

Residuated partially ordered algebras, logic and preclones: Part 2. Structure theory via Płonka sums of residuated po-semigroups

Stefano Bonzio, José Gil-Férez, Peter Jipsen,
Adam Prenosil, Melissa Sugimoto and Giuseppe Zecchini

ICS, Czech Academy of Sciences, Chapman University
and University of Cagliari

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Residuated partially ordered semigroups

A **residuated partially ordered semigroup** (or **rpo-semigroup**) is a structure

$\mathbf{A} = \langle A, \leq, \cdot, \backslash, / \rangle$ where:

- $\langle A, \leq \rangle$ is a poset,
- $\langle A, \cdot \rangle$ is a semigroup,
- $x \cdot y \leq z \iff x \leq z/y \iff y \leq x \backslash z$ (**residuation**).

If an identity 1 exists, $\widehat{\mathbf{A}} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ is a **residuated po-monoid**

$\langle A, \leq, \cdot, \sim, -, 0 \rangle$ is an **involutive po-monoid** if $\cdot$ is associative and

$$x \leq y \iff x \cdot \sim y \leq 0 \iff -y \cdot x \leq 0.$$

Then $1 := \sim 0$ is the identity element, $x \backslash y = \sim(-y \cdot x)$ and $x/y = -(y \cdot \sim x)$.

Examples:

- Groups with antichain order ($x \backslash y = x^{-1}y$, $x/y = xy^{-1}$),
- Partially ordered groups, Brouwerian semilattices, hoops, MV-algebras
- Subreducts of residuated lattices, complex algebras of monoids

Involutive po-semilattices

$(A, \leq, \cdot, \sim, -)$ is an **ipo-magma** if $\sim, -$ are order-reversing, $\sim -x = x = -\sim x$ and $xy \leq z \iff x \leq -(y \cdot \sim z) \iff y \leq \sim(-z \cdot x)$.

An **ipo-semilattice** $(A, \leq, \cdot, -)$ is an ipo-magma where $\cdot$ is associative, commutative and idempotent.

In an ipo-semilattice there is another partial order $\sqsubseteq$ called the **multiplicative order**, defined by $x \sqsubseteq y \iff xy = x$.

Examples of ipo-semilattices: **Boolean algebras** $(A, \leq, \cdot, -)$, where join is $-(-x \cdot -y)$.

They form a **po-subvariety** defined by $x \cdot -x \leq y \cdot -y$.

ipo-semilattices can be visualized by the **two Hasse diagrams** for $\leq, \sqsubseteq$

Visualizing ipo-semilattices

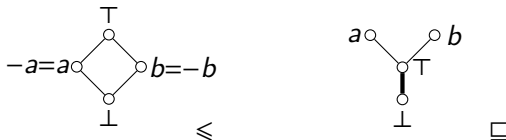


Figure: Partial order and multiplicative order of the smallest ipo-semilattice that does not have an identity element.

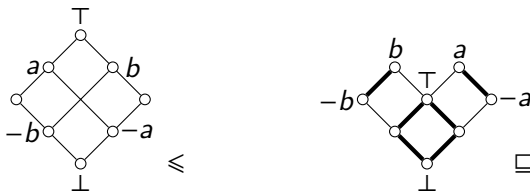


Figure: Smallest ipo-semilattice that is not lattice-ordered.

Structure of ipo-semilattices (later of some rpo-semigroups)

We describe ipo-semilattices based on **Płonka sums of Boolean algebras**.

Similar methods are used by Jenei [2022] to describe the structure of **even and odd involutive commutative residuated chains**.

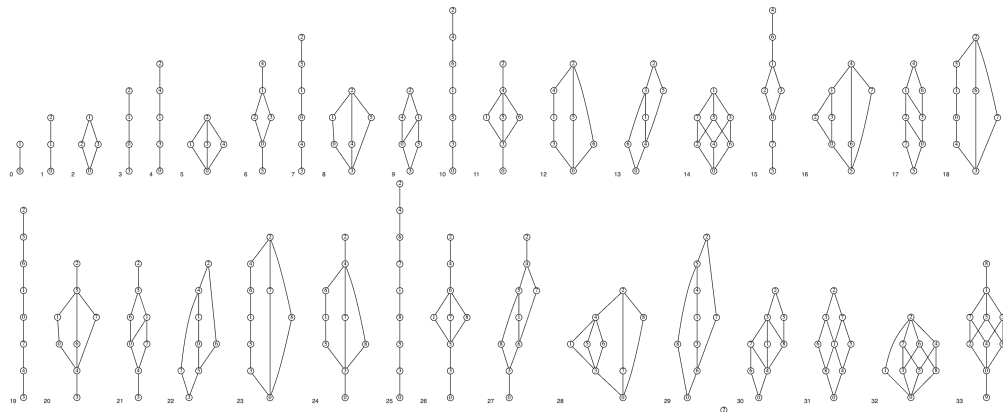
Inspired by a duality for involutive bisemilattices by Bonzio, Loi, Peruzzi [2019], we also give a more compact dual description of finite ipo-semilattices based on **semilattice direct systems of partial maps between sets**.

Why start with ipo-**semilattices**? Actually the project started with commutative idempotent involutive residuated lattices.

# of elem. $n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
CIdInRL	1	1	1	2	2	4	4	9	10	21	22	49	52	114	121	270
CId-ipo-monoids	1	1	1	2	2	4	4	9	10	22	24	53	61	134	157	343
ipo-semilattices	1	1	1	3	4	10	17	43	82	201						

rpo-semigroups 1, 3, 16, 154, 2100

What lattices appear as reducts of unital ipo-semilattices?



Open problem: What variety of lattices is generated the lattice-reducts of lattice-ordered unital ipo-semilattices?

