

Residuated partially ordered algebras, logic and preclones

Part 3. S -preclones, S -relations, and the ${}^S\text{Pol}$ – ${}^S\text{Inv}$ Galois connection

Peter Jipsen

joint work with Erkki Lehtonen and Reinhard Pöschel

Chapman University, University of Linz, University of Dresden

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Outline

- Partially ordered algebras
- Clones and preclones
- S -preclones
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- The lattice of S -preclones
- Maximal and minimal Boolean $\pm$ preclones
- Preimage classes of clones

Partially ordered algebras

Unsorted universal algebra has been developed over the last century

Central concepts: signature, algebras, homomorphisms, congruences, subalgebras, products, HSP, varieties, free algebras, clones, ...

Every algebra $\mathbf{A} = (A, (f^{\mathbf{A}} \mid f \in \mathcal{F}))$ has an underlying **set** A as its universe

A **partially ordered algebra** $\mathbf{A} = (A, \leq^{\mathbf{A}}, (f^{\mathbf{A}} \mid f \in \mathcal{F}))$ has an underlying **poset** $A = (A, \leq^{\mathbf{A}})$, as its universe; the **dual poset** A^{∂} is $(A, \geq^{\mathbf{A}})$

Each **fundamental operation symbol** $f \in \mathcal{F}$ corresponds to an operation $f^{\mathbf{A}}$ of $\mathbf{A}$ that is order-preserving **or order-reversing** in each argument

This information is part of the **signature** f^{λ} where $\lambda \in \bigcup_{n \geq 1} \{+, -\}^n$

E.g. f^{+-} means f is binary and $f^{\mathbf{A}}: A \times A^{\partial} \rightarrow A$ is order-preserving

Partially ordered algebras

In standard universal algebra the base category is the category of **sets**

For **partially ordered algebras (po-algebras)** the base category is **Pos** = the category of posets with order-preserving maps as morphisms

But term-operations on **A** are not necessarily morphisms in **Pos**

For $/^{+-}$, $x/\mathbf{A}x$ may **not** be order-preserving or order-reversing

Varieties of algebras with order-preserving operations have been studied by [Bloom 1976], [Bloom and Wright 1983], [Kurz and Velebil 2017], ...

However for algebraic logic, **negation**, **implication** and **residuation** are important operations, and they are **order-reversing** in some arguments

[Pigozzi 2004] treats po-algebras where arguments may be order-reversing.

Clones of algebras

Boolean algebras $(A, \wedge, \vee, \neg, 0, 1)$ are term-equivalent to Boolean rings $(A, +, \cdot, -, 0, 1)$ (= rings where $x \cdot x = x$).

I.e., they have the same clone of term-operations.

The **term clone** of an algebra **A** is the set of term-operations built from the basic operations listed in the algebra.

In algebra we study clones of algebras to **avoid choosing specific basic operations** that generate the clone.

Definition of clone

For a set A , let $\text{Op}(A) = \bigcup_{n \geq 1} A^{A^n}$ = the set of all finitary operations on A

A set $F \subseteq \text{Op}(A)$ is a **clone** on A if

F contains the **identity operation** id_A and

F is closed under $\zeta, \tau, \nabla, \Delta, \circ$ where

- ζ **cycles** arguments: $(\zeta f)(x_1, \dots, x_n) = f(x_2, \dots, x_n, x_1)$
- τ **permutes** first arguments: $(\tau f)(x_1, \dots, x_n) = f(x_2, x_1, x_3, \dots, x_n)$
- ∇ **adds a fictitious** argument: $(\nabla f)(x_1, \dots, x_n) = f(x_2, \dots, x_n)$
- Δ **identifies first 2** arguments: $(\Delta f)(x_1, \dots, x_n) = f(x_1, x_1, x_2, \dots, x_n)$
- $\circ$ **composes** f, g : $(f \circ g)(x_1, \dots, x_{m+n}) = f(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n})$

The lattice of all clones on a set

The collection $\mathcal{L}_A$ of all clones on a set A is closed under intersections.

So it forms a **complete lattice** under subset inclusion.

