

Functional Relations

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Operational and abstract clones

Definition

An **operational clone** is a set A together with a set $\mathcal{F}$ consisting of operations on A such that $\mathcal{F}$ contains all operations and is closed to composition.

Definition

An **abstract clone** is a heterogenous algebra with underlying sets $A_0, A_1, \dots$, constants $\pi_i^{(n)} \in A_n$ for all $1 \leq i \leq n$ and composition operators $c_k^{(n)} : A_n \times A_k^n \rightarrow A_k$ for each k and n satisfying

- $c_k^{(n)}(\pi_i^{(n)}, g_1, \dots, g_n) = g_i$ for all $g_1, \dots, g_n \in A_k$,
- $c_m^{(n)}(f, c_k^{(m)}(g_1, h_1, \dots, h_m), \dots, c_k^{(m)}(g_n, h_1, \dots, h_m)) = c_k^{(m)}(c_m^{(n)}(f, g_1, \dots, g_n), h_1, \dots, h_m)$ for all $f \in A_n, g_1, \dots, g_n \in A_m, h_1, \dots, h_m \in A_k$.

Clone homomorphisms

Definition

A **clone homomorphism** between abstract clones $\mathbf{A} = (A_0, A_1, \dots)$ and $\mathbf{B} = (B_0, B_1, \dots)$ is a sequence of maps $t_j : A_j \rightarrow B_j$ such that

- $t_n((\pi_i^{(n)})^{\mathbf{A}}) = (\pi_i^{(n)})^{\mathbf{B}},$
- $t_k(c_k^{(n)}(f, g_1, \dots, g_n)) = c_k^{(n)}(t_n(f), t_k(g_1), \dots, t_k(g_n))$ for all $f \in A_n, g_1, \dots, g_n \in A_k.$

For operational clones $(A, \mathcal{F})$ and $(B, \mathcal{G})$, a clone homomorphism is a map $* : \mathcal{F} \rightarrow \mathcal{G}$ that preserves arity, projections and satisfies

$$(f(g_1, \dots, g_n))^* = f^*(g_1^*, \dots, g_n^*).$$

Clone homomorphisms

Examples:

- For any algebra $\mathbf{A}$ and any $\mathbf{B} \leq \mathbf{A}$, $f \mapsto f|_{\mathbf{B}}$ is a clone homomorphism from the $\text{Term } \mathbf{A}$ (the clone of term functions of $\mathbf{A}$) to $\text{Term } \mathbf{B}$.
- If θ is a congruence of $\mathbf{A}$, then $f^*(a_1/\theta, \dots, a_n/\theta) := f(a_1, \dots, a_n)/\theta$ defines a clone homomorphism from $\text{Term } \mathbf{A}$ to $\text{Term}(\mathbf{A}/\theta)$.
- The term clone of an algebra $\mathbf{A}$ lying in a variety $\mathcal{V}$ is a homomorphic image of $\text{Term } \mathcal{V}$ by $f \mapsto f^{\mathbf{A}}$.
- The inclusion mapping is a clone homomorphism from the term clone of abelian groups to the term clone of rings.
- There is also a clone homomorphism from the term clone of rings to the term clone of abelian groups.
- But there is no clone homomorphism from unitary rings to abelian groups.

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Clone homomorphisms

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- For any algebra $\mathbf{A}$ and any $\mathbf{B} \leq \mathbf{A}$, $f \mapsto f|_B$ is a clone homomorphism from the Term $\mathbf{A}$ (the clone of term functions of $\mathbf{A}$) to Term $\mathbf{B}$.
- If θ is a congruence of $\mathbf{A}$, then $f^*(a_1/\theta, \dots, a_n/\theta) := f(a_1, \dots, a_n)/\theta$ defines a clone homomorphism from Term $\mathbf{A}$ to Term $(\mathbf{A}/\theta)$.
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Clone homomorphisms

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- For any algebra $\mathbf{A}$ and any $\mathbf{B} \leq \mathbf{A}$, $f \mapsto f|_{\mathbf{B}}$ is a clone homomorphism from the Term $\mathbf{A}$ (the clone of term functions of $\mathbf{A}$) to Term $\mathbf{B}$.
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Interpretability, Mal'tsev conditions