Decomposition into Boolean algebras

An element p in an rpo-magma is **positive** if $x \leq px$ and $x \leq xp$ for all x .

Define the term $1_x = x/x$ and let $E^+(\mathbf{A}) = \{1_x : x \in A\}$.

The fibres of the decomposition are $A_p := \{a \in A : 1_a = p\}$.

Theorem (Partition by Boolean Algebras)

Let $\mathbf{A}$ be an ipo-semilattice. Then

- 1 E^+ is a joinsemilattice of all positive idempotents with $p \vee q = pq$,
- 2 the fibers $(A_p, \cdot, -, p)$ are Boolean algebras with top p and they partition $\mathbf{A}$,
- 3 $1_x \cdot 1_y = 1_{xy}$,

$\mathbf{A}$ is a **Plonka sum** of $\{\varphi_{pq} : A_p \rightarrow A_q \mid p \leq q \in E^+(\mathbf{A})\}$ where $\varphi_{pq}(x) = xq$.

Łonka sums of semigroups

An interesting way to build an algebra from a **union of algebras**:

Let $(I, \vee)$ be a semilattice and $\{\varphi_{ij} : \mathbf{A}_i \rightarrow \mathbf{A}_j \mid i \leq j \in I\}$ a set of homomorphisms of **disjoint** semigroups $\mathbf{A}_i$ such that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ and $\varphi_{ii} = id_{A_i}$.

Then $\mathbf{A} = (\bigcup_{i \in I} A_i, \cdot)$ is a semigroup if for $x \in A_i, y \in A_j$ and $k = i \vee j$

we define $x \cdot y = \varphi_{ik}(x) \cdot_k \varphi_{jk}(y)$.

$\mathbf{A}$ is called the **Łonka sum** of the family of homomorphisms.

Regular identities (all variables on both sides) are preserved by Łonka sums.

ipo-semilattices are Płonka sums of Boolean algebras

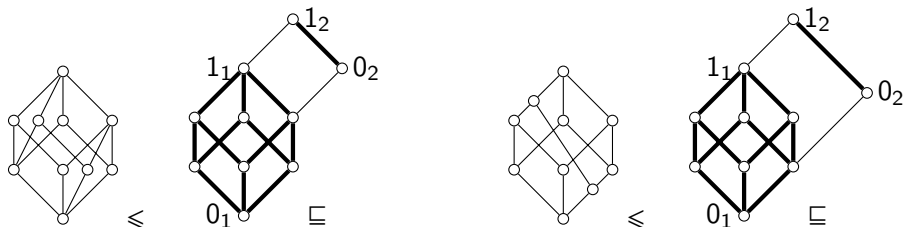


Figure: (Right) unital ipo-semilattice that is **not** an il -semilattice.

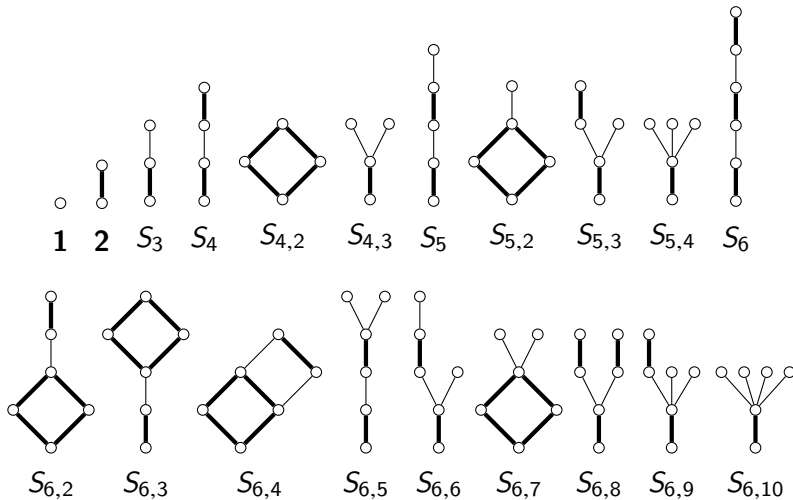
The Boolean fibers are denoted by **thick lines** and are connected by GBA-homomorphisms (thin lines).

For finite CIdInRL the previous theorem is due to [J., Tuyt, Valota 2021].

For $x \in A_p, y \in A_q, r = pq$ define $x \leq y \iff \varphi_{pr}(x) \cdot_r \varphi_{qr}(-y) = 0_r$.

Boolean algebras are the building blocks for ipo-semilattices

Now we can construct all these ipo-semilattices (only $\sqsubseteq$ is shown):



Dual Representation by Partial Functions Between Sets

A **proper partial function** $f : X \rightarrow Y$ is a function from U to Y where $U \subsetneq X$ is the domain of f .

Given a finite ipo-semilattice $\mathbf{A}$, it is a partition of Boolean fibres.

Each (finite) Boolean fiber is determined by its set of atoms.

The partial functions map between sets of atoms (opposite to homomorphisms).

A dual representation of Boolean algebras gives a much more compact way of drawing finite ipo-semilattices.

For $p \leq q$, the **generalized BA homomorphism** φ_{pq} corresponds to the **partial map** $f_{pq} : X_q \rightarrow X_p$ defined by

$$f_{qp}(a) = b \iff a \leq \varphi_{pq}(b) \text{ and } a \not\leq \varphi_{pq}(-p).$$

Dual Representation by Partial Functions Between Sets

The dual representations show clear differences even when the corresponding semilattice orders are visually similar.

