

Dual spaces of ortholattices

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References

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Definition (Ploščica 1995)

Let L be a lattice with bounds and $D(L)$ the set of MDFIPs of L . The dual digraph of L is defined by

$$G_L = (D(L), E) \quad \text{where} \quad \langle F, I \rangle E \langle G, J \rangle \iff F \cap J = \emptyset$$

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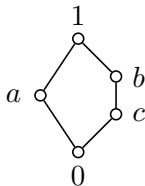
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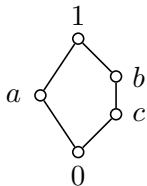
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For $x, y \in L$ we write xy for the MDFIP $\langle \uparrow x, \downarrow y \rangle$.

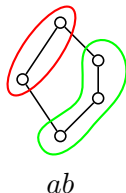


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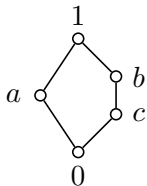


Dual digraph G_{N_5} :

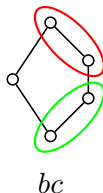
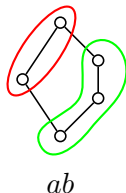
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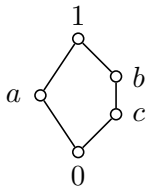
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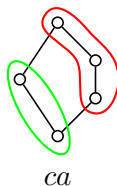
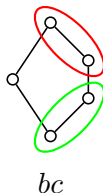
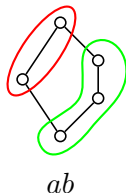
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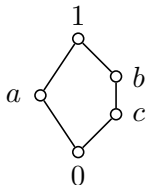


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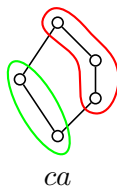
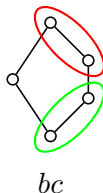
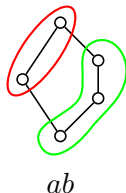
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MDFIPs:



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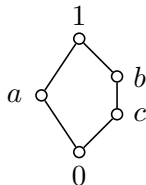
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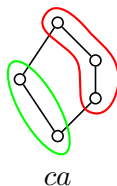
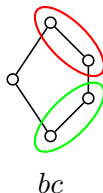
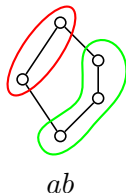
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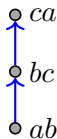
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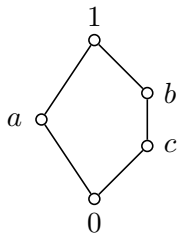
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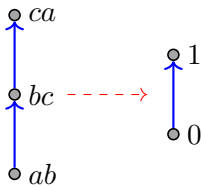
Recovering L from G_L and Ploščica's representation

For any $G = (X, E)$, the set of **maximal partial edge-preserving maps** from G to $(\{0, 1\}, \leq)$ forms a complete lattice when ordered by $\varphi \leq \psi \iff \varphi^{-1}(1) \subseteq \psi^{-1}(1)$.

Denote this lattice by $\mathbb{L}(G)$.



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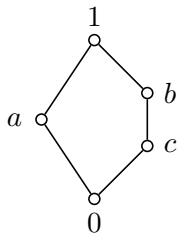


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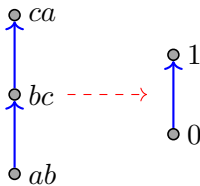
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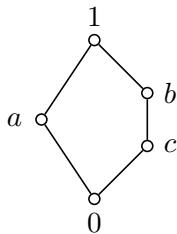


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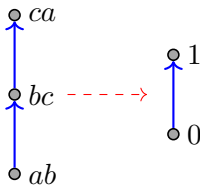
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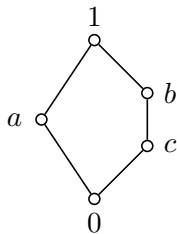


