

On algebras with easy direct limits, part 2

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Outline

- ① **Term operations**
- ② **Mono-unary algebras**
- ③ **Main construction and main theorem**
- ④ **Applications**
- ⑤ **Unary terms which are endomorphisms**

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Term operations

An algebra is a pair $A = (A, F)$ where A is a non-empty underlying set and F is a set of finitary fundamental operations on A .

The set $T(F)$ of term operations over F is the smallest set satisfying the following conditions:

- ▶ it contains each basic operation $f \in F$;
- ▶ it contains all coordinate projections

$$p_i^n(a_1, \dots, a_n) = a_i;$$

- ▶ it is closed under compositions.



Mono-unary algebras

Directed graphs with exactly one outgoing arrow from each vertex

Definition: mono-unary algebra

Definition

A mono-unary algebra is a pair

$$(A, h),$$

where $h : A \rightarrow A$ is a unary operation.

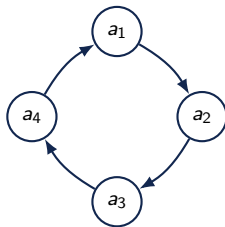
It can be represented as a directed graph:

- ▶ each element $a \in A$ is a vertex;
- ▶ for each $a \in A$, draw one directed edge $a \rightarrow h(a)$;
- ▶ every vertex has exactly one outgoing arrow.

Cycles

A cycle of length k consists of elements $a_1, \dots, a_k$ such that

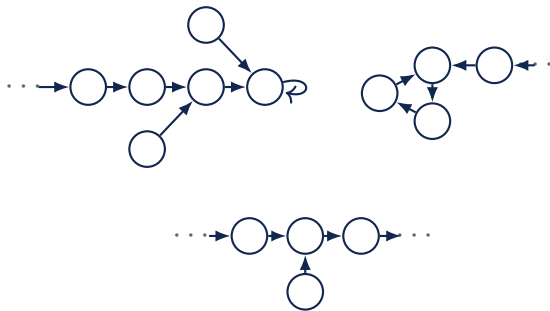
$$h(a_i) = a_{i+1} \quad (1 \leq i < k), \quad h(a_k) = a_1.$$



A fixed point of h is a cycle of length 1.

A connected mono-unary algebra may contain a cycle; if it does, the cycle is unique inside that component.

Figure 1: a mono-unary algebra



The figure visualizes a mono-unary algebra which consists of 3 connected components.

Figure 2: the algebra $\mathbb{Z}$, a line

The mono-unary algebra $\mathbb{Z}$ is defined on the set of all integers with the successor operation.



Definition

A line is a mono-unary algebra isomorphic to $\mathbb{Z}$ with the successor operation.



Lemma 1

Let (A, h) be a mono-unary algebra. Then the following statements are equivalent:

- ① (A, h) is a line.
- ② (A, h) is connected, A is infinite, and the operation h is bijective.
- ③ (A, h) is connected without a cycle and the operation h is bijective.

Disjoint sums of components

If I is a nonempty set and (B_i, h) are mono-unary algebras, their disjoint union is denoted by

$$\sum_{i \in I} (B_i, h).$$

Every mono-unary algebra can be decomposed into a disjoint union of connected components:

$$(A, h) = \sum_{i \in I} (B_i, h).$$

Diamond construction: definition

Let

$$(A, h) = \sum_{i \in I} (B_i, h),$$

where every (B_i, h) is connected. For each $i \in I$:

- ▶ if (B_i, h) contains a cycle of length k , let (C_i, h) be a cycle of length k ;
- ▶ otherwise, let (C_i, h) be a line.

The diamond algebra is

$$(A, h)^\diamond = \sum_{i \in I} (C_i, h).$$

(The diamond construction is applied component by component.)



Main construction and main theorem

Lemma 2

Let φ be a homomorphism from an algebra (A, F) into (A', F) and let $g \in T(F)$. Then φ is a homomorphism from the algebra (A, g) into (A', g) .

Meaning:

$$\varphi(g(a_1, \dots, a_n)) = g(\varphi(a_1), \dots, \varphi(a_n)).$$

Compatibility with the fundamental operations automatically implies compatibility with every term operation.