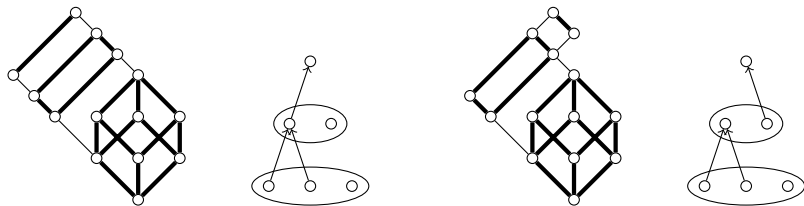


Figure: Two fourteen-element unitary ipo-semilattices and their dual representations.

Construction of Płonka sum of two ipo-monoids

Let $\mathbf{A}, \mathbf{B}$ be disjoint involutive residuated lattices or ipo-monoids, and $\varphi : \mathbf{A} \rightarrow \mathbf{B}$ a monoid homomorphism such that φ preserves existing joins (hence preserves the order) and $\varphi(x) \neq 0^{\mathbf{B}}$ for all $x \in A$.

On $A \cup B$, define $\sim = \sim^{\mathbf{A}} \cup \sim^{\mathbf{B}}$, $- = -^{\mathbf{A}} \cup -^{\mathbf{B}}$,

$$x \cdot y = \begin{cases} x \cdot^{\mathbf{A}} y & \text{if } x, y \in A, \\ x \cdot^{\mathbf{B}} y & \text{if } x, y \in B, \\ \varphi(x) \cdot^{\mathbf{B}} y & \text{if } x \in A \text{ and } y \in B, \\ x \cdot^{\mathbf{B}} \varphi(y) & \text{if } x \in B \text{ and } y \in A, \end{cases} \quad x \leqslant y \iff x \cdot \sim y = 0^{\mathbf{A}}$$

and let $\mathbf{A} \uplus_{\varphi} \mathbf{B} = (A \cup B, \leqslant, \cdot, \sim, -, 0^{\mathbf{A}})$ be the **binary Płonka sum**.

Theorem

For any ipo-monoids $\mathbf{A}, \mathbf{B}$ the po-algebra $\mathbf{A} \uplus_{\varphi} \mathbf{B}$ is an ipo-monoid.

Decomposition into Płonka sum of two ipo-monoid

The term 1_x is defined as $-(x \cdot \sim x) = x/x$.

An ipo-monoid is **balanced** if $x/x = x \backslash x$ and **steady** if $1_x \cdot 1_y = 1_{xy}$.

Theorem

Let $\mathbf{C}$ be a balanced steady involutive residuated lattice or ipo-monoid and assume

(p, q) is a splitting pair of **central idempotent** elements in $\mathbf{C}^+$, i.e., $\mathbf{C}^+ = \downarrow p \uplus \uparrow q$.

Let $A = \{x \in C \mid 1_x \leq p\}$, $B = \{x \in C \mid q \leq 1_x\}$ with restricted operations from $\mathbf{C}$.

Then $\mathbf{A}$ is a subalgebra, $\mathbf{B}$ is a 1-free subalgebra of $\mathbf{C}$ with identity q and $C = A \uplus B$.

Define $\varphi : A \rightarrow B$ by $\varphi(x) = x \cdot q$.

Then φ is a monoid homomorphism that preserves all existing joins and $\mathbf{C} = \mathbf{A} \uplus_{\varphi} \mathbf{B}$.

Commutative ipo-monoids where $0 \leq 1$ and A_e idempotent

Lemma

Let $\mathbf{A}$ be a commutative ipo-monoid with a unital component that is idempotent. Then

- A_e is a locally-integral ipo-monoid. Positive elements A^+ of A_e form a join-semilattice.
- $\mathbf{A}$ is steady: $1_x \cdot 1_y = 1_{xy}$

Proof (sketch).

$\mathbf{A}$ is commutative, and therefore cyclic and balanced. The units $1_x = x \setminus x \geq 1$ are positive, $1_x \cdot x = x$. If $x \in A_e$, then $x \leq 1_x$ and $x \setminus 1_x = 1_x$ by idempotence.

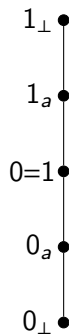
By Thm. 5.1 of the paper below, **fibers $A_{1_x} = \{y \mid 1_x = 1_y\}$ of A_e are Boolean.**

Proof of steadiness uses commutativity in an essential way. □

A_e can be constructed as a **Płonka sum** of its fibers A_p , indexed by the join semilattice A^+ . **Which ipo-monoids $\mathbf{A}$ can we construct using Płonka sums?**

J. Gil-Férez, P. Jipsen, M. Sugimoto: Locally integral involutive PO-semigroups, Fundamenta Informaticae, Volume 195, 1-4, 7, (2025) 1-29

Example: S_5



$$S_5 = 1 \uplus_{\varphi_1} 2 \uplus_{\varphi_2} 2$$

Commutative ipo-monoids with Boolean unital component

Lemma

Let $\mathbf{A}$ be a commutative ipo-monoid with a unital component that is a Boolean algebra $\mathbf{B}$ and denote the group $\mathbf{A}/\sim$ by $\mathbf{G}$. Then $\mathbf{A}$ is a conuclear image of $\mathbf{G} \times \mathbf{B}$ and all components of $\mathbf{A}$ are isomorphic to principal ideals of $\mathbf{B}$.

Proof (sketch).

The conuclear map on $\mathbf{G} \times \mathbf{B}$ is defined by $\delta((g, x)) = (g, \top_g \cdot \top_{g^{-1}} \cdot x)$. The proof uses that $\top_g \cdot \top_{g^{-1}} \cdot \top_g = \top_g$. □

In case every element of $\mathbf{A}$ has a finite order, we obtain the following insight: let order of $x, y \in A_g$ be n . Then

- $x^n = y^n$ iff $x = y$, and
- $x = x^{n+1} = x \cdot \top_g \cdot \top_{g^{-1}}$.