Definition

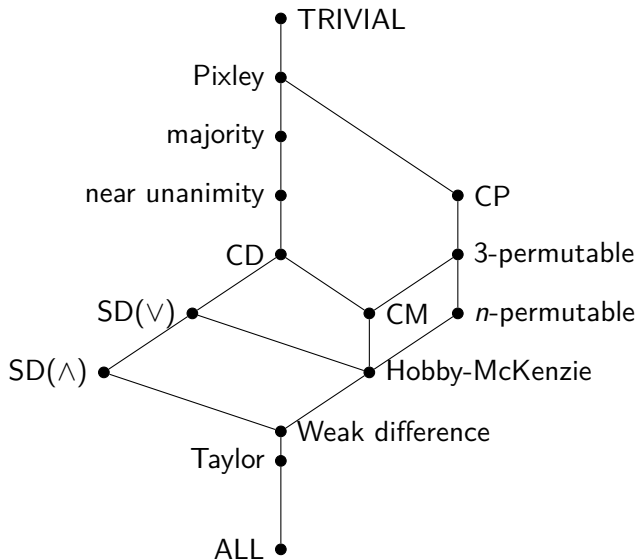
Let $\mathcal{V}$ and $\mathcal{W}$ be varieties, not necessarily of the same signature. Then $\mathcal{V}$ is said to be **interpretable** in $\mathcal{W}$ if there is a clone homomorphism $\text{Term } \mathcal{V} \rightarrow \text{Term } \mathcal{W}$.

Definition

A **strong Mal'tsev condition** is a finite axiomatization of a (finitely based) variety $\mathcal{V}$. An algebra $\mathbf{A}$ satisfies this condition if $\mathcal{V}$ is interpretable in $V(\mathbf{A})$, equivalently, if there is a clone homomorphism from $\text{Term } \mathcal{V}$ to $\text{Term } \mathbf{A}$.

A **Mal'tsev condition** is the union of a weakening chain of strong Mal'tsev conditions i.e. a sequence of finitely based varieties $\mathcal{V}_1, \mathcal{V}_2, \dots$ such that each $\mathcal{V}_{i+1}$ is interpretable in $\mathcal{V}_i$. Then $\mathcal{W}$ satisfies this condition if $\mathcal{V}_i$ is interpretable in $\mathcal{W}$ for some i .

Hierarchy of some Mal'tsev conditions



Definition

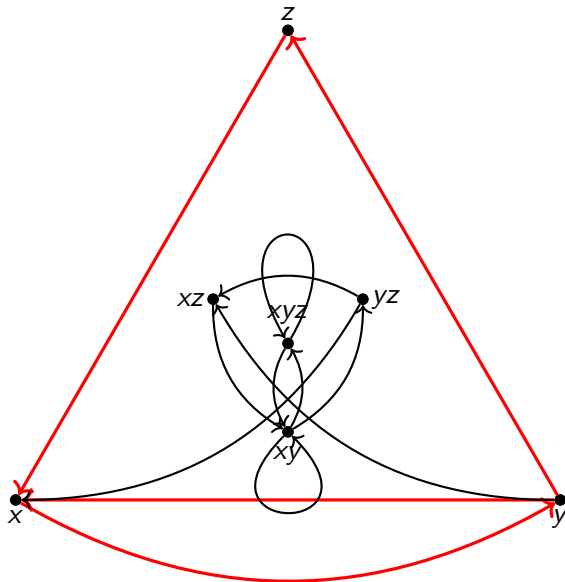
Let $\mathbb{D} = (D, R_1^{\mathbb{D}}, R_2^{\mathbb{D}}, \dots)$ be a relational structure and $\mathcal{V}$ a variety. Then $F_{\mathcal{V}}(\mathbb{D})$, the **free relational structure generated by $\mathbb{D}$ in $\mathcal{V}$** is defined by

- having the same signature as $\mathbb{D}$,
- having underlying set $F_{\mathcal{V}}(D)$, the algebra freely generated by the set D in $\mathcal{V}$,
- for each relation $R_i^{\mathbb{D}}$ of $\mathbb{D}$, the corresponding relation of the free structure being

$$R_i^{F_{\mathcal{V}}(\mathbb{D})} := Sg_{F_{\mathcal{V}}(D)}^{k_i} \langle R_i^{\mathbb{D}} \rangle,$$

k_i being the arity of R_i .

Example for a free digraph in the variety of semilattices



Retraction and interpretability

Definition

Let $\mathbb{D}$ and $\mathbb{D}'$ be relational structures of the same arity. If $f : \mathbb{D} \rightarrow \mathbb{D}'$ and $g : \mathbb{D}' \rightarrow \mathbb{D}$ are homomorphisms such that $f \circ g = id_{\mathbb{D}'}$, then f is called a **retraction** and g a **coretraction**.