Lemma 3

Let $D = (D, F)$ be an algebra and $g \in T(F)$. Suppose that $\{P, A_p, \varphi_{pq}\}$ is a direct family and

$$\{P, A_p, \varphi_{pq}\} \longrightarrow \bar{\mathcal{A}} = (\bar{A}, F).$$

If $\bar{\mathcal{A}}$ is isomorphic to a retract of D , then $(\bar{A}, g)$ is isomorphic to a retract of (D, g) .

This lemma allows us to transfer retraction information from the original algebra to the mono-unary reduct determined by a term.

Lemma 4

Let $A = (A, F)$ and let f be an endomorphism of A . Define, for each $i \in \mathbb{N}$,

$$A_i = \{(a, i) : a \in A\}.$$

The operations on A_i are copied from A : $g((a_1, i), \dots, (a_n, i)) = (g(a_1, \dots, a_n), i)$.

For $i \leq j$, define

$$\varphi_{ij}(a, i) = (f^{j-i}(a), j).$$

Then $\{\mathbb{N}, A_i, \varphi_{ij}\}$ is an A -uniform direct system.

Let

$$\{\mathbb{N}, A_i, \varphi_{ij}\} \longrightarrow \bar{A} = (\bar{A}, F).$$

This direct limit is built only from copies of A , so

$$\bar{A} \in \varinjlim A.$$

If A has easy direct limits, then $\bar{A}$ is isomorphic to a retract of A .

Main Theorem

Let $A = (A, F)$. Suppose that $\{\mathbb{N}, A_i, \varphi_{ij}\}$ is the A -uniform direct system from Lemma of the previous slide and $\{\mathbb{N}, A_i, \varphi_{ij}\} \longrightarrow \bar{A} = (\bar{A}, F)$.

Lemma 5

If $F = \{h\}$, where h is unary then

$$(\bar{A}, h) \cong (A, h)^\diamond.$$

Main Theorem

Let $f \in T(F)$ be a unary term such that f is an endomorphism of (A, F) . Then

$$(\bar{A}, f) \cong (A, f)^\diamond.$$

If (A, F) is an algebra with easy direct limits, then the mono-unary algebra $(A, f)^\diamond$ is isomorphic to a retract of (A, f) .

Corollary

Let $A = (A, F)$ and let $f \in T(F)$ be a unary term such that f is an endomorphism of the algebra A . Then:

- ① there exists $B = (B, F) \in \underline{L}_A$ such that f is bijective on B ;
- ② if A is an algebra with easy direct limits, then there exists $B = (B, F) \in R_A$ such that f is bijective on B .

How to use the theorem

The theorem provides a practical easy direct limit test for algebras which have a unary term that is an endomorphism.

- 1 Choose f which is a unary term over F and an endomorphism of (A, F) .
- 2 Draw or analyze the mono-unary algebra (A, f) .
- 3 Construct $(A, f)^\diamond$.
- 4 Check whether $(A, f)^\diamond$ is a retract of (A, f) .
- 5 If not, then (A, F) is not an algebra with easy direct limits.

$$(A, f)^\diamond \not\cong \text{a retract of } (A, f) \implies (A, F) \text{ is not with easy direct limits.}$$

IV

Applications

Integers, reals, and rationals through mono-unary graphs

Example 1: additive group of integers

Consider the additive group

$$(\mathbb{Z}, +, -, 0).$$

Let

$$f(x) = x + x = 2x.$$

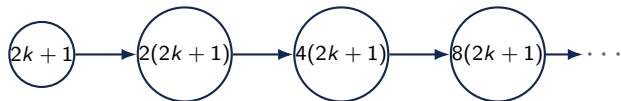
The operation f is a term operation of the group language. Since the group is abelian, f is an endomorphism:

$$f(x + y) = 2(x + y) = 2x + 2y = f(x) + f(y).$$

Now study the mono-ary algebra $(\mathbb{Z}, f)$.

Figure 4: infinite components of $(\mathbb{Z}, f)$

For $f(x) = 2x$, every nonzero infinite component is generated by an odd integer $2k + 1$:



There is also the one-element component $\{0\}$.