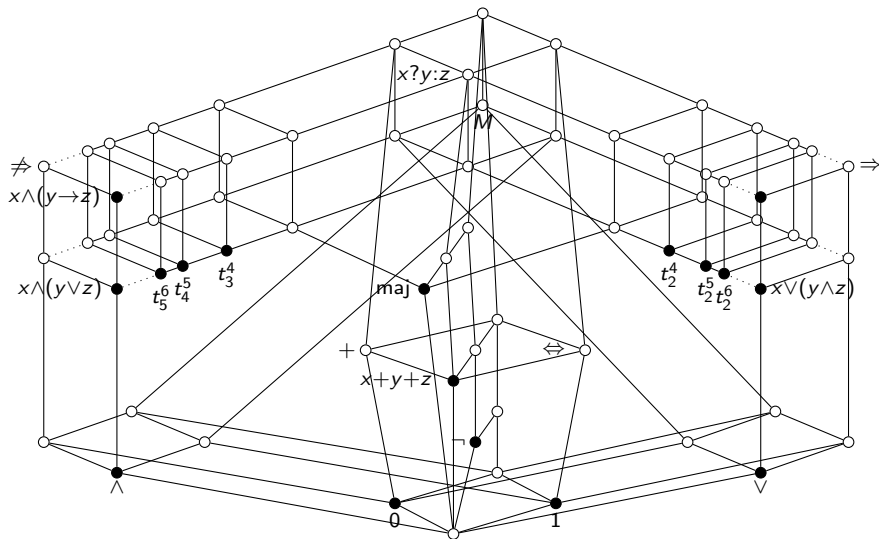
$\langle X \rangle = \bigcap \{F \mid X \subseteq F \text{ a clone}\} = \text{clone generated by } X.$

$$F \vee G = \langle F \cup G \rangle$$

The top is $\text{Op}(A)$ and the bottom is $\{p_i^n \mid i \leq n \in \mathbb{Z}^+\}$

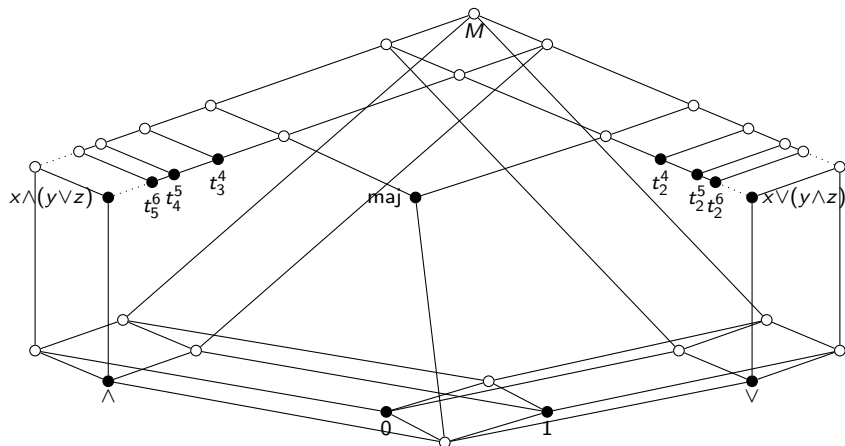
where $p_i^n(x_1, \dots, x_n) = x_i$ is the i^{th} **projection**.

The Post lattice of all clones on a 2-element set



$$t_k^n(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } |\{i \mid x_i = 1\}| \geq k \\ 0 & \text{otherwise.} \end{cases}$$

The monotone clones in the Post lattice



$$t_k^n(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } |\{i \mid x_i = 1\}| \geq k \\ 0 & \text{otherwise.} \end{cases}$$

Partially ordered clones

Nov 13, 2021 in the Panglobal Algebra and Logic Seminar (PALS) I gave a talk on po-algebras

For a poset $(P, \leq)$, a **po-clone** $(\mathcal{C}, \leq)$ is a clone $\mathcal{C}$ on P that is generated by operations that are order-preserving or order-reversing in each argument

In the Post lattice the clones $\langle x + y + z \rangle$, $\langle + \rangle$, $\langle \Leftrightarrow \rangle$ are **not** po-clones

What about $\langle x + y + z, \neg \rangle$, $\langle x + y + z, \text{maj} \rangle$ or $\langle +, 1 \rangle$?

The remaining clones on 2 elements are po-clones

Reinhard Pöschel and Martin Goldstern emailed me an answer: The 3 are also not po-clones.

Motivation: Beyond Clones

- In classical clones, operations are closed under arbitrary composition, variable permutation, and **identification of arguments** (Δ).

Failure of full clone closure for po-algebras

- In algebraic logic and partially ordered algebras $(A, \leq)$, fundamental operations (e.g., negation $\neg x$, implication $x \rightarrow y$, residuals $x \backslash y$) are **not** order-preserving in all arguments.
- Typically, each coordinate is either **order-preserving** $(+)$ or **order-reversing** $(-)$.
- Identifying a preserving argument $(+)$ with a reversing argument $(-)$ can destroy the order-preserving or reversing property.
- Thus, such function collections do **not** form a clone.
- They form an **operad / preclone** with **prescribed argument properties** governed by a monoid S .

