

Characterization of Archimedean t-Norms and Copulas with Linear Spline Generators

Based on Halaš – Mesiar – Pócs (2026 paper)

Radomír Halaš

Department of Algebra and Geometry, Palacký University Olomouc, Czech Republic

September 4, 2026

Outline

- 1 Motivation and overview
- 2 Background: definitions and representation
- 3 Linear splines in 1D and 2D
- 4 Example: a spline generator
- 5 Key geometric tool: the upper corner condition
- 6 Main theorem: spline t-norm implies spline generator
- 7 Summary and practical takeaways

Why t-norms and copulas?

- **t-norms** model conjunction in fuzzy logic and aggregation of degrees of truth.
- **Copulas** model dependence between random variables (separating margins from dependence).

Why t-norms and copulas?

- **t-norms** model conjunction in fuzzy logic and aggregation of degrees of truth.
- **Copulas** model dependence between random variables (separating margins from dependence).
- **Archimedean** families are very important: simple parametrizations, tractable formulas.

Why t-norms and copulas?

- **t-norms** model conjunction in fuzzy logic and aggregation of degrees of truth.
- **Copulas** model dependence between random variables (separating margins from dependence).
- **Archimedean** families are very important: simple parametrizations, tractable formulas.
- This talk describes a special structured subclass: **piecewise linear (spline) forms**.

Core question

When is an Archimedean t-norm / copula a **2D linear spline**, and how does that relate to its **1D additive generator**?

Main results (summary)

- Classical representation: Archimedean t-norms are generated by an **additive generator** f .

Main results (summary)

- Classical representation: Archimedean t-norms are generated by an **additive generator** f .
- Known direction: if f is a **1D linear spline**, then T is a **2D linear spline**.

Main results (summary)

- Classical representation: Archimedean t-norms are generated by an **additive generator** f .
- Known direction: if f is a **1D linear spline**, then T is a **2D linear spline**.
- Main contribution: the **converse** direction:

T is a 2D linear spline $\Rightarrow f$ and its pseudo-inverse g are 1D linear splines.

- A geometric tool: the **upper corner condition**.

t-norms: definition and natural examples

A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a **t-norm** if, for all $u, v, w \in [0, 1]$:

- **Identity:** $T(u, 1) = T(1, u) = u$
- **Commutativity:** $T(u, v) = T(v, u)$
- **Associativity:** $T(u, T(v, w)) = T(T(u, v), w)$
- **Monotonicity:** $u \leq u' \Rightarrow T(u, v) \leq T(u', v)$

Natural examples

$$T_M(u, v) = \min\{u, v\}, \quad T_P(u, v) = uv$$

$$T_L(u, v) = \max\{0, u + v - 1\}$$

$$T_D(u, v) = 0 \text{ if } u \text{ or } v \neq 1$$

$$T_D(u, v) \leq T(u, v) \leq T_M(u, v)$$

t-norms: definition and natural examples

A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a **t-norm** if, for all $u, v, w \in [0, 1]$:

- **Identity:** $T(u, 1) = T(1, u) = u$
- **Commutativity:** $T(u, v) = T(v, u)$
- **Associativity:** $T(u, T(v, w)) = T(T(u, v), w)$
- **Monotonicity:** $u \leq u' \Rightarrow T(u, v) \leq T(u', v)$

Natural examples

$$T_M(u, v) = \min\{u, v\}, \quad T_P(u, v) = uv$$

$$T_L(u, v) = \max\{0, u + v - 1\}$$

$$T_D(u, v) = 0 \text{ if } u \text{ or } v \neq 1$$

$$T_D(u, v) \leq T(u, v) \leq T_M(u, v)$$

Bounds for continuous t-norms

$$T_L(u, v) \leq T(u, v) \leq T_M(u, v).$$

In particular: $T_L \leq T_P \leq T_M$.

In this talk: T is (at least) **continuous**.

A continuous t-norm T is **Archimedean** if

$$T(u, u) < u \quad \text{for all } u \in (0, 1).$$

A continuous t-norm T is **Archimedean** if

$$T(u, u) < u \quad \text{for all } u \in (0, 1).$$

Intuition:

- Repeated conjunction strictly decreases truth unless already at 0 or 1.
- Excludes idempotent behavior like $\min(u, v)$ (which is not Archimedean).

