

# Representations of distributive residuated lattice expansions (I)

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# The team



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# Outline of Talk 1

- Relation algebras (RAs)
- Representable RAs & group representable RAs
- Residuated lattice expansions
- Representable RL expansions
- Representations via pregroups

# Operations on binary relations

Let  $X$  be a set and  $R, S \subseteq X^2$ .

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Also: consider  $\wp(X^2)$  and  $R, S, T \subseteq X^2$ . Then

$$R ; S \subseteq T \iff S \subseteq (R^\smile ; T^c)^c \iff R \subseteq (T^c ; S^\smile)^c$$

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and

- 1  $x^{\sqcup\sqcup} = x$
- 2  $(xy)^{\sqcup} = y^{\sqcup}x^{\sqcup}$
- 3  $x(y \vee z) = xy \vee xz$
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Let  $X$  be a set and  $\text{id}_X = \{(x, x) \mid x \in X\}$ . Then

$$\mathcal{A}(X) = \langle \wp(X^2), \cap, \cup, ^c, \emptyset, X^2, \cdot, \text{id}_X, ^\smile \rangle$$

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# Representable relation algebras

Let  $X$  be a set and  $\text{id}_X = \{(x, x) \mid x \in X\}$ . Then

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Relation algebra  $\mathbf{A}$  is **representable** if  $\mathbf{A} \in \text{ISP}(\{\mathcal{A}(X) \mid X \text{ a set}\})$

89*t.* Bjarni Jónsson and Alfred Tarski: *Representation problems for relation algebras.*

A *relation algebra* (RA)  $A$  is a Boolean algebra with a binary associative operator  $|$  and a unary operator  $\sim$  such that:  $|$  has the unit element  $e$ ;  $a \sim \sim a$ ;  $(a|b) \sim = b \sim | a \sim$ ;  $a | \overline{a|b} \leq \overline{b}$ . (See Tarski, *Journal of Symbolic Logic* vol. 6, and Abstract 88.) Examples: I. *Proper relation algebras* (PRA's)—families  $A'$  of subrelations of a binary relation  $U$  closed under set addition  $R+S$ , complementation  $U-R$ , relative multiplication  $R|S$ , conversion  $R \sim$ ; the set of all  $(a, a)$  with  $a \in U$  is in  $A'$ . II (by McKinsey). *Frobenius algebras* (FA's)—families  $A''$  of subsets of a group  $G$  closed under set addition and complementation, complex multiplication and inversion; the smallest subgroup of  $G$  is in  $A''$ . Results: 1.  $A$  is extendible to a complete atomistic RA. 2.  $A$  is isomorphic with an algebra where  $R+S, R|S, R \sim$  (but not complementation) have the meaning in I. 3. An atomistic  $A$  where  $a \sim | a \leq e$  (or  $a \sim | a = e$ ) for every atom  $a$  is isomorphic with a PRA (or FA). 4 (by McKinsey-Tarski).  $A$  is simple if and only if  $a \neq 0 \rightarrow 1 | a | 1 = 1$ . 5. A PRA which is simple is isomorphic with a PRA  $A'$  where  $U$  is a Cartesian set-square; and conversely. **Problems: Are all RA's isomorphic with PRA's? Are all RA's with  $a | b = 0 \rightarrow (a = 0 \vee b = 0)$  isomorphic with FA's?** (Received October 21, 1947.)

# Group representable RAs

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## Theorem (McKinsey 1940s)

*If  $\mathbf{G}$  is a group then  $\mathcal{R}(\mathbf{G}) = \langle \wp(G), \cap, \cup, ^c, \emptyset, G, \bullet, \{e\}, ^{-1} \rangle$  is a relation algebra.*

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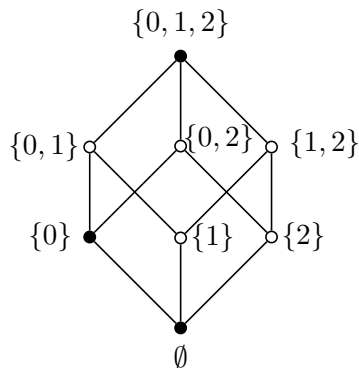
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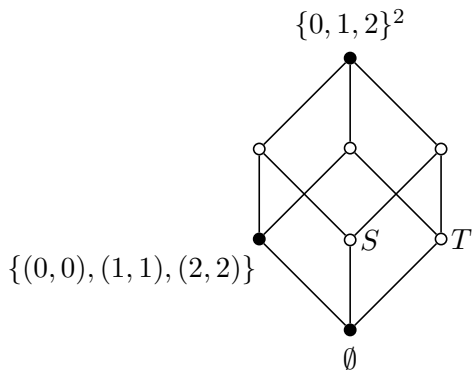
**Proof.**