Motivating Examples: From Posets to 3-Element Monoids

Example (The 2-Element Group $S = \{+, -\}$):

- Consider operations on $(A, \leq)$ that are coordinate-wise order-preserving (+) or order-reversing (-).
- Composition rule:

$$\text{reversing} \circ \text{reversing} = \text{preserving} \implies (- \cdot - = +).$$

- $S = \{+, -\} \cong \mathbb{Z}_2$ is the two-element group under multiplication.

Example (Monoids with Zero / Inactive Coordinates):

- Let $S' = \{+, o\}$ (2 elements) or $\hat{S} = \{+, -, o\}$ (3 elements) with zero o :
- Interpretation on posets $(A, \leq)$:
 - $+$: order-preserving,
 - $-$: order-reversing,
 - o : constant on each connected component (both preserving and reversing).

$\cdot$	$+$	$-$	o
$+$	$+$	$-$	o
$-$	$-$	$+$	o
o	o	o	o

Operations with signa from a monoid S

The set of term-operations on an algebra form a **clone**

What is a good notion of **clone for partially ordered algebras**?

For a set A , let $\text{Op}(A) = \bigcup_{n \geq 1} A^{A^n}$ = the set of all finitary operations on A

For a (fixed) **finite monoid** $(S, \cdot, e)$, let ${}^S\text{Op}(A) = \bigcup_{n \geq 1} A^{A^n} \times S^n$

A pair $(f, \lambda) \in {}^S\text{Op}(A)$ is an **S -operation** with **signum** λ , also written f^λ

E.g. $S = \{+, -\}$ with $e = +$ and $- \cdot - = +$

f^{+-} is a binary S -operation with two arguments labelled $+$ and $-$

S-preclones

A set $F \subseteq {}^S\text{Op}(A)$ is an **S-preclone** if

- F contains the **identity S-operation** $(\text{id}_A, (e))$ and
- F is closed under $\zeta, \tau, \nabla^s, \Delta, \circ$ where $s \in S$ and
 - ζ **cycles** arguments:
 $(\zeta f^\lambda)(x_1, \dots, x_n) = f(x_2, \dots, x_n, x_1)$ has signum $(\lambda_n, \lambda_1, \dots, \lambda_{n-1})$
 - τ **permutes** the first two arguments:
 $(\tau f^\lambda)(x_1, \dots, x_n) = f(x_2, x_1, x_3, \dots, x_n)$ has signum $(\lambda_2, \lambda_1, \lambda_3, \dots, \lambda_n)$
 - ∇^s **adds a fictitious** argument with signum s at first place:
 $(\nabla^s f^\lambda)(x_1, \dots, x_{n+1}) = f(x_2, \dots, x_{n+1})$ has signum $(s, \lambda_1, \dots, \lambda_n)$
 - Δ **identifies first two** arguments **if they have equal signs**:
 $(\Delta f^\lambda)(x_1, \dots, x_{n-1}) = f(x_1, x_1, x_2, \dots, x_{n-1})$ has signum $(\lambda_2, \dots, \lambda_n)$ if $\lambda_1 = \lambda_2$, otherwise $\Delta f^\lambda = f^\lambda$
 - $\circ$ **composes** f^λ with g^μ and uses the monoid to get the signum:
 $(f^\lambda \circ g^\mu)(x_1, \dots, x_{m+n-1}) = f(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n-1})$ with **signum** $(\mu_1 \lambda_1, \dots, \mu_m \lambda_1, \lambda_2, \dots, \lambda_n)$

Examples of S -preclones $F \subseteq {}^S\text{Op}(A)$

If $S = \{e\}$ then F is an S -preclone **iff** $\{f \mid f^\lambda \in F\}$ is a clone

If $S = \{+, -\}$ with $e = +$, $- \cdot - = +$ then an S -operation is also called a **$\pm$ -operation** and an S -preclone is also called a **$\pm$ -preclone**

${}^S\text{Op}(A)$ is the **largest** S -preclone on a set A

Let $p_i^n(x_1, \dots, x_n) = x_i$. Then $\{(p_i^n, \lambda) \mid \lambda_i = e, 1 \leq i \leq n \in \mathbb{Z}^+\}$ is the **smallest** S -preclone of **trivial S -operations** or **projections**

$S\langle F \rangle$ is the S -preclone **generated by** F (= least one containing F)

Relations, polymorphisms and invariant relations

For $f : A^n \rightarrow A$, $\rho \subseteq A^m$ and $\mathbf{a}_1, \dots, \mathbf{a}_n \in \rho$, consider the column vector

$$f(\mathbf{a}_1, \dots, \mathbf{a}_n) = (f(a_{11}, a_{21}, \dots, a_{n1}), \dots, f(a_{m1}, a_{m2}, \dots, a_{mn}))^T.$$

f **preserves** ρ , written $f \triangleright \rho$, if $\mathbf{a}_1, \dots, \mathbf{a}_n \in \rho \implies f(\mathbf{a}_1, \dots, \mathbf{a}_n) \in \rho$.