A function $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a **copula** if:

- **Boundary conditions:** $C(u, 0) = C(0, v) = 0$ and $C(u, 1) = u$, $C(1, v) = v$.
- **2-increasing:** for $u_1 \leq u_2$, $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0.$$

- **Archimedean copula:** C is associative and $C(u, u) < u$ for $u \in (0, 1)$.

A function $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a **copula** if:

- **Boundary conditions:** $C(u, 0) = C(0, v) = 0$ and $C(u, 1) = u$, $C(1, v) = v$.
- **2-increasing:** for $u_1 \leq u_2$, $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0.$$

- **Archimedean copula:** C is associative and $C(u, u) < u$ for $u \in (0, 1)$.
- **Sklar's theorem (idea):** any multivariate distribution can be decomposed into its marginal distributions and a copula capturing the **dependence structure**:

$$H(x, y) = C(F(x), G(y)),$$

and if F, G are continuous, the copula C is unique.

Additive generator representation

A (continuous) Archimedean t-norm can be represented as:

$$T(u, v) = g(f(u) + f(v)),$$

where:

- $f : [0, 1] \rightarrow [0, \infty]$ is continuous, strictly decreasing, and $f(1) = 0$;
- g is the **pseudo-inverse** of f :

$$g(x) = \begin{cases} f^{-1}(x), & x \in [0, f(0)], \\ 0, & x \in [f(0), \infty). \end{cases}$$

Additive generator representation

A (continuous) Archimedean t-norm can be represented as:

$$T(u, v) = g(f(u) + f(v)),$$

where:

- $f : [0, 1] \rightarrow [0, \infty]$ is continuous, strictly decreasing, and $f(1) = 0$;
- g is the **pseudo-inverse** of f :

$$g(x) = \begin{cases} f^{-1}(x), & x \in [0, f(0)], \\ 0, & x \in [f(0), \infty). \end{cases}$$

Uniqueness

The generator is unique **up to a positive multiplicative constant**.

Useful equivalent identity

From the generator form one obtains:

$$g(x + y) = T(g(x), g(y)) \quad \text{for } x, y \in [0, \infty).$$

Useful equivalent identity

From the generator form one obtains:

$$g(x + y) = T(g(x), g(y)) \quad \text{for } x, y \in [0, \infty).$$

Why it matters:

- If T is piecewise linear, this identity strongly constrains the shape of g .
- It is the crucial property to *propagate* linearity of g from small x to larger intervals.

When is an Archimedean t-norm a copula?

A key criterion:

Convexity criterion

An Archimedean t-norm is 2-increasing (hence an Archimedean copula) **iff** its additive generator f is **convex**.

When is an Archimedean t-norm a copula?

A key criterion:

Convexity criterion

An Archimedean t-norm is 2-increasing (hence an Archimedean copula) **iff** its additive generator f is **convex**.

Interpretation:

- Spline generators can be convex or non-convex depending on slopes.
- Thus, spline t-norms are not automatically copulas; convexity decides.

What is a linear spline? (intuitive)

- **1D linear spline:** a continuous function whose graph is made of finitely many line segments.
- **2D linear spline:** a continuous surface that is planar on each triangle of some triangulation of $[0, 1]^2$.

What is a linear spline? (intuitive)

- **1D linear spline:** a continuous function whose graph is made of finitely many line segments.
- **2D linear spline:** a continuous surface that is planar on each triangle of some triangulation of $[0, 1]^2$.

Triangulation viewpoint

On each triangle $\Delta \subset [0, 1]^2$, a 2D linear spline satisfies

$$T(u, v) = au + bv + c \quad \text{for constants } a, b, c \text{ (depending on } \Delta \text{).}$$

Known implication

If the generator f is a **1D linear spline**, then the induced Archimedean t-norm

$$T(u, v) = g(f(u) + f(v))$$

is a **2D linear spline**.