The map  $\sigma : \wp(G) \rightarrow \wp(G \times G)$  defined by  $\sigma(Y) = \{ (g, g \cdot y) \mid g \in G, y \in Y \}$  is an embedding of  $\mathcal{R}(\mathbf{G})$  into  $\mathcal{A}(G)$ . □

# Example: group representable relation algebra



$\mathcal{R}(\mathbb{Z}_3)$



$$S = \{(0, 1), (1, 2), (2, 0)\}$$

$$T = \{(0, 2), (1, 0), (2, 1)\}$$

$\sigma(\mathcal{R}(\mathbb{Z}_3))$

# Generalisations of relation algebras

$\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$  is a **residuated lattice** (RL) if  $\langle A, \wedge, \vee \rangle$  is a lattice and  $\langle A, \cdot, 1 \rangle$  is a monoid such that:

$$a \cdot b \leq c \iff b \leq a \backslash c \iff a \leq c / b.$$

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$$\sim a = a \backslash 0 \quad \text{and} \quad -a = 0 / a.$$

If  $\sim -a = a = -\sim a$  for all  $a \in A$ , then  $\langle A, \wedge, \vee, \cdot, 1, 0, \backslash, / \rangle$  is an **InFL-algebra**.

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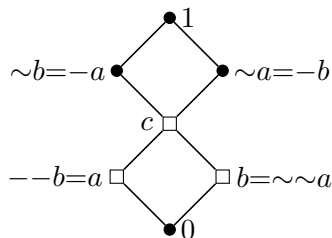
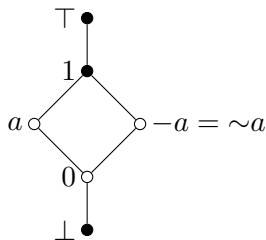
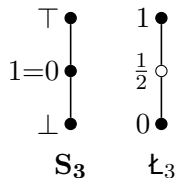
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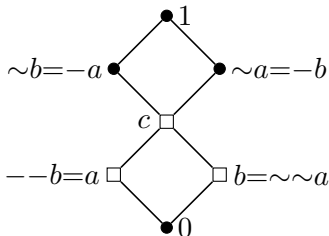
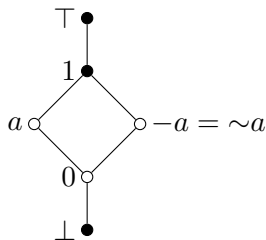
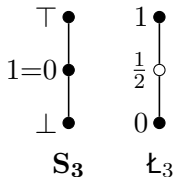
If  $\sim -a = a = -\sim a$  for all  $a \in A$ , then  $\langle A, \wedge, \vee, \cdot, 1, 0, \backslash, / \rangle$  is an **InFL-algebra**. An InFL-algebra is **cyclic** if  $\sim a = -a$  for all  $a \in A$ . A **distributive** InFL-algebra (**DInFL-algebra**) has a distributive lattice reduct.

NB: can axiomatize InFL-algebras with  $\langle A, \wedge, \vee, \cdot, 1, \sim, - \rangle$ .

# Examples of DInFL-algebras



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## Example

For  $\mathbf{A} = \langle A, \wedge, \vee, ', \perp, \top, \cdot, 1, \sqcup \rangle$  an RA, let  $\sim a = -a = (a')^\sqcup$ . Then  $\langle A, \wedge, \vee, \cdot, 1, \sim, - \rangle$  is a DInFL-algebra.

# Quasi relation algebras

## Definition (Jipsen & Galatos 2013)

A **quasi relation algebra** (qRA) is an InFL-algebra with a unary operation  $\neg$  satisfying:

$$\neg(a \vee b) = \neg a \wedge \neg b$$

$$\neg\neg a = a$$

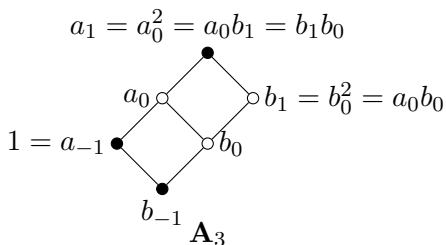
$$\neg(a \cdot b) = \sim(-\neg b \cdot -\neg a)$$

A qRA with a distributive lattice reduct will be called a DqRA.