The mono-unary algebra $(\mathbb{Z}, f)$ has one component $\{0\}$ and infinitely many infinite components generated by odd numbers.

- ▶ The diamond algebra $(\mathbb{Z}, f)^\diamond$ consists of one one-element cycle and infinitely many lines.
- ▶ But $(\mathbb{Z}, f)$ contains no line as a subalgebra.
- ▶ Therefore $(\mathbb{Z}, f)^\diamond$ is not isomorphic to a retract of $(\mathbb{Z}, f)$.

By Main Theorem,

$(\mathbb{Z}, +, -, 0)$ is not an algebra with easy direct limits.

Example 2: multiplicative real monoid

Consider the multiplicative monoid

$$(\mathbb{R}, \cdot, 1).$$

Let

$$g(x) = x \cdot x = x^2.$$

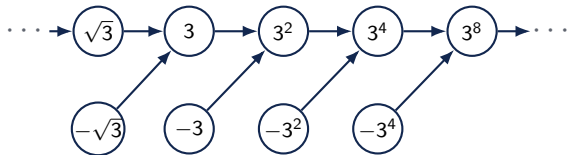
The operation g is a term operation. Commutativity implies that g is an endomorphism:

$$g(xy) = (xy)^2 = x^2y^2 = g(x)g(y).$$

The mono-unary algebra $(\mathbb{R}, g)$ has uncountably many components.

Figure 5: the component of $(\mathbb{R}, g)$ containing 3

In the real case, square roots create backward branches. Every infinite component contains a line.



Finite components include

$$\{0\} \quad \text{and} \quad \{-1, 1\}.$$

Other components are infinite and mutually isomorphic. Every infinite component contains a line as a subalgebra.

- ▶ The relevant cycles and lines can be retracts.
- ▶ Thus $(\mathbb{R}, g)^\diamond$ is isomorphic to a retract of $(\mathbb{R}, g)$.

Therefore Main Theorem does not decide whether $(\mathbb{R}, \cdot, 1)$ has easy direct limits.

Example 3: multiplicative rational monoid

Now consider the monoid

$$(\mathbb{Q}, \cdot, 1).$$

Again use

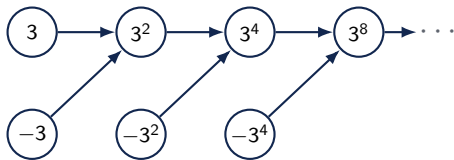
$$g(x) = x^2.$$

The term is an endomorphism for the same commutativity reason:

$$g(xy) = x^2 y^2.$$

Figure 6: the component of $(\mathbb{Q}, g)$ containing 3

In the rational case, missing square roots prevent the existence of a line as a subalgebra.



This missing backward chain is exactly the obstruction detected by Main Theorem.

The mono-unary algebra $(\mathbb{Q}, g)$ has infinitely many components, and the infinite components are isomorphic to each other.

- ▶ There is no line as a subalgebra of $(\mathbb{Q}, g)$.
- ▶ The diamond algebra $(\mathbb{Q}, g)^\diamond$ contains lines.
- ▶ Therefore $(\mathbb{Q}, g)^\diamond$ is not isomorphic to a retract of $(\mathbb{Q}, g)$.

By Main Theorem, $(\mathbb{Q}, \cdot, 1)$ is not an algebra with easy direct limits.

Comparison of the three main examples

Structure	Term	Mono-unary feature	Diamond/retract test	Conclusion
$(\mathbb{Z}, +, -, 0)$	$f(x) = 2x$	no line in $(\mathbb{Z}, f)$	$(\mathbb{Z}, f)^\diamond$ has lines, not a retract	not easy
$(\mathbb{R}, \cdot, 1)$	$g(x) = x^2$	infinite components contain lines	diamond algebra is a retract	inconclusive
$(\mathbb{Q}, \cdot, 1)$	$g(x) = x^2$	no line in $(\mathbb{Q}, g)$	$(\mathbb{Q}, g)^\diamond$ has lines, not a retract	not easy

V

Unary terms that are endomorphisms

Which classes supply useful terms for the theorem?