In this case f is a **polymorphism** of ρ , and ρ is an **invariant relation** of f .

For $F \subseteq \text{Op}(A)$ and $R \subseteq \text{Rel}(A) = \bigcup_{m \geq 1} \mathcal{P}(A^m)$, define

$\text{Pol}(R) = \{f \in \text{Op}(A) \mid \forall \rho \in R, f \triangleright \rho\}$ = all **polymorphisms of R** and

$\text{Inv}(F) = \{\rho \in \text{Rel}(A) \mid \forall f \in F, f \triangleright \rho\}$ = all **invariant relations of F**

E.g. $\text{Pol}(\{\leq\})$ = all **order-preserving operations** on a poset $(A, \leq)$

Relational clones and the Pol–Inv Galois connection

A set $R \subseteq \text{Rel}(A)$ is a **relational clone** if

- R contains the **identity relation** $\delta_A = \{(x, x) \mid x \in A\}$ and
- R is **closed under** $\zeta, \tau, \text{pr}, \times, \wedge$ where for $\rho, \rho' \in R$

$\zeta\rho = \{(a_2, a_3, \dots, a_m, a_1) \mid (a_1, a_2, \dots, a_m) \in \rho\}$ **cyclic shift**

$\tau\rho = \{(a_2, a_1, a_3, \dots, a_m) \mid (a_1, a_2, \dots, a_m) \in \rho\}$ **transposition**

$\text{pr}\rho = \{(a_2, \dots, a_m) \mid (a_1, a_2, \dots, a_m) \in \rho\}$ **deletion of 1st row**

$\rho \times \rho' = \{(a_1, \dots, a_m, b_1, \dots, b_{m'}) \mid (a_1, \dots, a_m) \in \rho, (b_1, \dots, b_{m'}) \in \rho'\}$

$\rho \wedge \rho' = \rho \cap \rho'$ **intersection** **Cartesian product**

$[R]$ is the relational clone **generated by** $R \subseteq \text{Rel}(A)$ (least one $\supseteq R$)

Theorem (Pol–Inv Galois connection for clones)

$\text{Pol}(R)$ is a **clone** and $\text{Inv}(F)$ is a **relational clone**.

If A is **finite**, $\langle F \rangle = \text{Pol}(\text{Inv}(F))$ and $[R] = \text{Inv}(\text{Pol}(R))$.

S -relations, S -polymorphisms and S -invariant relations

An m -ary **S -relation** is of the form $\rho = (\rho_s)_{s \in S}$ where $\rho_s \subseteq A^m$

The set of **all S -relations** is ${}^S\text{Rel}(A) = \bigcup_{m \geq 1} (P(A^m))^S$

Define $f^\lambda \triangleright \rho$ if $\forall s \in S \ (\mathbf{a}_1 \in \rho_{\lambda_1 s}, \dots, \mathbf{a}_n \in \rho_{\lambda_n s} \implies f(\mathbf{a}_1, \dots, \mathbf{a}_n) \in \rho_s)$

Abbreviated as: $\forall s \in S, f(\rho_{\lambda_1 s}, \dots, \rho_{\lambda_n s}) \subseteq \rho_s$

E.g. $S = \{+, -\}$: $f^\lambda \triangleright \rho$ if $f(\rho_{\lambda_1}, \dots, \rho_{\lambda_n}) \subseteq \rho_+$ and $f(\rho_{\lambda_1 -}, \dots, \rho_{\lambda_n -}) \subseteq \rho_-$

$f^\lambda \triangleright (\leq, \geq)$ **iff** f is order-preserving in $+$ args and order-reversing in $-$ args

For $F \subseteq {}^S\text{Op}(A)$ and $R \subseteq {}^S\text{Rel}(A)$ define

${}^S\text{Pol}(R) = \{f^\lambda \in {}^S\text{Op}(A) \mid \forall \rho \in R, f^\lambda \triangleright \rho\}$ = all **S -polymorphisms of R**

${}^S\text{Inv}(F) = \{\rho \in {}^S\text{Rel}(A) \mid \forall f^\lambda \in F, f^\lambda \triangleright \rho\}$ = all **S -invariant relations of F**