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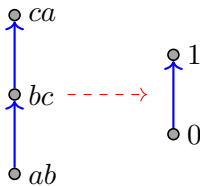
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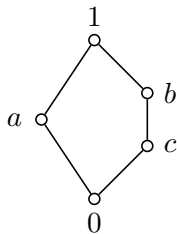
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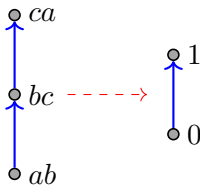
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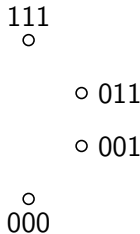
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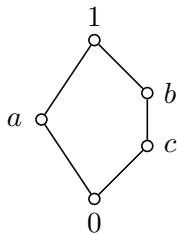
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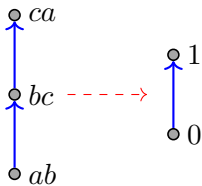
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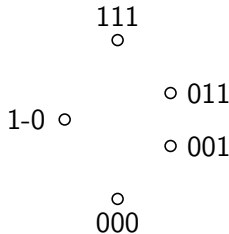
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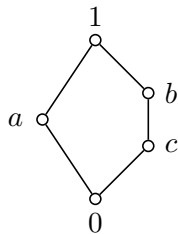
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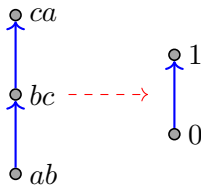
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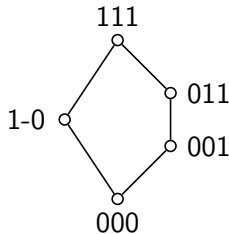
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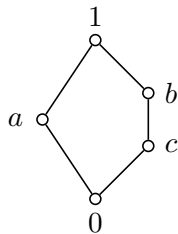


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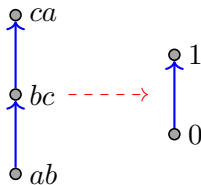
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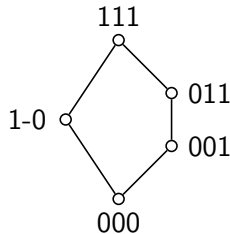
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Theorem (Ploščica 1995)

If L is a finite lattice, then $L \cong \mathbb{L}(G_L)$.

TiRS digraphs as Ploščica's duals of lattices

Let $xE = \{y \in V \mid xEy\}$ and $Ex = \{y \in V \mid yEx\}$.

Definition (Craig, Gouveia, Haviar 2015)

A **TiRS digraph** is a digraph $G = (V, E)$ such that $E \subseteq V \times V$ is reflexive and:

$$(S) \quad u \neq v \implies (uE \neq vE \text{ or } Eu \neq Ev).$$

$$(R) \quad \begin{aligned} uE \subset vE &\implies (u, v) \notin E \\ Ev \subset Eu &\implies (u, v) \notin E. \end{aligned}$$

$$(Ti) \quad uEv \implies \exists w (wE \subseteq uE \ \& \ Ew \subseteq Ev)$$

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Theorem (Craig, Gouveia, Haviar 2015)

There is a **one-to-one correspondence between finite lattices and finite TiRS digraphs** (which **generalises Birkhoff's representation**).
If G is a finite TiRS digraph and L is a finite lattice, then

$$L \cong \mathbb{L}(G_L) \quad \text{and} \quad G \cong G_{\mathbb{L}(G)}.$$

Ploščica's dual representation of (infinite) lattices (1995)

- Ploščica's dual of a *general lattice $\mathbf{L}$ with bounds* is $\mathcal{D}(\mathbf{L}) = (P_{\mathbf{L}}, E, \mathcal{T}_{\mathbf{L}})$, i.e. the digraph of the MDFIPs, or *the set $P_{\mathbf{L}}$ of all MPHs from $\mathbf{L}$ into $\underline{2}$* , with topology $\mathcal{T}_{\mathbf{L}}$ having as a subbasis of closed sets all sets of the form

$$V_a = \{ f \in P_{\mathbf{L}} \mid f(a) = 0 \} \quad \text{and} \\ W_a = \{ f \in P_{\mathbf{L}} \mid f(a) = 1 \}$$

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- The lattice L is represented as the lattice $\mathcal{G}_{\mathcal{T}}^{\text{mp}}(\mathcal{D}(\mathbf{L}), \mathbf{2}_{\mathcal{T}})$ of *maximal partial E -preserving continuous functions* (morphisms) from the dual space $\mathcal{D}(\mathbf{L})$ to the two-element poset with the discrete topology $\mathbf{2}_{\mathcal{T}}$.

Ploščica spaces

- In the paper Craig–Haviar–São João: *Dual spaces of lattices and semidistributive lattices (2026)* we introduced **Ploščica spaces** as abstractions of the duals $\mathcal{D}(\mathbf{L}) = (P_{\mathbf{L}}, E, \mathcal{T}_{\mathbf{L}})$ of general lattices $\mathbf{L}$ with bounds.

