

Retracts of degenerate solutions of the Yang-Baxter equation

Přemysl Jedlička
with Agata Pilitowska

Department of Mathematics and Physics
Faculty of Engineering (former Technical Faculty)
Czech University of Life Sciences (former Czech University of Agriculture) in Prague

7th September 2026



Faculty of
Engineering

SSAOS 2026





Yang–Baxter equation

Definition

Let X be a set. A mapping $r : X \times X \rightarrow X \times X$ is called a *set-theoretic solution of Yang–Baxter equation* if it satisfies

$$(r \times \text{id}_X)(\text{id}_X \times r)(r \times \text{id}_X) = (\text{id}_X \times r)(r \times \text{id}_X)(\text{id}_X \times r).$$

A solution $r : (x, y) \mapsto (\sigma_x(y), \tau_y(x))$ is called *bijective* if r is a bijection. It is called *left non-degenerate* if σ_x are bijections, for all $x \in X$. The solution is *non-degenerate* if it is left and right non-degenerate. A solution is called *involution* if $r^2 = \text{id}_{X^2}$.

Observation

If r is involutive and non-degenerate then $\sigma_x(y) = \tau_{\tau_y(x)}^{-1}(y)$.

Yang–Baxter equation

Definition

Let X be a set. A mapping $r : X \times X \rightarrow X \times X$ is called a *set-theoretic solution of Yang–Baxter equation* if it satisfies

$$(r \times \text{id}_X)(\text{id}_X \times r)(r \times \text{id}_X) = (\text{id}_X \times r)(r \times \text{id}_X)(\text{id}_X \times r).$$

A solution $r : (x, y) \mapsto (\sigma_x(y), \tau_y(x))$ is called *bijective* if r is a bijection. It is called *left non-degenerate* if σ_x are bijections, for all $x \in X$. The solution is *non-degenerate* if it is left and right non-degenerate. A solution is called *involutive* if $r^2 = \text{id}_{X^2}$.

Observation

If r is involutive and non-degenerate then $\sigma_x(y) = \tau_{\tau_y(x)}^{-1}(y)$.

Cayley congruence

Definition (P. Etingof, T. Schedler, A. Soloviev)

Let (X, σ, τ) be a left non-degenerate involutive solution. We define a relation $\sim$ on X
 $x \sim y$ if and only if $\tau_x = \tau_y$.

Theorem (P. Etingof, T. Schedler, A. Soloviev)

Let (X, σ, τ) be a left non-degenerate involutive solution. Then $\sim$ is a congruence compatible with σ , τ , σ^{-1} and τ^{-1} .

Definition

Let (X, σ, τ) be a non-degenerate involutive solution. The factor $X / \sim$ is called the *retract* of X and denoted by $\text{Ret}(X)$.

Cayley congruence

Definition (P. Etingof, T. Schedler, A. Soloviev)

Let (X, σ, τ) be a left non-degenerate involutive solution. We define a relation $\sim$ on X
 $x \sim y$ if and only if $\tau_x = \tau_y$.

Theorem (P. Etingof, T. Schedler, A. Soloviev)

Let (X, σ, τ) be a left non-degenerate involutive solution. Then $\sim$ is a congruence compatible with σ , τ , σ^{-1} and τ^{-1} .

Definition

Let (X, σ, τ) be a non-degenerate involutive solution. The factor $X / \sim$ is called the *retract* of X and denoted by $\text{Ret}(X)$.

Retracts of involutive solutions

σ	0	1	2	3
0	1	2	3	0
1	3	0	1	2
2	1	2	3	0
3	3	0	1	2

τ	0	1	2	3
0	1	3	1	3
1	2	0	2	0
2	3	1	3	1
3	0	2	0	2

σ	02	13
02	13	02
13	13	02

τ	02	13
02	13	13
13	02	02

Definition

We say that an involutive solution (X, σ, τ) has *multipermutation level* k if k is the smallest integer such that $|\text{Ret}^k(X)| = 1$.

Retracts of involutive solutions

σ	0	1	2	3
0	1	2	3	0
1	3	0	1	2
2	1	2	3	0
3	3	0	1	2

τ	0	1	2	3
0	1	3	1	3
1	2	0	2	0
2	3	1	3	1
3	0	2	0	2

σ	02	13
02	13	02
13	13	02

τ	02	13
02	13	13
13	02	02

Definition

We say that an involutive solution (X, σ, τ) has *multipermutation level* k if k is the smallest integer such that $|\text{Ret}^k(X)| = 1$.