S -relational clones

A set $R \subseteq {}^S\text{Rel}(A)$ is a **S -relational clone** if

- R contains the S -relation $(\delta_A)_{s \in S} = (\delta_A, \dots, \delta_A)$ and
- R is **closed under** $\zeta, \tau, \text{pr}, \times, \wedge, \wedge_{St}, \mu_t$ where for $\rho, \rho' \in R$ and $t \in S$

$$\zeta(\rho) = (\zeta \rho_s)_{s \in S} \quad \text{cyclic shift}$$

$$\tau(\rho) = (\tau \rho_s)_{s \in S} \quad \text{transposition}$$

$$\text{pr}(\rho) = (\text{pr} \rho_s)_{s \in S} \quad \text{deletion of 1}^{\text{st}} \text{ row}$$

$$\rho \times \rho' = (\rho_s \times \rho'_s)_{s \in S} \quad \text{Cartesian product}$$

$$\rho \wedge_{St} \rho' = (\rho_s \cap \rho'_s \text{ if } s \in St \text{ else } \rho_s)_{s \in S} \quad \text{St-intersection, } St = \{s \cdot t \mid s \in S\}$$

$$\mu_t(\rho) = (\rho_{st})_{s \in S} \quad \text{index translation by } t$$

${}^S[R]$ is the S -relational clone **generated by** $R \subseteq {}^S\text{Rel}(A)$ (least one $\supseteq R$)

S-Relations and S-Diagonals

In clone theory, diagonal relations $\delta_\epsilon^m = \{(a_1, \dots, a_m) : i \in j \implies a_i = a_j\}$ are invariant under all projections. In the S -sorted world:

Definition (S -Diagonal, Definition 2.9)

An S -relation $\varrho = (\delta_{\epsilon_s}^m)_{s \in S}$ is an **S -diagonal** (${}^S D_A$) if each ϱ_s is a diagonal relation and:

$$Ss \subseteq St \implies \delta_{\epsilon_s}^m \subseteq \delta_{\epsilon_t}^m,$$

where $St = \{st : s \in S\}$ is the principal left ideal generated by t .

Remark

If S is a group, $Ss = St = S$ for all s, t , so S -diagonals simply satisfy $\epsilon_s = \epsilon_t$ ($\varrho_s = \varrho_t$ is constant across S).

Characterizing S -preclones and S -relational clones

An **m -ary diagonal relation** $\delta_\epsilon^m = \{a \in A^m \mid a_i = a_j \text{ for all } (i, j) \in \epsilon\}$ where ϵ is an equivalence relation on $\{1, \dots, m\}$

The set of all diagonal relations on A is the smallest relational clone $[\delta_A]$

$(\delta_{\epsilon_s}^m)_{s \in S}$ is an **S -diagonal relation** if $\forall s, t \in S, Ss \subseteq St \implies \delta_{\epsilon_s}^m \subseteq \delta_{\epsilon_t}^m$

Let ${}^S D_A$ be the set of all S -diagonal relations on A (all $m \geq 1$)

Lemma

${}^S[(\delta_A)_{s \in S}] = {}^S D_A$, ${}^S \text{Pol}(R) = {}^S \text{Pol}({}^S[R])$ and ${}^S \text{Inv}(F) = {}^S \text{Inv}({}^S\langle F \rangle)$.

Theorem

Let $F \subseteq {}^S \text{Op}(A)$. Then $\langle \{f \mid f^\lambda \in F\} \rangle = \text{Pol}\{\rho \mid (\rho)_{s \in S} \in {}^S \text{Inv}(F)\}$.

Characterizing S -preclone generation

For $F \subseteq {}^S\text{Op}(A)$ and $\rho \in (\mathcal{P}(A^m))^S$, define

$$\Gamma_F(\rho) = \bigcap \{ \sigma \in (\mathcal{P}(A^m))^S \mid \rho \subseteq \sigma \in {}^S\text{Inv}(F) \}$$

where $\bigcup, \subseteq$ are applied coordinatewise.

For $|A| = k < \omega$, let $(\varkappa_1, \dots, \varkappa_n)$ be the $k^n \times n$ -matrix with columns $\varkappa_i$ such that the rows are **all** n -tuples in A^n .

For a signum $\lambda \in S^n$ define $\chi^\lambda = (\{ \varkappa_i \mid \lambda_i = s, i \in \{1, \dots, n\} \})_{s \in S}$

Theorem

- For finite A , $F \subseteq {}^S\text{Op}(A)$, $\lambda \in S^n$ and $g^\lambda \in {}^S\text{Op}(A)$,

$$g^\lambda \in {}^S\langle F \rangle \iff g^\lambda \triangleright \Gamma_F(\chi^\lambda).$$

- For finite A and $F \subseteq {}^S\text{Op}(A)$, ${}^S\langle F \rangle = {}^S\text{Pol}({}^S\text{Inv}(F)).$

The Main Galois Duality Theorems

Theorem (4.1 Galois Closure for Operations)

Let S be an arbitrary monoid and A a finite set. Then for any $F \subseteq {}^S\text{Op}(A)$:

$${}^S\langle F \rangle = {}^S\text{Pol } {}^S\text{Inv } F.$$

*That is, the Galois closed sets of operations are precisely the **S -preclones**.*