We will use **conuclear images of $\mathbf{G} \times \mathbf{B}$** as **fibers** of **Plonka sums** to construct commutative ipo-monoids (e.g. those with Sugihara unital component).

Ipo-monoids where $0 \leq 1$, and $x \leq 0$ or $1 \leq x$ for all $x \in A_e$

Note that the assumptions hold if $A_e \simeq \mathbf{S}_k$.

Lemma

Let A be an ipo-monoid such that $0 \leq 1$ and $(x \leq 0 \text{ or } 1 \leq x)$ for all $x \in A_e$.

If $x, y \in A_g$ have order n then $f : A_g \rightarrow A_e$ given by $f(x) = x^n$ is an order embedding.

Proof.

Assume $x^n \leq y^n$. We want to show $x \leq y$.

Suppose $x \not\leq y$, then $x \cdot \sim y \not\leq 0$. Note $x, y \in A_g \implies x \cdot \sim y \in A_e$.

From $(z \leq 0 \text{ or } 1 \leq z)$ for $z \in A_e$ we conclude $1 \leq x \cdot \sim y$.

Hence $1 \leq x \cdot \sim y \leq x \cdot 1 \cdot \sim y \leq x \cdot x \cdot \sim y \cdot \sim y \leq \dots \leq x^n \cdot (\sim y)^n$.

By the preceding lemma and $x^n \leq y^n$, $x^n \cdot (\sim y)^n \leq x^n \cdot y^n \leq 0$.

Therefore $x \cdot \sim y \leq 0$, a contradiction. □

Structure of ipo-monoids with $A_e \cong S_k$

Theorem

Let $\mathbf{A}$ be an ipo-monoid with $A_e \cong S_k$ where every element is of a finite order. Then $\mathbf{A}$ is a disjoint union of chains, each with $\leq k$ elements, and $\cdot$ determined by subgroups of $\mathbf{A}/\sim$. Moreover, $\mathbf{A}$ is cyclic and balanced.

Proof (outline).

Let $\mathbf{G} = \mathbf{A}/\sim$. The map $f : A_g \rightarrow A_e$ is an **order embedding**. Therefore all connected components are chains with $\leq k$ elements.

For $1 \leq l \leq k$, let $\mathbf{H}_l = \{g \in \mathbf{G} \mid l \leq |A_g|\}$.

Then $\mathbf{H}_k \leq \dots \leq \mathbf{H}_2 \leq \mathbf{H}_1 = \mathbf{G}$ is a nested sequence of subgroup of $\mathbf{G}$ and these subgroups uniquely determine the multiplication on the algebra. $\square$

Structure of ipo-monoids with $A_e \cong \mathbf{S}_k$

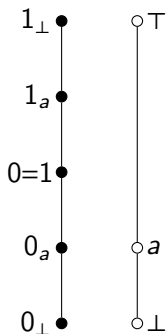


Figure: The ipo-monoid $\mathbf{1} \uplus_{\varphi_1} \mathbf{2}_{\mathbb{Z}_2, \{e\}} \uplus_{\varphi_2} (\mathbb{Z}_2 \times \mathbf{2})$ with S_5 as unital component

Structure of commutative ipo-monoids with $A_e \cong \mathbf{S}_k$

Every **commutative** ipo-monoid $\mathbf{A}$ with $A_e \cong \mathbf{S}_k$ can also be constructed using a **Płonka sum** over

- a linear join-semilattice of positive elements A^+ with the bottom element 1,
- $E^+(\mathbf{A})$ directed system of ipo-monoids $A_p = \{x \in A \mid 1_x = p\}$ (fibers)
- $|A_p \cap A_g| \leq 2$
- fibers are ipo-monoids of the form $2_{\mathbf{G}, \mathbf{H}}$.

$\Rightarrow$ **Every commutative ipo-monoid $\mathbf{A}$ with $A_e \cong \mathbf{S}_k$ and $\mathbf{G} = \mathbf{A}/\sim$ is a conuclear image of $\mathbf{G} \times \mathbf{S}_k$** (due to X. Zhuang for odd k).

J. Gil-Férez, P. Jipsen, M. Sugimoto: Locally integral involutive PO-semigroups, *Fundamenta Informaticae*, Volume 195, 1-4, 7, (2025) 1–29.

S. Bonzio, J. Gil-Férez, P. Jipsen, A. Přenosil, M. Sugimoto: On the structure of balanced residuated partially ordered monoids, *RAMiCS 2024*, LNCS Vol 14787, Springer 2024, 83–100.

X. Zhuang: Unilinear residuated lattices, (2023) Dissertation, digitalcommons.du.edu/etd/2335/

Structure of commutative ipo-monoids with $A_e \cong \mathbf{S}_n$

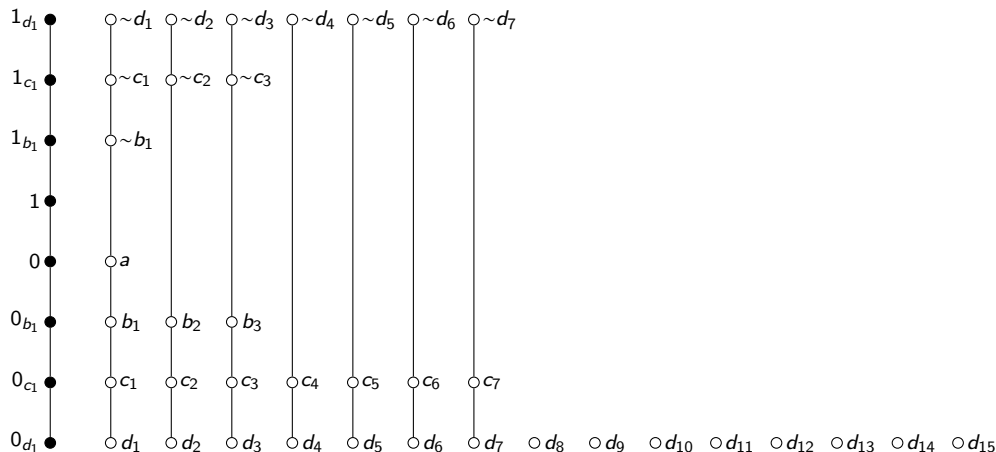
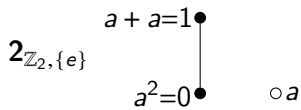
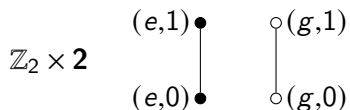