Retracts of involutive solutions

σ	0	1	2	3
0	1	2	3	0
1	3	0	1	2
2	1	2	3	0
3	3	0	1	2

τ	0	1	2	3
0	1	3	1	3
1	2	0	2	0
2	3	1	3	1
3	0	2	0	2

σ	02	13
02	13	02
13	13	02

τ	02	13
02	13	13
13	02	02

Definition

We say that an involutive solution (X, σ, τ) has *multipermutation level* k if k is the smallest integer such that $|\text{Ret}^k(X)| = 1$.

Retracts of non-involutive solutions

Theorem

The equivalence

$$x \sim y \quad \Leftrightarrow \quad \sigma_x = \sigma_y \quad \text{and} \quad \tau_x = \tau_y.$$

is a congruence of a non-degenerate solution (X, σ, τ) if

- *X is a finite bijective injective solution [V. Lebed, L. Vendramin, 2019]*
- *X is arbitrary [P.J., A.P., A. Zamojska-Dzienio, 2019]*
- *X is a bijective solution [F. Cedó, E. Jespers, Ł. Kubat, A. van Antwerpen, C. Verwimp, 2021]*

Example of a degenerate solution

σ	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

τ	0	1	2	3
0	1	1	3	1
1	2	2	2	2
2	1	1	1	1
3	2	2	2	2

Question:

Does there exist any notion of retract for degenerate solutions so that the retract of a solution of “multipermutation” level k is a solution of “multipermutation” level $k - 1$?

Example of a degenerate solution

σ	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

τ	0	1	2	3
0	1	1	3	1
1	2	2	2	2
2	1	1	1	1
3	2	2	2	2

Question:

Does there exist any notion of retract for degenerate solutions so that the retract of a solution of “multipermutation” level k is a solution of “multipermutation” level $k - 1$?

Example of a degenerate solution

σ	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

τ	0	1	2	3
0	1	1	3	1
1	2	2	2	2
2	1	1	1	1
3	2	2	2	2

Question:

Does there exist any notion of retract for degenerate solutions so that the retract of a solution of “multipermutation” level k is a solution of “multipermutation” level $k - 1$?

Example of a degenerate solution

σ	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

τ	0	1	2	3
0	1	1	3	1
1	2	2	2	2
2	1	1	1	1
3	2	2	2	2

Question:

Does there exist any notion of retract for degenerate solutions so that the retract of a solution of “multipermutation” level k is a solution of “multipermutation” level $k - 1$?

General retract congruence

Definition

Let (X, Γ) be an algebra with a set Γ of binary operations. For any $a, b \in X$ and each binary operation $\gamma \in \Gamma$, the image of a pair (a, b) will be denoted as $\gamma_a(b)$. Hence we will consider γ as a mapping $X \rightarrow X^X$ rather than $X \times X \rightarrow X$.

We define the equivalence $\sim$ on X as

$$x \sim y \quad \text{if and only if} \quad \gamma_x = \gamma_y, \text{ for all } \gamma \in \Gamma.$$

Let $\approx$ be the biggest congruence of (X, Γ) smaller than or equal to $\sim$.

We define the *retract* $\text{Ret}(X)$ as the factor $(X, \Gamma)/\approx$ and the *iterated retraction* $\text{Ret}^{i+1}(X)$ as $\text{Ret}^i(X)/\approx$, for $i \in \mathbb{N}$. If there exists an integer k such that $\text{Ret}^k(X)$ has one element only then we will say that the algebra (X, Γ) has *multipermutation level* k . We say that an algebra (X, Γ) is *irretractable* if $\text{Ret}(X) = (X, \Gamma)$.

Searching for the biggest congruence

σ	a	b	c	d	e
a	a	e	d	c	a
b	a	a	d	c	a
c	a	e	d	c	a
d	a	a	d	c	a
e	a	e	d	c	a

τ	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

Searching for the biggest congruence

σ	a	b	c	d	e
a	a	e	d	c	a
b	a	a	d	c	a
c	a	e	d	c	a
d	a	a	d	c	a
e	a	e	d	c	a

τ	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

σ	a	c	e	b	d
I	a	I	II	I	I
	c	I	II	I	I
	e	I	II	I	I
II	b	I	II	I	I
	d	I	II	I	I

τ	a	c	e	b	d
I	a	I	I	I	I
	c	I	I	I	I
	e	I	I	I	I
II	b	II	II	II	II
	d	II	II	II	II

Searching for the biggest congruence

σ	a	b	c	d	e
a	a	e	d	c	a
b	a	a	d	c	a
c	a	e	d	c	a
d	a	a	d	c	a
e	a	e	d	c	a

τ	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

	σ	a	e	c	b	d
III	a	III	III	II	III	IV
	e	III	III	II	III	IV
IV	c	III	III	II	III	IV
II	b	III	III	II	III	IV
	d	III	III	II	III	IV