Theorem (4.6 Galois Closure for Relations)

Let S be an arbitrary monoid and A a finite set. Then for any $Q \subseteq {}^S\text{Rel}(A)$:

$${}^S[Q] = {}^S\text{Inv } {}^S\text{Pol } Q.$$

*That is, the Galois closed sets of S -relations are precisely the **S -relational clones**.*

This generalizes the celebrated Bodnarčuk–Kalužnin–Kotov–Romov (1969) / Geiger (1968) theorem from clones to S -preclones over any monoid S

The S -preclone lattice is dual to S -relational clone lattice

The set ${}^S\mathcal{L}_A$ of all S -preclones on A is a **complete lattice** with intersection as meet and S -preclone generated by union as join.

Corollary

*For a finite monoid S , the lattice ${}^S\mathcal{L}_A$ of S -preclones on a **finite** set A and the lattice of S -relational clones on A are **dually isomorphic**.*

Embeddings of the Clone Lattice into ${}^S\mathcal{L}_A$

How does the classical clone lattice $\mathcal{L}_A$ sit inside ${}^S\mathcal{L}_A$?

Theorem

There are two canonical lattice embeddings $\Psi, \Phi: \mathcal{L}_A \rightarrow {}^S\mathcal{L}_A$:

- ① *Bottom Embedding Ψ (into a Principal Ideal):*

$$\Psi(F) := {}^S\langle \{f \in {}^S\text{Op}(A) \mid \mathring{f} \in F \text{ and } \text{Sgn}(f) = \{e\}\} \rangle.$$

Ψ embeds $\mathcal{L}_A$ into $[{}^SJ_A, \Psi(\text{Op}(A))]_{{}^S\mathcal{L}_A}$.

- ② *Top Embedding Φ (into a Principal Filter):*

$$\Phi(F) := \{f \in {}^S\text{Op}(A) \mid \mathring{f} \in F\}.$$

Φ embeds $\mathcal{L}_A$ into $[\Phi(J_A), {}^S\text{Op}(A)]_{{}^S\mathcal{L}_A}$.

If S is a **group**, then Φ is an isomorphism **onto** the principal filter generated by ${}^S\langle \{\text{id}^s \mid s \in S\} \rangle$

Symmetries and Automorphisms of ${}^S\mathcal{L}_A$

The lattice ${}^S\mathcal{L}_A$ enjoys two distinct families of automorphisms:

1. Inner Automorphisms (Permutations of A):

- For any permutation $\pi \in \text{Sym}(A)$, define $f^\pi(x_1, \dots, x_n) = \pi(f(\pi^{-1}x_1, \dots, \pi^{-1}x_n))$ with $\text{sgn}(f^\pi) = \text{sgn}(f)$.
- The mapping $F \mapsto F^\pi$ is a lattice automorphism of ${}^S\mathcal{L}_A$.

2. Monoid Automorphisms (Symmetries of S):

- For any monoid automorphism $h \in \text{Aut}(S)$, define:

$$\text{sgn}(f^h) = (h(s_1), \dots, h(s_n)) \quad \text{and} \quad (f^h)^\circ = \mathring{f}.$$

- The mapping $F \mapsto F^h$ is a lattice automorphism of ${}^S\mathcal{L}_A$.
- **Note:** In classical clone theory ($S = \{e\}$), only inner automorphisms exist. Monoid automorphisms are a new feature of S -preclones.

Connection to Minions and Clones

Definition (Minion / Minor-Closed Class)

A set of function $M \subseteq \bigcup_n B^{A^n}$ is a **minion** if it is closed under forming minors:

$$g(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(m)}) \quad \text{for any } \sigma : \{1, \dots, m\} \rightarrow \{1, \dots, n\}.$$

For $s \in S$, the **s -signed part** of an S -preclone F is

$$F^{[s]} := \{f \in F \mid \text{Sgn}(f) = \{s\}\}.$$

Proposition (4.7)

Let F be an S -preclone on A .

- 1 For each $s \in S$, $F^{[s]}$ is a **minion** (closed under variable identifications and deletions)
- 2 For the identity $e \in S$, $F^{[e]}$ is a **clone** (also closed under composition)
- 3 If $F = {}^S \text{Pol } Q$, then $F^{[s]} = {}^S \text{Pol} \{(\varrho_{st}, \varrho_t) \mid \varrho \in Q, t \in S\}$, characterizes $F^{[s]}$ as polymorphisms of relation pairs (Pippenger 2002)

Finiteness Properties of ${}^S\text{Op}(A)$ and ${}^S\text{Rel}(A)$

Proposition (5.1)

For any finite set A and finite monoid S :

- 1 The full S -preclone ${}^S\text{Op}(A)$ is **finitely generated**:

$${}^S\text{Op}(A) = {}^S\langle \{m^{(e,e)}\} \cup \{\text{id}^s \mid s \in S\} \rangle,$$

where m is a 2-ary Sheffer function for A , and id^s has signum (s) .