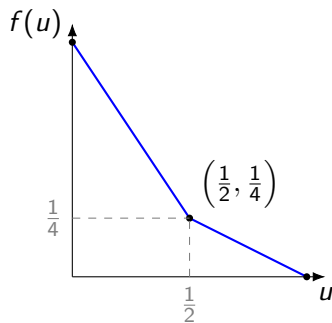
Idea (intuition):

- $f(u) + f(v)$ is piecewise linear on regions where f is linear in each variable.
- Composing with a piecewise linear g preserves the piecewise linear structure.

Example: a nontrivial spline generator

Consider the piecewise linear additive generator

$$f(u) = \begin{cases} 1 - \frac{3}{2}u, & 0 \leq u \leq \frac{1}{2}, \\ \frac{1}{2} - \frac{1}{2}u, & \frac{1}{2} < u \leq 1. \end{cases}$$

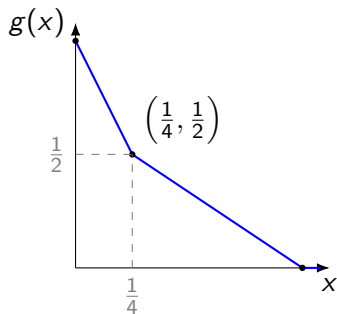


- Two linear pieces: finite and easy to compute.
- Slopes increase from $-\frac{3}{2}$ to $-\frac{1}{2}$, so f is convex.
- Hence the induced Archimedean t-norm is also a copula.

Example: the pseudo-inverse

The pseudo-inverse is again piecewise linear:

$$g(x) = \begin{cases} 1 - 2x, & 0 \leq x \leq \frac{1}{4}, \\ \frac{2}{3} - \frac{2}{3}x, & \frac{1}{4} < x \leq 1, \\ 0, & 1 < x. \end{cases}$$



- g is a compactly supported 1D spline.
- The first linear piece determines the upper corner.
- Here

$$q = g\left(\frac{1}{4}\right) = \frac{1}{2}.$$

Example: the generated t-norm

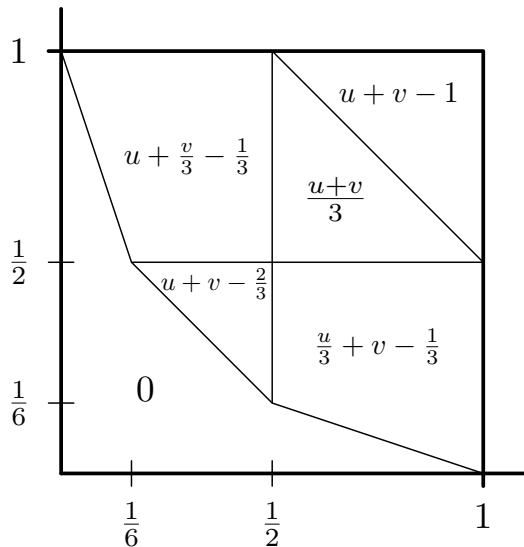
Generated operation

$$T(u, v) = g(f(u) + f(v)).$$

- f and g are 1D linear splines.
- Hence T is a 2D linear spline.
- The upper corner is explicit:

$$u + v \geq \frac{3}{2} \Rightarrow T(u, v) = u + v - 1.$$

- Since f is convex, this example is also an Archimedean copula.



Definition

A t-norm T satisfies the **upper corner condition** if there exists $q < 1$ such that

$$T(u, v) = u + v - 1 \quad \text{for all } u, v \in [0, 1] \text{ with } u + v \geq q + 1.$$

Definition

A t-norm T satisfies the **upper corner condition** if there exists $q < 1$ such that

$$T(u, v) = u + v - 1 \quad \text{for all } u, v \in [0, 1] \text{ with } u + v \geq q + 1.$$

Interpretation:

- In the top-right corner of the unit square, T coincides with the Lukasiewicz t -norm.
- Geometrically: the surface is exactly the plane $u + v - 1$ on that triangular region.

