $x, w_1 \leq z_1$, so apply Lemma (1) to obtain $x_2 \leq x, w_2$:



Now $x_2, w_2 \leq z_2$, so apply Lemma (1) again to obtain $x_3 \leq x_2, w_2$:



Use induction to get a lower bound of x, y in n steps.

Similarly use Lemma (2) to obtain an upper bound.



What posets occur for rpo-magmas?

In a poset define $x \rightsquigarrow y$ if x and y are in the same connected component.

Recall a **quasigroup** $\mathbf{Q} = (Q, \cdot, \backslash, /)$ is a set with 3 binary operations such that

$$xy = z \iff x = z/y \iff y = x \backslash z.$$

Theorem (Exercise 3)

For any po-algebra the relation $\rightsquigarrow$ is a **congruence**.

For a rpo-magma $\mathbf{A}$ the quotient algebra $\mathbf{A}/\rightsquigarrow$ is a **quasigroup** with $\leq$ as equality.

Conversely, from any quasigroup $\mathbf{Q}$ and a pairwise disjoint family of **bounded** posets A_g for $g \in \mathbf{Q}$, one can construct an rpo-magma with poset $\bigcup_{g \in \mathbf{Q}} A_g$.

Hint: for a quasigroup $\mathbf{Q}$ and $x \in A_g, y \in A_h$ define

$$x \cdot y = \perp_{gh}, \quad x \backslash y = \top_{g \backslash h}, \quad x / y = \top_{g/h}.$$

Residuated po-semigroups

A **rpo-semigroup** or **Lambek algebra** is a rpo-magma where $\cdot$ is associative. These are algebraic models of Lambek Calculus.

Lemma

*Any rpo-semigroup for which the poset is an antichain, is a **group**.*

Proof.

An associative quasigroup is a group: $x = y \setminus yx = y(y \setminus x)$.

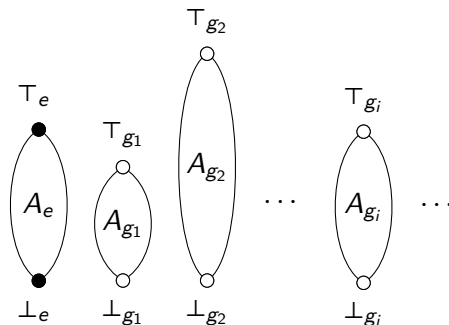
Hence $x = x(x \setminus x) \Rightarrow xy = x(x \setminus x)y \Rightarrow y = x \setminus xy = (x \setminus x)y \Rightarrow y/y = x \setminus x$.

So $x \setminus x = e$ is constant, $xe = x$ and $ey = y$, i.e., e is an identity.

Now $(e/x)x = e \Rightarrow x(e/x)x = x \Rightarrow x(e/x) = x/x = e$, so $x^{-1} = e/x$. □

$\Rightarrow$ For any rpo-semigroup **A** (and therefore **for any ipo-monoid**) the quotient **A**/ $\sim$ is a **group**.

Poset of a rpo-semigroup with a lower bound somewhere



(black dots denote idempotent elements)

Lemma (Exercise 4)

If any connected component is lower bounded, they are all **bounded**.

Unital rpo-monoids with trivial unital connected component

The connected component that contains the identity element is denoted by A_e .

Lemma

Suppose $\mathbf{A}$ is a rpo-magma with an identity element 1.

If $A_e = \{1\}$ then $\mathbf{A}$ is a quasigroup with antichain order.

Proof.

Assume $A_e = \{1\}$. Then $z \leq 1 \Rightarrow 1 \leq z$ and it suffices to show $x \leq y \Rightarrow y \leq x$.

From (res), $1 \leq x/x$ implies $1/(x/x) \leq (1/(x/x)) \cdot (x/x) \leq 1$.

Since 1 is minimal, we have $1 \leq 1/(x/x)$, hence $x/x \leq 1$.

Now assume $x \leq y$. Then $(x/y)x \leq (x/y)y \leq x$. By residuation $x/y \leq x/x \leq 1$.

Again from minimality of 1, we deduce $1 \leq x/y$, hence $y \leq x$. □

It follows that any rpo-monoid with $A_e = \{1\}$ is a **group** with the antichain order, and any ipo-monoid with $A_e = \{1\}$ is a **pointed group** with the antichain order.

Ipo-monoids with additional assumptions

The connected components of $\mathbf{A}$ are denoted A_g for $g \in \mathbf{A}/\sim$, with the **unital component** $1 \in A_e$. All components are up-and-down-directed.

The **order** of $x \in \mathbf{A}$ is the smallest positive $n \in \mathbb{N}$ s.t. $x^n \in A_e$, or ∞ if no n exists.