Proposition

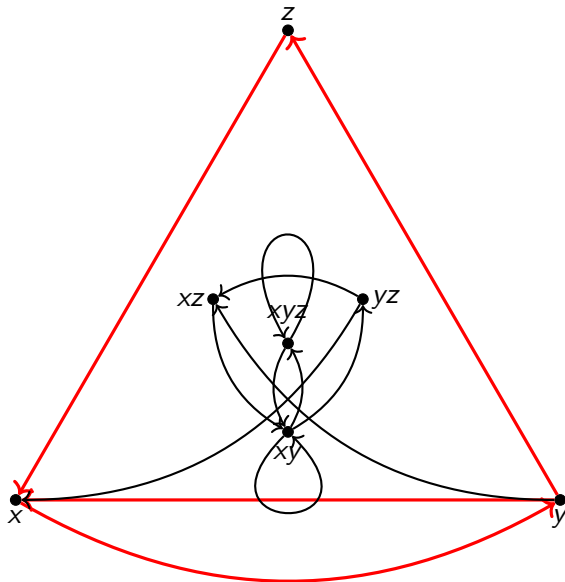
For a linear variety $\mathcal{V}$ and relation structure $\mathbb{D}$, TFAE

- ① *the inclusion mapping $\mathbb{D} \rightarrow F_{\mathcal{V}}(\mathbb{D})$ is a coretraction,*
- ② *there is a clone homomorphism from $\text{Term } \mathcal{V}$ to the polymorphism clone of $\mathbb{D}$*

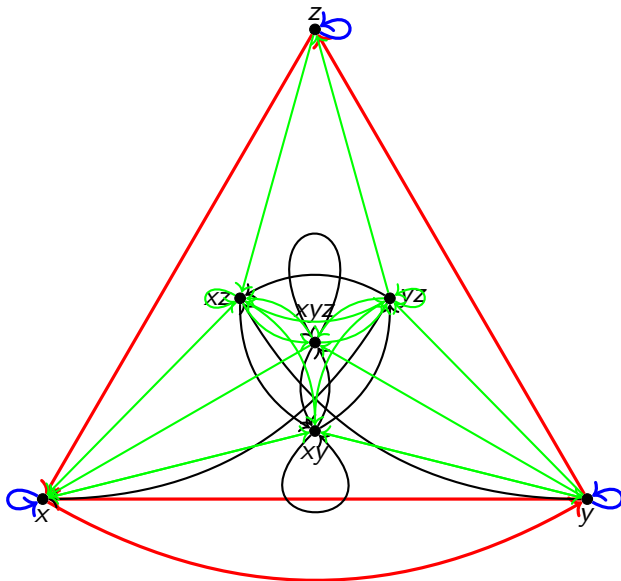
Problem

Is this true without linearity, at least if we add idempotency to $\mathcal{V}$ and reflexivity to $\mathbb{D}$?

No retraction from the free digraph to the generator



Retraction existing from the free digraph to the generator



Existence of retraction, Galois connection no. 1 (varieties and relational structures)

Consider the correspondence between the class of relational structures and the class of varieties given by the inclusion mapping $\mathbb{D} \rightarrow F_{\mathcal{V}}(\mathbb{D})$ being a coretraction, and its induced Galois connection.

Theorem (GyMZ)

The class of non-Taylor, and the class of non-Hobby–McKenzie varieties are closed under this Galois connection (both corresponding to a 3-element digraph).

Conjecture

On the varietal side, all closed classes are complements of Mal'tsev classes.

Closures of reflexive boolean relation clones

Each boolean reflexive relation clone $\mathbb{D}$ is generated by a single relation R . The inclusion mapping is a coretraction from $\mathbb{D}$ to $F_{\mathcal{V}}(\mathbb{D})$

- ① for $R = \{000, 011, 100, 110, 111\}$ (clone of constants): if $\mathcal{V}$ is not Taylor,
- ② for $R = \{000, 001, 011, 100, 110, 111\}$ (clone of unary functions): if $\mathcal{V}$ is not Taylor,
- ③ for $R = \{000, 010, 100, 111\}$ (clone generated by the constants and conjunction): if $\mathcal{V}$ is not Hobby-McKenzie,
- ④ for $R = \{000, 011, 101, 111\}$ (clone generated by the constants and disjunction): if $\mathcal{V}$ is not Hobby-McKenzie,
- ⑤ for $R = \{00, 01, 11\}$ (clone of monotone operations): if $\mathcal{V}$ is not Hagemann-Mitschke,
- ⑥ for $R = \{(a, b, c, d) \mid a + b + c + d = 0\}$ (clone of affine operations): if ??? (this is an unfamiliar Mal'tsev condition related to $\wedge$ -semidistributivity),
- ⑦ for $R = \{00, 11\}$ (clone of all operations): if $\mathcal{V}$ is nontrivial.