	τ	a	e	c	b	d
III	a	III	III	III	III	III
	e	III	III	III	III	III
IV	c	IV	IV	IV	IV	IV
II	b	II	II	II	II	II
	d	II	II	II	II	II

Searching for the biggest congruence

σ	a	b	c	d	e
a	a	e	d	c	a
b	a	a	d	c	a
c	a	e	d	c	a
d	a	a	d	c	a
e	a	e	d	c	a

τ	a	b	c	d	e
a	a	a	a	a	a
b	b	b	b	b	b
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

	σ	a	e	c	b	d
III	a	III	III	VI	III	IV
	e	III	III	VI	III	IV
IV	c	III	III	VI	III	IV
V	b	III	III	VI	III	IV
VI	d	III	III	VI	III	IV

	τ	a	e	c	b	d
III	a	III	III	III	III	III
	e	III	III	III	III	III
IV	c	IV	IV	IV	IV	IV
V	b	V	V	V	V	V
VI	d	VI	VI	VI	VI	VI

Algorithm for finding the biggest congruence

Observation

The congruence $\approx$ always exists and it is unique.

Algorithm

Let (X, Γ) be an algebra with a set Γ of binary operations and let $a, b \in X$. We inductively construct relations $\sim_i$, for any $i \in \mathbb{N} \cup \{\infty\}$:

- Initial step: $\sim_0 := \sim$.*
- Inductive step: $a \sim_{i+1} b$ if and only if $a \sim_i b$ and $\gamma_x(a) \sim_i \gamma_x(b)$, for all $x \in X$, $\gamma \in \Gamma$ and $i \in \mathbb{N}$.*
- Limit step: $a \sim_\infty b$ if and only if $a \sim_i b$, for all $i \in \mathbb{N}$.*

Algorithm for finding the biggest congruence

Observation

The congruence $\approx$ always exists and it is unique.

Algorithm

Let (X, Γ) be an algebra with a set Γ of binary operations and let $a, b \in X$. We inductively construct relations $\sim_i$, for any $i \in \mathbb{N} \cup \{\infty\}$:

- *Initial step:* $\sim_0 := \sim$.
- *Inductive step:* $a \sim_{i+1} b$ if and only if $a \sim_i b$ and $\gamma_x(a) \sim_i \gamma_x(b)$, for all $x \in X$, $\gamma \in \Gamma$ and $i \in \mathbb{N}$.
- *Limit step:* $a \sim_\infty b$ if and only if $a \sim_i b$, for all $i \in \mathbb{N}$.

Infinite example

Example

Let $Y = \mathbb{N} \cup (\infty - \mathbb{N}) = \{0, 1, 2, 3, \dots\} \cup \{\dots, \infty - 2, \infty - 1, \infty\}$. Let $X = Y \cup \{\overline{\infty - 1}\}$ where $\overline{\infty - 1}$ is a twin copy of the element $\infty - 1$. Let

$$\sigma_x(y) = 0 \quad \text{for all } x, y \in X,$$

$$\tau_y(x) = \begin{cases} \overline{\infty - 1} & \text{if } x = \infty \text{ and } y = 0, \\ \infty - 2 & \text{if } x = \overline{\infty - 1}, \\ \max(0, x - 1) & \text{otherwise.} \end{cases}$$

Then (X, σ, τ) is a solution and

$$x \sim_i y \quad \text{if and only if} \quad x = y \text{ or } (x > i \text{ and } y > i).$$

In the limit step we get

$$x \approx y \quad \text{if and only if} \quad x = y \text{ or } (x \notin \mathbb{N} \text{ and } y \notin \mathbb{N}).$$

Axiomatization of involutive multipermutation solutions

Proposition (T. Gateva-Ivanova)

A non-degenerate involutive solution (X, σ, τ) is of multipermutation level at most $k \in \mathbb{N}$ if and only if

$$\sigma_{\sigma \dots \sigma_y(x_1)}(x_{k-1})(x_k) = \sigma_{\sigma \dots \sigma_z(x_1)}(x_{k-1})(x_k)$$

for all $x_1, \dots, x_k, y, z \in X$.

Axiomatization of non-degenerate multipermutation solutions

Proposition (P.J., A.P.)

A non-degenerate solution (X, σ, τ) is of multipermutation level at most $k \in \mathbb{N}$ if and only if

$$\gamma_{\gamma^{(k-1)}(x_{k-1})}^{(k)}(x_k) = \gamma_{\gamma^{(k-1)}(x_{k-1})}^{(k)}(x_k)$$

for all $x_1, \dots, x_k, y, z \in X$ and all $\gamma^{(1)}, \dots, \gamma^{(k)} \in \{\sigma, \tau\}$.