Figure: Example of a Płonka sum $2_{\mathbb{Z}_2, \{e\}} \uplus_{\varphi_1} 2_{\mathbb{Z}_2^2, H_1} \uplus_{\varphi_2} 2_{\mathbb{Z}_2^3, H_2} \uplus_{\varphi_3} 2_{\mathbb{Z}_2^4, H_3}$

$\mathbb{Z}_2 \times 2$ and $2_{\mathbb{Z}_2, \{e\}}$ -generated (po-)varieties



The order is definable: $x \leq y \iff 1 = x \setminus y$.

The term $\tau(x) := \sim x^2 \cdot (x + x)$ equals 1 if $x = a$, and equals 0 otherwise.

(quasi-)inequational bases (CONJECTURE) for the varieties generated by

- $\mathbb{Z}_2 \times 2$

- Boolean axioms instances in $(\cdot, +, \sim, 0)$ for x^2
- ipo-monoid axioms
- $x^3 = x$
- $\sim(x^2) = (\sim x)^2$

- $2_{\mathbb{Z}_2, \{e\}}$

- Boolean axioms instances in $(\cdot, +, \sim, 0)$ for x^2
- ipo-monoid axioms
- $x^3 = x$
- $\tau(x) \cdot \tau(y) \leq \tau(x) \cdot (\sim(xy))^2$

Positive elements and balanced residuated semigroups

An element $p \in A$ is **positive** if for all $a \in A$:

$$a \leq pa \quad \text{and} \quad a \leq ap \quad (\iff p \backslash a \leq a \text{ and } a / p \leq a).$$

A^+ is an upset. If 1 exists, $A^+ = \{a \in A : 1 \leq a\}$.

$E^+ \mathbf{A} = \{p \in A : p^2 = p \text{ and } p \text{ is positive}\}$ is the set of **positive idempotents**.

A residuated semigroup is **balanced** if:

- It satisfies $x \backslash x = x / x$ (abbreviated by 1_x), and
- All self-residuals are positive ($y \leq 1_x \cdot y$ and $y \leq y \cdot 1_x$).

Then 1_x is a **local identity**: $1_x \cdot x = x = x \cdot 1_x$.

Non-commutativity of positive idempotents

In a general residuated semigroup, $E^+ \mathbf{A}$ need not be closed under multiplication, nor commutative.

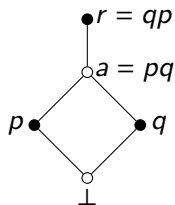


Figure: The product of idempotent positives is not commutative in general.

Here $E^+ \mathbf{A} = \{p, q, r\}$. The element $a = pq$ is positive, but not idempotent:

$$aa = (pq)(pq) = p(qp)q = prq = rq = r.$$

Centrality of positive idempotents

(Numbered results refer to <https://arxiv.org/abs/2505.12024>)

Proposition (2.5)

For every residuated semigroup $\mathbf{A}$ and $p \in E^+ \mathbf{A}$, the following are equivalent:

- ① p is **central** ($pa = ap$ for all $a \in A$),
- ② $pa = a \iff ap = a$,
- ③ $p \setminus a = a/p$ for all $a \in A$,
- ④ $p \setminus A = A/p$, where $p \setminus A = \{p \setminus a : a \in A\}$, $A/p = \{a/p : a \in A\}$.

Proof Sketch: (2) $\implies$ (1) and (3) $\implies$ (4).

For (2) $\implies$ (1): $p(pa) = pa \implies (pa)p = pa$.

Also $p(ap) = ap$. Thus $pa = pap = ap$.

For (3) $\implies$ (4): $p \setminus (p \setminus a) = (pp) \setminus a = p \setminus a$, so $(p \setminus a)/p = p \setminus a = a/p$. □

Centrality of positive idempotents

(2) holds in every **balanced** residuated semigroup:

$$a \leq pa \iff a \leq ap \text{ by positivity}$$

$$pa \leq a \iff p \leq a \backslash a \iff p \backslash a / a \iff ap \leq a \text{ since } x \backslash x = x / x.$$

$$\text{Let } ZE^+ \mathbf{A} = \{p \in E^+ \mathbf{A} : px = xp \text{ for all } x \in A\}$$

Then in every **balanced** residuated semigroup, $E^+ \mathbf{A} = ZE^+ \mathbf{A}$

and it forms a join-semilattice $E^+ \mathbf{A} = \langle E^+ \mathbf{A}, \cdot \rangle$.

Subalgebra partition: Condition (H)

For $p \in E^+ \mathbf{A}$, define the fiber:

$$A_p = \{a \in A : a \setminus a = p\} = \{a \in A : a / a = p\}.$$

The family $\{A_p : p \in E^+ \mathbf{A}\}$ partitions A .