- 2 The full S -relational clone ${}^S\text{Rel}(A)$ is **finitely generated**.

For $|A| \geq 3$: ${}^S\text{Rel}(A) = {}^S[(\Delta, \nabla, \dots, \nabla), (\leq, \dots, \leq), (\neq, \dots, \neq)]$.

Corollary (Atoms and Coatoms)

The lattice ${}^S\mathcal{L}_A$ of all S -preclones on A

- is **atomic** and **coatomic**;
- has **finitely many atoms** (minimal S -preclones) and **finitely many coatoms** (maximal S -preclones).

Characterization of Maximal S-Preclones

Proposition (5.3)

Each maximal S-preclone $F \subseteq {}^S \text{Op}(A)$ is of the form $F = {}^S \text{Pol } \varrho$ for some S-relation $\varrho = (\varrho_s)_{s \in S}$ such that for each $s \in S$:

ϱ_s is a diagonal relation, or $\text{Pol}(\varrho_s)$ is a maximal clone on A .

The relations ϱ_s that yield maximal clones are known from **Rosenberg's Classification Theorem** (1970):

- 1 Unary relations (central / non-trivial),
- 2 Partial orders with 0 and 1,
- 3 Equivalence relations,
- 4 Permutations,
- 5 Affine / linear relations,
- 6 Homogeneous / central relations.

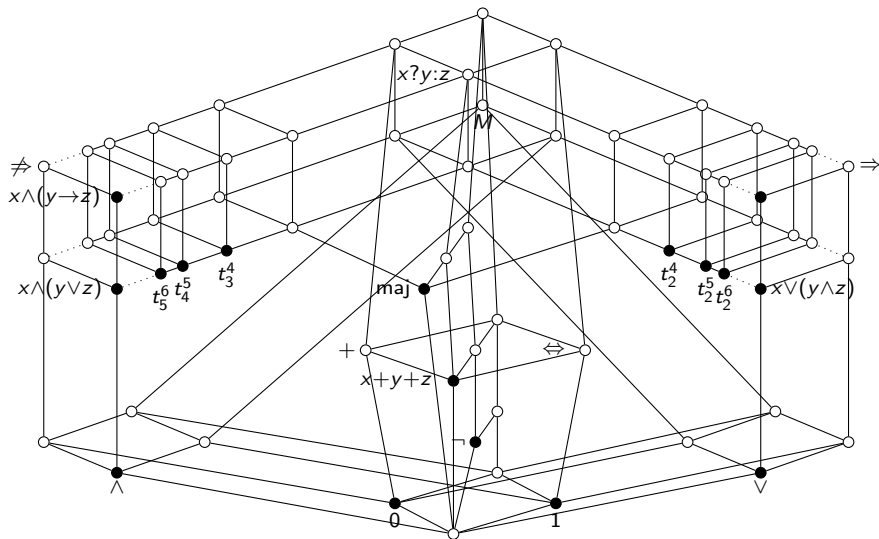
Maximal and minimal Boolean clones

If $A = \{0, 1\}$ then clones on A are called **Boolean clones**

The lattice $\mathcal{L}_{\{0,1\}}$ of Boolean clones is called the **Post lattice** since it was first described by **Emil Post [1941]**.

The Post lattice has **5 maximal** clones and **7 minimal** clones.

The Post lattice of all clones on a 2-element set



$$t_k^n(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } |\{i \mid x_i = 1\}| \geq k \\ 0 & \text{otherwise.} \end{cases}$$

All nine maximal Boolean $\pm$ -preclones

Theorem

There are **nine** maximal Boolean $\pm$ -preclones listed below. Each such preclone is of the form $F = {}^{\pm}\text{Pol } \rho$ for some $\pm$ -relation $\rho = (\rho_+, \rho_-)$:

- a ${}^{\pm}\text{Pol}(\sigma, \sigma)$ with $\sigma \in \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ where $\text{Pol } \sigma_i$ is maximal in $\mathcal{L}_2$ (0-preserving, 1-preserving, monotone, self-dual, linear operations)
 $\sigma_0 = \{0\}$, $\sigma_1 = \{1\}$, $\sigma_2 = \leq = \{(0, 0), (0, 1), (1, 1)\}$,
 $\sigma_3 = \{(0, 1), (1, 0)\}$, $\sigma_4 = \{(x, y, z, u) \in A^4 \mid x + y + z + u = 0\}$.
- b ${}^{\pm}\text{Pol}(\leq, \geq) =$ all functions where each $+$ -argument is order-preserving and each $-$ -argument is order-reversing.
- c ${}^{\pm}\text{Pol}(A, \emptyset) =$ all functions with positive or mixed signum.
- d ${}^{\pm}\text{Pol}(A^2, \delta_A) =$ all Boolean $\pm$ -functions, where each negative argument is fictitious
- e ${}^{\pm}\text{Pol}(\{0\}, \{1\})$.