A counterexample

Example

Let $k > 2$, let $X = \{0, 1, \dots, k\}$ and let

$$\sigma_x(y) = \begin{cases} \min\{k, x + 1\} & \text{if } y = 0, \\ 1 & \text{if } y \neq 0, \end{cases} \quad \tau_y(x) = \begin{cases} 2 & \text{if } x = k \text{ and } y = 0, \\ 1 & \text{if } x \neq k \text{ or } y \neq 0. \end{cases}$$

We can note that $k \sim (k - 1)$ is the only pair of different elements in the relation $\sim$. But $\tau_0(k) = 2 \neq 1 = \tau_0(k - 1)$ and hence the solution (X, σ, τ) is irretractable.

However,

$$\gamma_{\gamma_{\gamma^{(k-1)}}^{(k)}(x_{k-1})}^{(k)}(x_k) = \gamma_{\gamma_{\gamma^{(k-1)}}^{(k-1)}(x_{k-1})}^{(k)}(x_k)$$

$$\vdots$$

$$\gamma_y^{(1)}(x_1) \qquad \qquad \gamma_z^{(1)}(x_1)$$

for all $\gamma^{(1)}, \dots, \gamma^{(k)} \in \Gamma = \{\sigma, \tau\}$ and $x_1, \dots, x_k, y, z \in X$.

Axiomatization of degenerate multipermutation solutions

Proposition (P.J., A.P.)

An algebra (X, Γ) is of multipermutation level at most $k \in \mathbb{N}$ if and only if

$$\begin{array}{c} \mathfrak{w}_k \gamma^{(k)} \\ \mathfrak{w}_{k-1} \gamma^{(k-1)} \\ \vdots \\ \mathfrak{w}_1 \gamma_y^{(1)}(x_1) \end{array} (x_{k-1}) (x_k) = \begin{array}{c} \mathfrak{w}_k \gamma^{(k)} \\ \mathfrak{w}_{k-1} \gamma^{(k-1)} \\ \vdots \\ \mathfrak{w}_1 \gamma_z^{(1)}(x_1) \end{array} (x_{k-1}) (x_k)$$

for all $x_1, \dots, x_k, y, z \in X$, all $\gamma^{(1)}, \dots, \gamma^{(k)} \in \Gamma$ and all $\mathfrak{w}_1, \dots, \mathfrak{w}_k \in \{\gamma_x \mid \gamma \in \Gamma, x \in X\}^*$.

Proposition (P.J., A.P.)

A solution (X, σ, τ) is of multipermutation level at most 2 if and only if, for all $x, y, z \in X$:

$$\sigma_{\sigma_x(z)} = \sigma_{\sigma_y(z)}, \quad \tau_{\tau_x(z)} = \tau_{\tau_y(z)}, \quad \sigma_{\tau_x(z)} = \sigma_{\tau_y(z)}, \quad \tau_{\sigma_x(z)} = \tau_{\sigma_y(z)}.$$

Axiomatization of degenerate multipermutation solutions

Proposition (P.J., A.P.)

An algebra (X, Γ) is of multipermutation level at most $k \in \mathbb{N}$ if and only if

$$\begin{array}{c} \mathfrak{w}_k \gamma^{(k)} \\ \mathfrak{w}_{k-1} \gamma^{(k-1)} \\ \vdots \\ \mathfrak{w}_1 \gamma_y^{(1)}(x_1) \end{array} (x_{k-1}) (x_k) = \begin{array}{c} \mathfrak{w}_k \gamma^{(k)} \\ \mathfrak{w}_{k-1} \gamma^{(k-1)} \\ \vdots \\ \mathfrak{w}_1 \gamma_z^{(1)}(x_1) \end{array} (x_{k-1}) (x_k)$$

for all $x_1, \dots, x_k, y, z \in X$, all $\gamma^{(1)}, \dots, \gamma^{(k)} \in \Gamma$ and all $\mathfrak{w}_1, \dots, \mathfrak{w}_k \in \{\gamma_x \mid \gamma \in \Gamma, x \in X\}^*$.

Proposition (P.J., A.P.)

A solution (X, σ, τ) is of multipermutation level at most 2 if and only if, for all $x, y, z \in X$:

$$\sigma_{\sigma_x(z)} = \sigma_{\sigma_y(z)}, \quad \tau_{\tau_x(z)} = \tau_{\tau_y(z)}, \quad \sigma_{\tau_x(z)} = \sigma_{\tau_y(z)}, \quad \tau_{\sigma_x(z)} = \tau_{\sigma_y(z)}.$$