To ensure each A_p is a **subalgebra**, we impose **Condition (H)**:

$$(H1) \quad 1_x = 1_y \implies 1_{xy} = 1_x,$$

$$(H2) \quad 1_x = 1_y \implies 1_{x/y} = 1_x,$$

$$(H3) \quad 1_x = 1_y \implies 1_{x \setminus y} = 1_x.$$

Proposition (3.4)

In residuated semigroups with central 1_x :

- $(H2) \iff (H3)$,
- Each of $(H2)$ and $(H3)$ implies $x \setminus x = x / x$,
- Both imply $(H1)$.

Proof Details: Properties of Condition (H)

Proof of Proposition 3.4 (selected steps).

Assume $1_x y = y 1_x$ and (H2).

(1) Since $a 1_a = 1_a a \leq a$, we have $1_a \leq a \backslash a$. Also $a(a \backslash a)(a \backslash a) \leq a \implies a \backslash a \leq 1_{a \backslash a}$.
Using (H2) and $1_a \leq a \backslash a$:

$$a \backslash a \leq 1_{a \backslash a} \leq (a/1_a)/a = a/(1_a a) = (a/a)/1_a = 1_{a/a} = 1_a.$$

Thus $a \backslash a = 1_a = a/a$, so **A** is balanced.

(2) To show (H2) $\implies$ (H1): Let $1_a = 1_b$. Then $(1_a/a)ab \leq b \implies ab \leq (1_a/a) \backslash b$.
Therefore:

$$1_{ab} = ab/ab \leq (1_a/a) \backslash 1_b/a = (1_a/a) \backslash (1_a/a) = 1_{1_a/a} = 1_a \leq 1_{ab}.$$

Hence $1_{ab} = 1_a$, which proves (H1). □

Decomposition into integrally closed monoids

A residuated monoid is **integrally closed** if $x \backslash x = 1$ (equivalently $x / x = 1$).

Theorem (3.5 Decomposition result)

Every balanced residuated semigroup $\mathbf{A}$ satisfying (H) decomposes into a disjoint union of integrally closed residuated monoids:

$$\mathbf{A}_p = \langle A_p, \leq_p, \cdot_p, \backslash_p, /_p, 1_p \rangle \quad (p \in E^+ \mathbf{A})$$

where:

- $p = 1_p$ is the global identity of $\mathbf{A}_p$ and its unique positive idempotent,
- The constant-free reduct of $\mathbf{A}_p$ is a subalgebra of $\mathbf{A}$,
- $\mathbf{A}$ is square-decreasing ($x^2 \leq x$) $\iff$ each $\mathbf{A}_p$ is integral ($a \leq p$).

Example: Residuated poset satisfying (H)

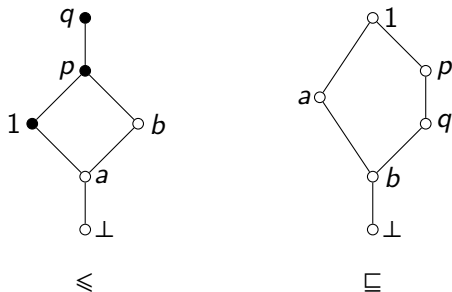


Figure: Residuated poset satisfying condition (H).

$\cdot$ is the meet $\sqcap$ of $\langle A, \sqsubseteq \rangle$, preserving joins in $\langle A, \leq \rangle$.

$E^+ \mathbf{A} = \{1, p, q\}$, with fibers $A_1 = \{1, a\}$, $A_p = \{p\}$, $A_q = \{q, b, \perp\}$.

Fibers are closed under $\cdot, \backslash, / \implies \mathbf{A}$ satisfies (H).

Łonka sums of residuated semigroups

Theorem (Łonka sum construction)

Let $(I, \vee)$ be a semilattice and $\{\varphi_{ij}, \psi_{ij} : \mathbf{A}_i \rightarrow \mathbf{A}_j \mid i \leq j \in I\}$ two families of maps of **disjoint** rpo-monoids $\mathbf{A}_i$ such that for all $i \leq j \leq k$

- $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$, $\varphi_{ii} = id_{A_i}$, $\varphi_{ij}(xy) = \varphi_{ij}(x)\varphi_{ij}(y)$, $\varphi_{ij}(1_i) = 1_j$
- $\psi_{jk} \circ \psi_{ij} = \psi_{ik}$, $\psi_{ii} = id_{A_i}$, $\psi_{ij}(x \setminus y) = \varphi_{ij}(x) \setminus \psi_{ij}(y)$, $\psi_{ij}(x / y) = \psi_{ij}(x) / \varphi_{ij}(y)$,

(S1) if $i \prec j, k$ and $\ell = j \vee k$, then $\varphi_{j\ell} \circ \psi_{ij} \leq_\ell \psi_{k\ell} \circ \varphi_{ik}$ pointwise, and

(S2) if $i \prec j$ and $x, y \in A_i$ and $\varphi_{ij}(x) \leq_j \psi_{ij}(y)$, then $x <_i y$.

Then $\mathbf{A} = (\bigcup_{i \in I} A_i, \cdot, /, \setminus)$ is an rpo-semigroup if for $k = i \vee j$ we define

$$x \cdot y = \varphi_{ik}(x) \cdot_k \varphi_{jk}(y), \quad x / y = \psi_{ik}(x) /_k \varphi_{jk}(y),$$

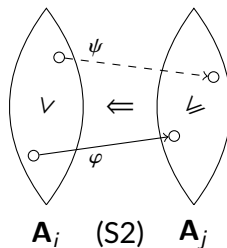
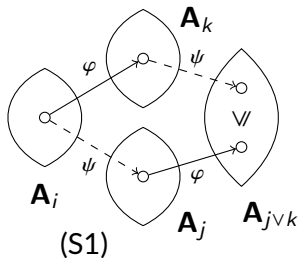
$$x \setminus y = \varphi_{ik}(x) \setminus_k \psi_{jk}(y) \text{ and } x \leq y \iff \varphi_{ik}(x) \leq_k \psi_{jk}(y).$$

$\mathbf{A}$ is the **po-Łonka sum** of the family of **meta**morphisms $\langle \varphi, \psi \rangle$.