If $\mathbf{A}/\sim$ is finite and every element has a finite order, they all divide the order of the group, so $\exists n \forall x (x^n \sim 1)$. We are interested in the map $f : A_g \rightarrow A_e$, $f(x) = x^n$, whether it is an order embedding and what is its image.

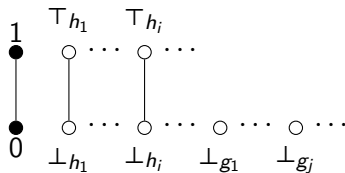
In particular we look at

- Ipo-monoids where $0 \leq 1$, and A_e is idempotent
- Ipo-monoids with $x^{n+1} = x$ for some n (**n -periodic contraction**)
- Ipo-monoids where $x \leq 0$ or $1 \leq x$ for all $x \in A_e$
- **weakly integral** ipo-monoids ($1 \leq x \Rightarrow 1 = x$)
- Ipo-monoids where A_e is an ipo-**Sugihara monoid**, or a **Boolean algebra**.

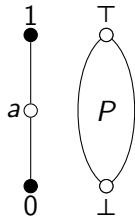
lpo-monoids with specified bounded unital component

$A_e = \{1\}$ implies $\mathbf{A}$ is a group

$A_e = \{0, 1\} = \mathbf{2}$ implies all components have 1 or 2 elements, hence we get $\mathbf{2}_{\mathbf{G}, \mathbf{H}}$



$A_e = \{0 < a < 1\} = \mathbf{L}_3$ with $a^2 = 0$
 $\mathbf{G} = \mathbb{Z}_2$, P is any self-dual poset



$$x \cdot y = \begin{cases} 0 & \text{if } x \leq -y \\ x & \text{if } x \neq 0, y = 1 \\ y & \text{if } x = 1, y \neq 0 \\ a & \text{if } x, y \in P, x \not\leq -y \\ \perp & \text{if } x \in \{0, a\}, y \in P \\ \perp & \text{if } x \in P, y \in \{0, a\} \end{cases}$$

What about cyclicity/commutativity?

- Ipo-monoids of unions of **finite chains** with $0 \leq 1$ are **cyclic** (i.e. $\sim x = -x$).
- **Cyclic** ipo-monoids with a **Boolean** A_e are **commutative** if they satisfy $x^{n+1} = x$ for some $n \geq 2$. Not all cyclic ipo-monoids with a Boolean A_e are commutative.
- If (every element of $\mathbf{A}$ has a finite order), $A_e = \{x \mid x \leq 0 \text{ or } 1 \leq x\}$, $0 \leq 1$ and A_e is idempotent, then
 - ▶ A_e is cyclic if and only if $\mathbf{A}$ is **balanced**
($x \cdot \sim x = -x \cdot x =: 0_x$ and $x \setminus x = x / x =: 1_x$)
 - ▶ If A_e is cyclic then A_e is a commutative subalgebra of $\mathbf{A}$,
 - ▶ If A_e is cyclic and A is finite, then $\mathbf{A}$ is cyclic.
- Ipo-monoids with a **Sugihara** $A_e = S_m$ are all **cyclic** (and therefore balanced).
- There are finite single-component non-cyclic ipo-monoids (6 elements), even chains (7 elements).

A non-cyclic ipo-monoid

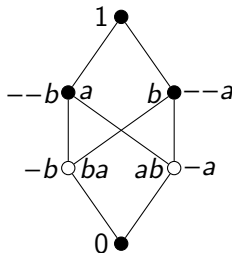


Figure: Smallest noncyclic ipo-monoid with a single connected component.

Ipo-monoids where $0 \leq 1$ (A_e is a subalgebra of $\mathbf{A}$)

Lemma

In an ipo-monoid with $0 \leq 1$, for any n , $(\sim x)^n \leq \sim x^n$ and $(-x)^n \leq -x^n$.

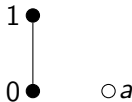
Proof.

By induction, assume $(\sim x)^n \leq \sim x^n$. Then $x^n(\sim x)^n \leq 0 \leq 1$.

Now $x \cdot x^n(\sim x)^n \cdot \sim x \leq x \cdot \sim x \leq 0$.

Hence $x^{n+1}(\sim x)^{n+1} \leq 0$ and therefore $(\sim x)^{n+1} \leq \sim x^{n+1}$. □

The converse inequality $\sim x^n \leq (\sim x)^n$ is not true in general: it fails for $x = a$ and $n = 2$ in the following three-element ipo-monoid with $a = \sim a$ acting as an infectious value, and $a^2 = 0$ and $a^3 = a$:





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