There are 20 minimal Boolean $\pm$ -preclones

Theorem

There are twenty minimal Boolean $\pm$ -preclones.

- Ⓐ $\pm\langle (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \rangle, \quad \pm\langle x + y + z \rangle$
where the generators have signum $\lambda = (+, +, +, -)$.
- Ⓑ $\pm\langle h_0^\lambda \rangle, \pm\langle h_1^\lambda \rangle, \pm\langle h_y^\lambda \rangle$ where $h_i(x, y, z, u) = \begin{cases} x & \text{if } x = y \text{ or } z = u, \\ i & \text{otherwise,} \end{cases}$
and $\lambda = (+, +, -, -)$.
- Ⓒ $\pm\langle x \wedge y \rangle, \quad \pm\langle x \wedge (y \vee z) \rangle, \quad \pm\langle x \wedge (y \vee \neg z) \rangle$
 $\pm\langle x \vee y \rangle, \quad \pm\langle x \vee (y \wedge z) \rangle, \quad \pm\langle x \vee (y \wedge \neg z) \rangle$
 $\pm\langle (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \rangle, \quad \pm\langle (x \wedge y) \vee (y \wedge \neg z) \vee (\neg z \wedge x) \rangle$
where the generators have signum $\lambda = (+, +, -)$
- Ⓓ $\pm\langle 0 \rangle, \pm\langle 1 \rangle, \pm\langle y \rangle, \pm\langle \neg y \rangle, \pm\langle \neg x \rangle, \pm\langle x \wedge y \rangle, \pm\langle x \vee y \rangle$
where the generators have signum $\lambda = (+, -)$

The preimage class of a Boolean clone

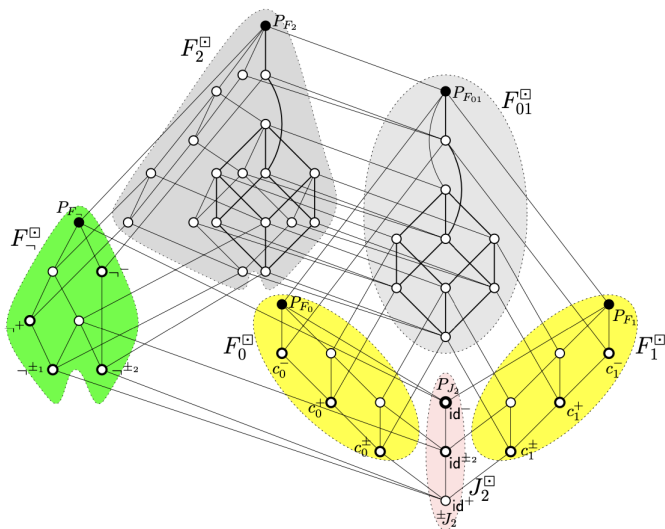
Let F be a Boolean clone, i.e., $F \in \mathcal{L}_2$.

A $\pm$ -preclone P is called a **preimage of F** if $\langle \mathring{P} \rangle = F$, i.e., the operations in P without signa generate all operations in F .

The **preimage class of F** , denoted $F^\square$, is the set of all preimages of F .

Note that $\{F^\square \mid F \in \mathcal{L}_2\}$ is a partition of ${}^\pm\mathcal{L}_2$ into preimage classes.

The lattice of all 51 unary $\pm$ -preclones



Unary $\pm$ -preclones in six preimage classes $F_{\text{id}}^{\square}$, $F_0^{\square}$, $F_1^{\square}$, $F_{01}^{\square}$, $F_{\neg}^{\square}$ and $F_2^{\square}$ (the greatest elements in each preimage class and the join-irreducibles are marked)

Some open(?) problems

Is the lattice of Boolean $\pm$ -preclones countable?

Theorem (Vojtěch David, a week ago)

*The lattice of Boolean $\pm$ -preclones is **uncountable**.*

The proof uses a result by Erkko on uncountably many Boolean clonoids.

Classify the maximal S -preclones for $|S| > 2$ and $|A| > 2$, i.e., generalize Rosenberg's classification to S -preclones.

Classify the maximal $\pm$ -preclones below ${}^{\pm}\text{Pol}((\leq, \geq))$.

What is the cardinality of the ideal below this $\pm$ -preclone?

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THANKS!