

On algebras with easy direct limits I

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Outline

- 1 Introduction
- 2 Basic definitions, examples
- 3 An overview of results on algebras with EDL

Section I

Introduction

an algebra with
easy direct limits
(in short: EDL)

=

an algebra from which
we can obtain by a
direct limit construction
a retract of itself only

Introduction

Background

The direct limits construction is basic in Universal Algebra. It builds up new algebras from directed systems of algebras (homomorphisms are included).

- ▶ A natural question is: if we start with one algebra A , how complicated can the algebras obtained from A by direct limits be?
- ▶ Algebras with EDL are the most lean case: every such direct limit is already represented by a retract of A

Motivation

- ① a general study of the construction
- ② a study of direct limit closed classes

Motivation 1: General properties of the construction

Grätzer G. Universal algebra, 2nd edition with updates, Springer Science+Business Media, LLC, 2008.

Chapter 3 - Constructions of algebras, p. 128:

§21. DIRECT AND INVERSE LIMITS OF ALGEBRAS

There are two well known methods to build up algebras from families of algebras, the so-called direct and inverse limits. No systematic account of the properties of these constructions can be found in the literature, although almost all the results given below belong to the “folklore”. Besides, most results are set theoretic in nature, and they can be derived from the case of groups; see for example S. Eilenberg and W. Steenrod, Foundations of algebraic topology, Princeton University Press, Princeton, N. J., 1952.†

Motivation 1: General properties of the construction

Grätzer G. Universal algebra, 2nd edition with updates, Springer Science+Business Media, LLC, 2008.

Problem on the page 161, still open:

26. For an algebraic class K , define K^α for every ordinal α as follows: $K^0 = K$, $K^{\alpha+1} = \mathbf{IL}(K^\alpha)$, $K^{\lim \alpha_\nu} = \bigcup (K^{\alpha_\nu} \mid \nu)$. Is it possible to find for every ordinal α a class such that $K^{\alpha+1} \neq K^\alpha$? Is it true that for every class K there exists an α with $K^\alpha = K^{\alpha+1}$? If this is so, which ordinals can occur as a smallest such α ? (See Ex. 82.)

Motivation 1: General properties of the construction

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Example: $K^2 \neq K^1$, algebras of K are isomorphic

E. Halušková: On iterated direct limits of a monounary algebra, Contributions to General Algebra, 10 (Klagenfurt, 1997), Austria, 189–195, Verlag Johannes Heyn, Klagenfurt, 1998.

Motivation 2: Direct limit closed classes

H. J. Keisler: Theory of Models with Generalized Atomic Formulas, The Journal of Symbolic Logic, Vol.25, No.1 (1960), 1-26.

J. Jakubík, G. Pringerová: Direct limits of cyclically ordered groups, Czechoslovak Math. J. 44 (1994), 231-250.

Lemma

If A has EDL, then the class of all isomorphic copies of retracts of A is direct limit closed.

Motivation 2: Direct limit closed classes

Theory of Models with Generalized Atomic Formulas

Author(s): H. Jerome Keisler

Source: *The Journal of Symbolic Logic*, Vol. 25, No. 1 (Mar., 1960), pp. 1-26

THEOREM 6. *If $\mathbf{K} \in \text{EC}_\delta$, then $[L]\text{-Sub}(F\text{-DirLim}(\mathbf{K})) = \exists \forall \wedge F\text{-Sub}(\mathbf{K}) = (\mathbf{K}^* \cap \forall \mathbf{B} \exists \forall \wedge F)^*$; that is, $\mathfrak{A}$ has an elementary extension isomorphic to an F -direct limit of an F -direct system of elements of $\mathbf{K}$ if and only if $\mathfrak{A}$ is an $\exists \forall \wedge F$ -subsystem of an element of $\mathbf{K}$, and this is true if and only if every sentence of $\forall \mathbf{B} \exists \forall \wedge F$ holding throughout $\mathbf{K}$ also holds in $\mathfrak{A}$.*

In this connection we observe that the sentences of $\forall \mathbf{B} \exists \forall \wedge F$ are precisely those sentences equivalent to a sentence of the form

$$\forall x_1 \dots x_s \left[\bigwedge_{j=1}^r (g_j(x_1, \dots, x_s) \supset \exists x_{s+1} \dots x_t h_j(x_1, \dots, x_t)) \right]$$

where $g_j, h_j \in \forall \wedge F$, ($j = 1, \dots, r$).

Motivation 2: Direct limit closed classes

Definition of the class Δ

all first-order sentences equivalent to a sentence of the form

$$\bigwedge_{i=1}^m (\forall x_1, \dots, x_s) (\alpha_i \rightarrow (\exists x_{s+1}, \dots, x_t) \beta_i),$$

where $m \in \mathbb{N}$, α_i, β_i are positive for every $i = 1, \dots, m$, $s, t \in \mathbb{N}_0$, $s \leq t$.

Theorem

Every first-order class of algebras is closed with respect to direct limits if and only if it is determined by some sentences from Δ .