General observations

The identity mapping is always a term operation and an endomorphism.

If φ and ψ are term operations and endomorphisms of A , then

$$\varphi \circ \psi$$

is again a term operation and an endomorphism of A .

The interesting question is whether nontrivial unary terms have this property. The answer depends strongly on the algebraic class.

Proposition 1: abelian groups

Proposition 1

Let $(G, +, -, 0)$ be an abelian group. Then every unary term is an endomorphism of $(G, +, -, 0)$.

Every unary term of an additive group has the form

$$f(x) = kx \quad \text{for some } k \in \mathbb{Z}.$$

For $f(x) = kx$ and $g_1, g_2 \in G$:

$$\begin{aligned} f(0) &= 0, \\ f(-g_1) &= k(-g_1) = -(kg_1) = -f(g_1), \\ f(g_1 + g_2) &= k(g_1 + g_2) = kg_1 + kg_2 = f(g_1) + f(g_2). \end{aligned}$$

(The last equality uses the assumption that G is abelian.)

Proposition 2: rings of characteristic zero

Proposition 2

Let $(R, +, -, \cdot, 0, 1)$ be a ring of characteristic zero with 1. Then a unary term is an endomorphism of $(R, +, -, \cdot, 0, 1)$ if and only if it is the identity operation.

Proof: Every unary term has polynomial form with integer coefficients:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Vandermonde argument in Proposition 2

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

If f is a ring endomorphism, then $f(m) = m$ for every $m \in \mathbb{Z}$. Hence $a_0 = f(0) = 0$ and the equations for $1, 2, \dots, n$ lead to

$$\begin{cases} a_n + a_{n-1} + \cdots + a_2 + a_1 = 1, \\ a_n 2^{n-1} + a_{n-1} 2^{n-2} + \cdots + a_2 \cdot 2 + a_1 = 1, \\ \vdots \\ a_n n^{n-1} + a_{n-1} n^{n-2} + \cdots + a_2 n + a_1 = 1. \end{cases}$$

The coefficient matrix is Vandermonde matrix:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 2^{n-1} & 2^{n-2} & \cdots & 2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ n^{n-1} & n^{n-2} & \cdots & n & 1 \end{bmatrix}.$$

Its determinant is nonzero, so the only solution is $a_1 = 1$, $a_2 = \cdots = a_n = 0$. Then $f(x) = x$.

Proposition 3: mono-unary algebras

Proposition 3

Let (A, h) be a mono-unary algebra. Then every unary term is an endomorphism of (A, h) .

Every unary term over one unary operation has the form

$$g = h^k \quad \text{for some } k \in \mathbb{N}_0.$$

Then, for each $a \in A$,

$$g(h(a)) = h^k(h(a)) = h^{k+1}(a) = h(h^k(a)) = h(g(a)).$$

Characterization for unary algebras

Proposition 4

Let $A = (A, F)$ be a unary algebra. Then $\varphi : A \rightarrow A$ is an endomorphism of A if and only if it is an endomorphism of the mono-unary algebra (A, h) for each $h \in F$.

Proposition 5

Let (A, F) be a unary algebra and $F = \{h_i : i \in I\}$. Suppose that $g \in T(F)$ and $a^* \in A$ are such that

- ① $h_i(a^*) = a^*$ for each $i \in I$;
- ② there exists $k \in \mathbb{N}$ such that $g^k(a) = a^*$ for each $a \in A$.

Then $g^k \in T(F)$ and g^k is an endomorphism of (A, F) .

Summary: where useful unary terms come from

Class	Unary term endomorphisms	Consequence for the theorem
Abelian groups	all unary terms $f(x) = kx$	many nontrivial tests are available
Rings of characteristic zero with 1	only the identity	the theorem gives no nontrivial test from terms
Mono-unary algebras	all terms h^k	the theory applies naturally
General unary algebras	depends on interactions among operations	requires structural analysis

References



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Additional references are listed in the first article.

Thank you

easy direct limits – retracts – mono-unary algebras – diamond construction