About the conditions (S1) and (S2) for posets $\mathbf{A}_i$ ($i \in I$)

Let $\langle \varphi, \psi \rangle$ be two directed systems of monotone maps $\varphi_{ij}, \psi_{ij}: \mathbf{A}_i \rightarrow \mathbf{A}_j$. Define the relation $\leq$ on $A = \biguplus \mathbf{A}_i$ by

$$x \leq y \iff \varphi_{ik}(x) \leq_k \psi_{jk}(y), \quad \text{where } k = i \vee j.$$



Theorem (5.1)

The relation $\leq$ is a partial order extending each $\leq_i$ **iff** $\langle \varphi, \psi \rangle$ satisfies:

(S1) $i \prec j, k \implies \varphi_{j\ell}\psi_{ij} \leq_\ell \psi_{k\ell}\varphi_{ik}$ pointwise ($\ell = j \vee k$),

(S2) $i \prec j, x, y \in A_i$ and $\varphi_{ij}(x) \leq_j \psi_{ij}(y) \implies x <_i y$.

All steady residuated po-semigroups are Płonka sums

In a balanced $(x/x = x \backslash x)$ rpo-semigroup, define the term $1_x = x/x$.

A balanced rpo-semigroup **A** is **steady** if it satisfies $1_{xy} = 1_x 1_y = 1_{x/y}$

Theorem (Płonka sum decomposition)

Let **A** be a steady *r*-semigroup, let $E^+(\mathbf{A}) = \{p \in A \mid x \leq px, x \leq xp, pp = p\}$,

define the fibers $A_p = \{x \in A : 1_x = p\}$,

and maps $\varphi_{pq}(x) = xq$, $\psi_{pq}(x) = x/q$.

Then the $\mathbf{A}_p = (A_p, \leq_p, \cdot_p, /_p, \backslash_p, p)$ are rpo-monoids with the induced operations, unique positive idempotents, the conditions of the previous theorem hold and the Płonka sum of these fibers reconstructs **A**.

Steady residuated semigroups

Let $\mathbf{A}$ be balanced over a subsemilattice $\mathbf{I} \leq \mathbf{ZE}^+\mathbf{A}$ and assume the following exist:

$$u_a := \max\{p \in I : pa = a\} = \max\{p \in I : ap = a\}.$$

$\mathbf{A}$ is **steady over $\mathbf{I}$** if for all $a, b \in A$:

(St1) $u_{ab} = u_a \cdot u_b,$

(St2) $u_{a/b} = u_a \cdot u_b,$

(St3) $u_{b \setminus a} = u_a \cdot u_b.$

$\mathbf{A}$ is **steady** if it is steady over $\mathbf{E}^+\mathbf{A}$, i.e., if the following identities hold:

$$1_{xy} = 1_x \cdot 1_y \quad 1_{x/y} = 1_x \cdot 1_y \quad 1_{y \setminus x} = 1_x \cdot 1_y.$$

S. Jenei has a recent preprint about how to avoid steadyness by defining coherent unions of fibres <https://arxiv.org/abs/2607.24689>.

Iterated decomposition: Example 1

When $\mathbf{A}$ is not steady over $\mathbf{E}^+\mathbf{A}$, it can (sometimes) be decomposed **iteratively** over subsemilattices. E.g., below (St1) fails: $1_a 1_p = 1p = p \neq q = 1_b = 1_{a.p}$

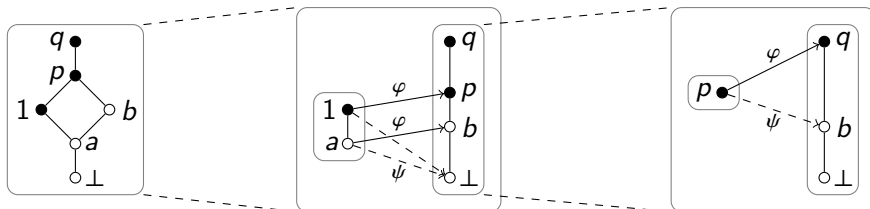
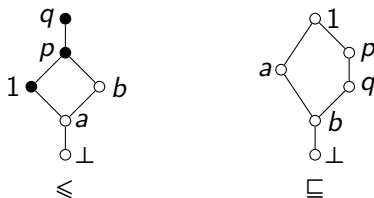


Figure: Iterated decomposition of the above Example over $\mathbf{I} = \{1, p\}$.

Iterated decomposition: Example 2

In the 4-element chain $\perp \leq 1 \leq p \leq q$, (St2) fails:

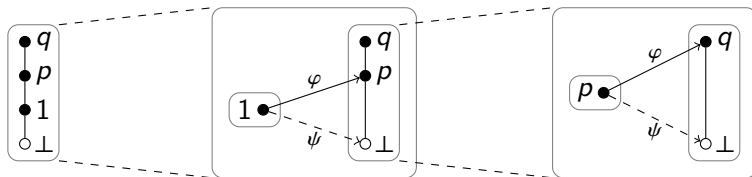


Figure: Iterated decomposition of the 4-element chain over $I = \{1, p\}$.

Balanced idempotent residuated semigroups

Proposition (7.1)

The following are equivalent for each idempotent residuated semigroup $\mathbf{A}$:

- ① $1_x \cdot y = y \cdot 1_x$,
- ② $x \backslash x = x / x$,
- ③ $\mathbf{A}$ is commutative ($xy = yx$).