Motivation 2: Direct limit closed classes

Definition of the class Γ

all sentences equivalent to a sentence of the form

$$(\forall x_0, \dots, x_{n-1})(\alpha \rightarrow (\exists y_\mu; \mu < \varrho) \beta),$$

where α is a quantifiers-free first-order, β is a positive, $n \in \mathbb{N}_0$ and $\varrho \in \text{Ord}$.

Proposition

If a class of algebras is determined by some sentences from Γ , then it is closed with respect to direct limits.

The question, whether every axiomatic direct limit closed class is determined by some sentences of Γ , is open.

Section II

Basic definitions, examples

Retracts

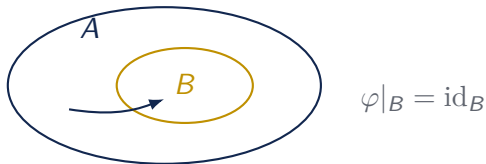
Suppose that $B = (B, F)$ is a subalgebra of $A = (A, F)$.

Definition

B is a **retract** of A if there exists an endomorphism φ of A such that

$$\varphi(A) = B \quad \text{and} \quad \varphi(b) = b \quad \text{for every } b \in B.$$

The map φ is called a **retraction** of A onto B .



Direct limits

Definition

A **direct system** of algebras $\{I, A_i, \varphi_{ij}\}$ contains

- ① upward directed poset $\langle I, \leq \rangle$, $I \neq \emptyset$;
- ② algebra (A_i, F) for each $i \in I$;
- ③ homomorphism φ_{ij} of A_i into A_j ($i < j$);
 φ_{ii} the identity on A_i ;
 $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ ($i < j < k$).

Put $x \equiv y$ if $\varphi_{ik}(x) = \varphi_{jk}(y)$

The **direct limit** of $\{I, A_i, \varphi_{ij}\}$ is $(\bar{A}, F)$, where

- ▶ $\bar{A} = \dot{\bigcup}_{i \in I} A_i / \equiv$
- ▶ $f(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) = \overline{f(\varphi_{i_1 k}(x_1), \dots, \varphi_{i_n k}(x_n))}$ for n -ary $f \in F$

Notation

For an algebra $A = (A, F)$:

$\varinjlim A$ the class of all isomorphic copies of direct limits obtainable from A

R_A the set of all retracts of A

Basic inclusion

$$R_A \subseteq \varinjlim A.$$

For a class $\mathcal{C}$ of algebras of the same type

$[\mathcal{C}]$ the class of all isomorphic copies of algebras from $\mathcal{C}$

Definition

Let $A = (A, F)$ be an algebra.

If every algebra from $\varinjlim A$ is isomorphic to a retract of A , i.e. $\varinjlim A = [R_A]$, then we say that A has **easy direct limits** (is with **easy direct limits**), in short: **EDL**.

EDL for basic algebraic structures on integer, rational, real and complex numbers

EDL	ring	additive group	multiplicative monoid
$\mathbb{Z}$	✓	✗	✗
$\mathbb{Q}$	✓	✓	✗
$\mathbb{R}$	✓	✗	✗
$\mathbb{C}$	✗	✗	✗

Section III

An overview of results on algebras with EDL

In general

Theorem [H., Ploščica 1999]

Every finite algebra has EDL.

Lemma

If every endomorphism of A is constant or bijective, then A has EDL.

Theorem [presented on SSAOS 2025]

A simple algebra has EDL iff every endomorphism of it is constant or bijective.

Open problems:

- ▶ Is the cardinality of algebras of a given type with EDL bounded? How?
- ▶ Is there an algebra with EDL which has a retract which is not with EDL?

In general

Denote by $\mathcal{P}_{(A,F)}$ the property that there exists $(B, F) \in \underline{\mathbf{L}}(A, F)$ such that $\|B\| > \|A\|$.

Proposition

Let (A, F) be an algebra such that $\mathcal{P}_{(A,F)}$ is valid. If $G \subseteq F$, then the algebra (A, G) is not with EDL.

Proposition

Let A_m be an algebra of type F which has a constant endomorphism e_m for each $m \in M$. Denote

$$A = \prod_{m \in M} A_m.$$

If $k \in M$ is such that $\|A_k\| = \|A\|$ and $\mathcal{P}_{A_k}$ is valid, then A is not with EDL.

Mono-unary algebras

Theorem [H. 2018]

If a mono-unary algebra has EDL, then it is countable and the number of all non-isomorphic retracts of it is not equal to $\aleph_0$.

Theorem [H. 2018]

Let h be a bijection on a set A . Then TFAE:

- (i) The mono-unary algebra (A, h) has EDL.
- (ii) The mono-unary algebra (A, h) does not contain infinitely many mutually isomorphic components.

Open problem:

A description of all mono-unary algebras with EDL.

Vector spaces and finitely generated abelian groups

Theorem

Let G be finitely generated abelian group. TFAE

- ① G is finite,
- ② G has EDL.

Theorem [presented on SSAOS 2025]

Let V be a vector space over a field. TFAE

- ① the dimension of V is finite,
- ② V has EDL.

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Thank you for your attention!