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Definition (Craig, Haviar, São João, 2026)

A **Ploščica space** is a structure $\mathbb{P} = (X, E, \mathcal{T})$ such that

- (1) (X, E) is a **TiRS digraph** and $(X, \mathcal{T})$ is a **comp. top. space**.
- (2) $\mathbb{P}$ is **totally edge-disconnected**, meaning that for all $x, y \in X$ such that $(x, y) \notin E$ there exists a morphism $\varphi \in \mathcal{G}_{\mathcal{T}}^{\text{mp}}(\mathbb{P}, \mathcal{Z}_{\mathcal{T}})$ with $\varphi(x) = 1$ and $\varphi(y) = 0$.
- (3) Certain specific sets are closed and the family

$$\{ X \setminus \varphi^{-1}(1), X \setminus \varphi^{-1}(0) \mid \varphi \in \mathcal{G}_{\mathcal{T}}^{\text{mp}}(\mathbb{P}, \mathcal{Z}_{\mathcal{T}}) \}$$

forms a **subbase for $\mathcal{T}$** (technicalities omitted).

- We proved that for every lattice $\mathbf{L}$ with bounds its dual $\mathcal{D}(\mathbf{L}) = (P_{\mathbf{L}}, E, \mathcal{T}_{\mathbf{L}})$ is a **Ploščica space**.

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- We claim that *recovering the lattice* from its **Ploščica's dual via the maximal partial morphisms is easier** and more natural than recovering the lattice from its dual L-space via the doubly closed stable sets in the Urquhart representation.

- We proved that for every lattice $\mathbf{L}$ with bounds its dual $\mathcal{D}(\mathbf{L}) = (P_{\mathbf{L}}, E, \mathcal{T}_{\mathbf{L}})$ is a **Ploščica space**.
- For a Ploščica space $\mathbb{P} = (X, E, \mathcal{T})$, we showed that $\mathcal{L}(\mathbb{P}) = (\mathcal{G}_{\mathcal{T}}^{\text{mp}}(\mathbb{P}, \mathcal{Z}_{\mathcal{T}}), \leq)$ is a bounded lattice, which we considered to be the **dual lattice** of $\mathbb{P}$.
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- Another **advantage of using Ploščica spaces** is that *in case the lattice $\mathbf{L}$ is distributive*, the edge relation E in a Ploščica space becomes *the Priestley order relation*.

Ortho-Ploščica spaces

Definition (Craig, Haviar, 2026)

The **dual of an ortholattice** $\mathbf{L}' = (L; \vee, \wedge, ', 0, 1)$ is the structure $\mathcal{D}(\mathbf{L}') = (P_{\mathbf{L}}, E, \mathcal{T}, g)$, where $(P_{\mathbf{L}}, E, \mathcal{T})$ is **the dual Ploščica space** of **the lattice reduct** $\mathbf{L}$ of $\mathbf{L}'$ and **the map** g is given as follows: for an MPH $f : \mathbf{L} \rightarrow \mathbf{2}$ and $a \in L$, let an MPH $g(f) \in \mathcal{L}^{\text{mp}}(\mathbf{L}, \underline{2})$ is

$$(g(f))(a) = \begin{cases} 0 & \text{if } f(a') = 1, \\ 1 & \text{if } f(a') = 0. \end{cases}$$

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Theorem

Let $\mathbf{L}'$ be an ortholattice. The map g of $\mathcal{D}(\mathbf{L}')$ satisfies:

- (M1) $g(g(x)) = x$,
- (M2) $x E \subseteq y E \implies E g(x) \subseteq E g(y)$,
- (M3) $E x \subseteq E y \implies g(x) E \subseteq g(y) E$,
- (O) $\forall x \exists y (y E \subseteq x E \ \& \ E y \subseteq E g(x))$.

Ortho-Ploščica spaces—cont.

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An **ortho-Ploščica space** is a structure $\mathbf{X} = (X, E, \mathcal{T}, g)$ such that $(X, E, \mathcal{T})$ is a *Ploščica space* and $g : X \rightarrow X$ is a continuous map satisfying (M1)–(M3), (O).

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Definition

Let $\mathbf{X} = (X, E, \mathcal{T}, g)$ be an *ortho-Ploščica space* and let φ_0 and φ_1 be the constant maps from X to 0 and 1, respectively. Then **the dual of $\mathbf{X}$ is the lattice** with bounds

$\mathcal{E}(\mathbf{X}) = (\mathcal{G}_{\mathcal{T}}^{\text{mp}}(\mathbf{X}, \mathcal{Z}_{\mathcal{T}}), \vee, \wedge, \neg, \varphi_0, \varphi_1)$, where $\neg$ assigns to every $\varphi \in \mathcal{G}_{\mathcal{T}}^{\text{mp}}(\mathbf{X}, \mathcal{Z}_{\mathcal{T}})$ a partial map $\neg\varphi : X \rightarrow \{0, 1\}$ given as follows:

$$(\neg\varphi)(x) = \begin{cases} 1 & \text{if } \varphi(g(x)) = 0, \\ 0 & \text{if } \varphi(g(x)) = 1, \\ - & \text{otherwise.} \end{cases}$$