Moreover, each of these conditions implies Condition (H).

Proof Sketch: (2) $\implies$ (3).

$bba = ba \implies b \leq ba / ba = ba \backslash ba$ and $bab \leq ba$.

Multiplying by a : $abab \leq aba \implies ab \leq aba$ by idempotence.

Similarly $baa = ba \implies a \leq ba \backslash ba = ba / ba \implies aba \leq ba \implies ab \leq ba$.

By symmetry $ba \leq ab$, so $ab = ba$.



Brouwerian components and meet-preservation

Corollary (7.3)

Every integrally closed idempotent residuated monoid is a **Brouwerian semilattice** (with $x \wedge y = xy$, $x \rightarrow y = x \backslash y$, and top element $1 = x \backslash x$).

Proposition (7.4)

In every commutative idempotent residuated lattice:

$$1_x = 1_y \leq 1_z \implies (x \wedge y)1_z = x1_z \wedge y1_z.$$

That is, the linking maps $\varphi_{pq}(x) = xq$ preserve binary meets.

Proof.

Fibers $\mathbf{A}_p$ are Brouwerian algebras, so $ab = a \wedge b$ on A_p . For $q = 1_c \geq 1_a = 1_b$, aq and bq lie in A_q , so $aqbq = aq \wedge bq$. By centrality and idempotence:

$$(x \wedge y)q = (xy)q = (xq)(yq) = xq \wedge yq.$$

□

Example: Two-fiber Płonka sum

Let $\mathbf{A}_1, \mathbf{A}_2$ be residuated posets, indexed over $1 < 2$, with:

$$\varphi_{12}(a) = 1^{\mathbf{A}_2} \quad \text{and} \quad \psi_{12}(a) = 0^{\mathbf{A}_2} \quad (0^{\mathbf{A}_2} < 1^{\mathbf{A}_2}).$$

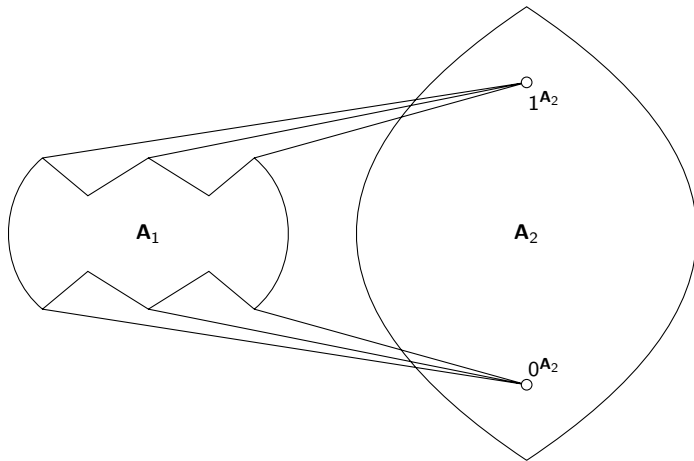


Figure: Płonka sum $\mathbf{A}_1 \uplus \mathbf{A}_2$ of two residuated posets.

Application: Relation algebra $\mathcal{P}(\mathbb{Z}_2)$

For any monoid $\mathbf{M} = \langle M, \cdot, e \rangle$, the **complex algebra** $\mathcal{P}(\mathbf{M})$ is a residuated lattice.

The 2-fiber Płonka sum decomposes $\mathcal{P}(\mathbb{Z}_2)$ into $\mathbb{Z}_2$ and the 2-element Boolean algebra $\mathbf{2}$:

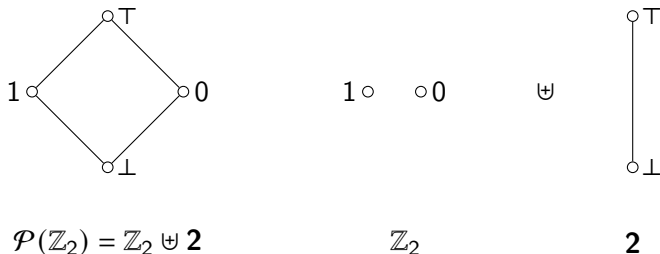


Figure: 4-element relation algebra $\mathcal{P}(\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbf{2}$.

$\mathcal{P}(\mathbb{Z}_2)$ is not locally integral, but it is steady.

Summary of structural results

Class	Defining Axioms	Decomposition and Fibers
Residuated semigroups	Poset $\langle A, \leq \rangle$, semigroup $\langle A, \cdot \rangle$, $xy \leq z \Leftrightarrow x \leq z/y \Leftrightarrow y \leq x \backslash z$	None in general.
Balanced resid. semigroups	$x \backslash x = x/x (= 1_x)$, $y \leq 1_x y, y \leq y 1_x$ (self-residuals positive)	Partitioned into disjoint sets A_p ($p \in E^+ \mathbf{A}$); not necessarily subalgebras.
Balanced with (H)	Balanced and (H1) $1_x = 1_y \Rightarrow 1_{xy} = 1_x$ (H2) $1_x = 1_y \Rightarrow 1_{x/y} = 1_x$	Disjoint subalgebras $\mathbf{A}_p$ ($p \in E^+ \mathbf{A}$) which are integrally closed residuated monoids.
Steady residuated semigroups	Balanced and (St1) $1_{xy} = 1_x 1_y$ (St2) $1_{x/y} = 1_x 1_y$ (St3) $1_{y \backslash x} = 1_x 1_y$	Generalized Płonka sum of metamorphisms of integrally closed residuated monoids over $E^+ \mathbf{A}$.
Idempotent steady resid. semigroups	Steady, $x^2 = x$, $1_x y = y 1_x$ (equivalently, commutativity)	Decomposes into Brouwerian semilattices.

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Thank you!