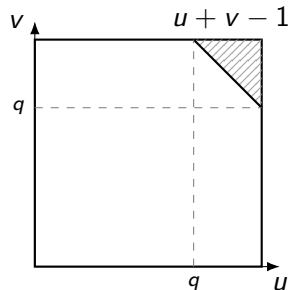
Upper corner condition: geometric picture

Upper corner condition

There exists $q < 1$ such that

$$T(u, v) = u + v - 1 \quad \text{whenever } u + v \geq q + 1.$$

- The line $u + v = q + 1$ cuts the unit square.
- In the shaded corner region, the surface of T is the plane $u + v - 1$.



Upper corner condition vs generator behavior

A central lemma in the paper links corner planarity to linearity of the pseudo-inverse:

Lemma (equivalence, informal)

T satisfies the upper corner condition



there exists $z > 0$ such that the pseudo-inverse g is **linear** on $[0, z]$.

Upper corner condition vs generator behavior

A central lemma in the paper links corner planarity to linearity of the pseudo-inverse:

Lemma (equivalence, informal)

T satisfies the upper corner condition



there exists $z > 0$ such that the pseudo-inverse g is **linear** on $[0, z]$.

- “Planar (Lukasiewicz) near $(1, 1)$ ” forces “linear near $x = 0$ ” in generator space.
- This is the “seed” linearity that will later be extended to all of $[0, \infty)$.

Step 1: spline T implies upper corner

Lemma (key step)

Every Archimedean **2D linear spline** t-norm satisfies the upper corner condition.

Step 1: spline T implies upper corner

Lemma (key step)

Every Archimedean **2D linear spline** t-norm satisfies the upper corner condition.

Proof idea:

- A triangulation gives finitely many planar patches.
- Boundary condition $T(u, 1) = u$ heavily constrains planes near $(1, 1)$.
- Associativity prevents arbitrary planes from meeting there, forcing the Lukasiewicz plane in the corner.

Step 2: propagate linearity of g

Assume now T is a 2D linear spline and Archimedean.

Step 2: propagate linearity of g

Assume now T is a 2D linear spline and Archimedean.

- From Step 1: upper corner condition holds.
- From the equivalence lemma: g is linear on some $[0, z]$.
- Use the functional identity

$$g(x + y) = T(g(x), g(y))$$

to extend piecewise linearity to sums of intervals.

Step 2: propagate linearity of g

Assume now T is a 2D linear spline and Archimedean.

- From Step 1: upper corner condition holds.
- From the equivalence lemma: g is linear on some $[0, z]$.
- Use the functional identity

$$g(x + y) = T(g(x), g(y))$$

to extend piecewise linearity to sums of intervals.

Why extension works

On each planar region of T , the right-hand side is linear in $g(x)$ and $g(y)$, so the shape of g is forced to remain piecewise linear.

Theorem (paper)

Let T be an Archimedean t-norm with additive generator f and pseudo-inverse g . If T is a **2D linear spline**, then both

f and g

are **1D linear splines**.

Theorem (paper)

Let T be an Archimedean t-norm with additive generator f and pseudo-inverse g . If T is a **2D linear spline**, then both

$$f \text{ and } g$$

are **1D linear splines**.

Consequences:

- 2D piecewise planarity of T is not arbitrary: it must come from a 1D piecewise linear generator.
- The entire class of such t-norms is parameterized by spline generators.

Corollary: full characterization (+ copulas)

Corollary

For Archimedean t-norms:

T is a 2D linear spline $\iff f$ is a 1D linear spline.

For Archimedean copulas

Additionally require: f is **convex**.

Corollary: full characterization (+ copulas)

Corollary

For Archimedean t-norms:

T is a 2D linear spline $\iff f$ is a 1D linear spline.

For Archimedean copulas

Additionally require: f is **convex**.

Practical message:

- Want a spline Archimedean copula? Design a **convex** spline generator f .

- The paper gives a clean equivalence between:
 - spline geometry of T on $[0, 1]^2$, and
 - spline geometry of the generator f on $[0, 1]$.
- The upper corner condition is the main geometric tool.
- Convexity of f determines when the t-norm is a copula (2-increasing).

Application

Spline generators are interpretable and computationally friendly, enabling fast evaluation and flexible modeling.

Thank you! Any questions?