Dual representation theorems for ortholattices

Proposition

Let $\mathbf{X} = (X, E, \mathcal{T}, g)$ be an ortho-Ploščica space. Then $\neg$ as defined above is an orthocomplement on the lattice reduct of $\mathcal{E}(\mathbf{X})$. Hence the dual $\mathcal{E}(\mathbf{X}) = (\mathcal{G}_{\mathcal{T}}^{\text{mp}}(\mathbf{X}, \mathfrak{L}_{\mathcal{T}}), \vee, \wedge, \neg, \varphi_0, \varphi_1)$ of an ortho-Ploščica space $\mathbf{X}$ is an ortholattice.

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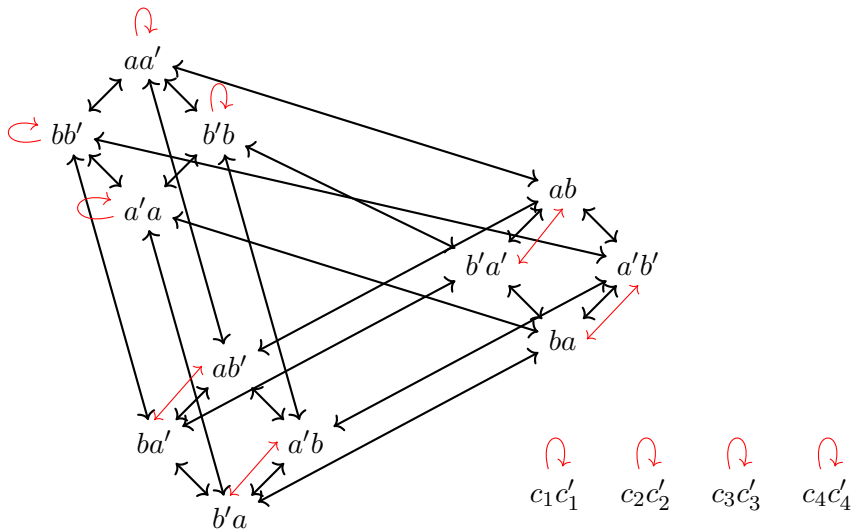
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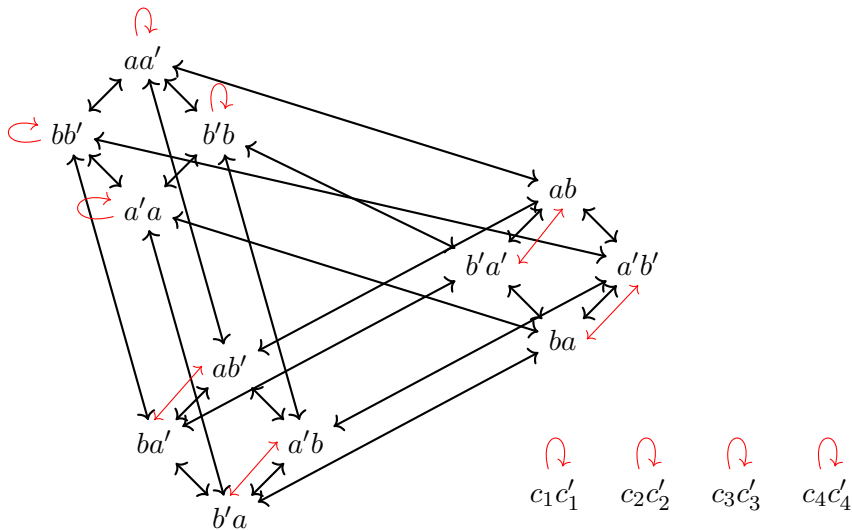
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- There are only two types of MDFIPs of $\mathcal{F}_{OM}(2)$:
the first type is $\langle \uparrow(x, \underline{0}), \downarrow(y, \underline{1}) \rangle$ where x, y are atoms of MO_2 (for simplicity we write xy);
the second type is $\langle \uparrow(\perp, c_i), \downarrow(\top, c'_i) \rangle$ (for simplicity we write $c_i c'_i$).
- The 16-element dual digraph of $\mathcal{F}_{OM}(2)$ is the disjoint union of the dual digraph of MO_2 and four isolated vertices with loops representing the dual of 2^4 .

The dual of $\mathcal{F}_{\mathcal{OM}}(2)$



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Thank you for your attention!