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Functionality, Galois connection no. 2 (varieties and relations)

Definition

An n -ary relation R is **functional** if it is the graph of an operation i.e. if for any $a_1, \dots, a_{n-1}$ there is a unique a_n such that $(a_1, \dots, a_n) \in R$.

Now for each functional R and each variety, we can ask whether $F_{\mathcal{V}}(R)$ is functional, obtaining another Galois connection.

Proposition (GyMZ)

The closed sets on the relational side are all collections of clones. That is, for each variety $\mathcal{V}$ and each underlying set A there is an operational clone with underlying set A so that for any operation f in A and its graph R_f , $F_{\mathcal{V}}(R_f)$ is functional iff f is in the clone.

Functionality, Galois connection no. 2 (varieties and relations)

Problem

Describe the clones that appear in closed sets. (Not all do, for example, they have to contain all unary operations.)

Conjecture

There are closed sets on the varietal side which are not complements of Mal'tsev classes.

Problem

Is it true that if $\mathcal{V}$ is Taylor then there is always a functional R with $F_{\mathcal{V}}(R)$ being nonfunctional?

Functional factor, Galois connection no. 3 (varieties, reflexive relational structures with pair of constants)

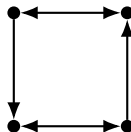
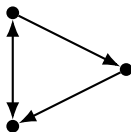
Proposition (GyMZ)

If $\mathbb{D}$ is a reflexive relational structure on A such that all its relations are functional, $a_1, a_2 \in A$ and $\mathcal{V}$ an idempotent variety, there is a smallest congruence θ of $F_{\mathcal{V}}(A)$ such that in $F_{\mathcal{V}}(\mathbb{D})/\theta$, all relations are functional and compatible with $F_{\mathcal{V}}(A)/\theta$.

Our third Galois connection asks whether θ collapses a_1 and a_2 .

Proposition (GyMZ)

The closed classes on the varietal side are Mal'tsev classes.



Definition

Let $\mathbb{D}$ and $\mathbb{E}$ be relational structures of the same signature.

- We say that two elements a and b of D are **$\mathbb{E}$ -connected** if there is a path $a = c_0, c_1, \dots, c_k = b$ such that for each i , there is a homomorphism from $\mathbb{E}$ to $\mathbb{D}$ with range containing both c_i and c_{i+1} . We denote this by $\overset{\mathbb{E}}{\longleftrightarrow}$. We call $\mathbb{D}$ itself $\mathbb{E}$ -connected if this relation is full.
- Consider the sequence of relational structures $\mathbb{D}, \mathbb{D}/\overset{\mathbb{E}}{\longleftrightarrow}, (\mathbb{D}/\overset{\mathbb{E}}{\longleftrightarrow})/\overset{\mathbb{E}}{\longleftrightarrow}, \dots$. If there is an $\mathbb{E}$ -connected structure (equivalently, a one-element one) among these, we say that $\mathbb{D}$ is $\mathbb{E}^*$ -connected.

Galois connections 4 and 5 (idempotent varieties, pair of reflexive relational structures)

Proposition

For an idempotent variety $\mathcal{V}$ and a reflexive relation structure $\mathbb{D}$, $F_{\mathcal{V}}(\mathbb{D})$ is $\mathbb{D}$ -connected.

We can ask whether, given an idempotent variety $\mathcal{V}$ and reflexive relational structures $\mathbb{D}$ and $\mathbb{E}$, $F_{\mathcal{V}}(\mathbb{D})$ is $\mathbb{E}$ -connected, and whether it is $\mathbb{E}^*$ -connected.

Theorem (GyMZ)

The following classes are closed under one of these Galois connections: all varieties, Taylor, Hobby-McKenzie, Hagemann-Mitschke.

Conjecture

There are no other closed classes.