# Examples of DqRAs

Let  $n \in \omega$  with  $n \geq 3$ . If  $n = 2k + 1$  for  $k \geq 1$ , set

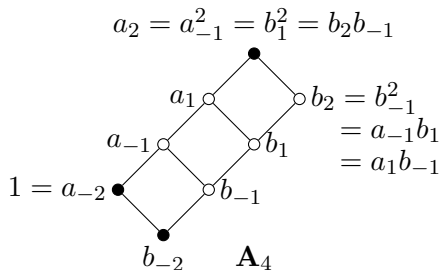
$$A_n = \{a_{-k}, \dots, a_{-1}, a_0, a_1, \dots, a_k, b_{-k}, \dots, b_{-1}, b_0, b_1, \dots, b_k\}.$$



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## Decidability of quasi relation algebras

### Theorem (Galatos, J. 2013)

If  $\mathcal{V} = \{\mathbf{A} \in \mathbf{InFL} \mid \mathbf{A} \models \mathcal{E}\}$  for  $\mathcal{E}$  a self-dual set of identities and  $\mathcal{V}$  is equationally decidable then  $\mathcal{V}' = \{\mathbf{A} \in \mathbf{qRA} \mid \mathbf{A} \models \mathcal{E}\}$  is also decidable.

Using results of [Holland, McCleary 1979], [Yetter 1990], [Wille 2005], [Kozak 2011], [Galatos, J. (Res. Frames) 2013]:

### Corollary

The equational theories of **qRA**, **cyclic qRA**, **cyclic distributive qRA**, **commutative qRA** and the variety of qRAs that have  $\ell$ -group reducts ( $= \mathbf{A} \in \mathbf{qRA} \mid \mathbf{A} \models x \cdot \sim x = 1$ ) are decidable.

### Theorem (Galatos, J. 2013)

**qRA**, **cyclic qRA** and **commutative qRA** have FMP.

**Problem 2:** Do (cyclic) **distributive qRA** have the FMP?

**Problem 3:** Define and investigate **representable qRA**.



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Define and study representations of DqRAs as algebras of binary relations.

# Representable DInFL-algebras

$\langle X, \leq \rangle$  a poset. Define  $\preceq$  on  $X^2$ :  $(x, y) \preceq (u, v)$  iff  $u \leq x$  &  $y \leq v$

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**Theorem (C., Robinson 2025)**

*Let  $\mathbf{X} = \langle X, \leq \rangle$  be a poset and  $\alpha : X \rightarrow X$  an order automorphism of  $\mathbf{X}$ . Set  $1 = \leq$  and for  $R \in \text{Up}(X^2, \preceq)$ , define  $\sim R = R^{c\smile}$ ;  $\alpha$  and  $-R = \alpha ; R^{c\smile}$ . Then*

$$\langle \text{Up}(X^2, \preceq), \cap, \cup, ;, 1, -, \sim \rangle$$

*is a DInFL-algebra and it is cyclic iff  $\alpha$  is the identity.*

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**Definition (C., Robinson 2025)**

A DInFL-algebra  $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, -, \sim, \rangle$  will be called **representable** if  $\mathbf{A} \in \text{ISP}(\text{FDInFL})$ .

# Distributive residuated lattices from pomonoids

## Definition

A partially ordered algebra  $\mathbf{P} = \langle P, \leq, \cdot, 1 \rangle$  is a **pomonoid** if

- $(P, \leq)$  is a poset
- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- $a \cdot 1 = a = 1 \cdot a$
- if  $a \leq b$  then  $a \cdot c \leq b \cdot c$  and  $c \cdot a \leq c \cdot b$

# Distributive residuated lattices from pomonoids

## Definition

A partially ordered algebra  $\mathbf{P} = \langle P, \leq, \cdot, 1 \rangle$  is a **pomonoid** if

- $(P, \leq)$  is a poset
- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- $a \cdot 1 = a = 1 \cdot a$
- if  $a \leq b$  then  $a \cdot c \leq b \cdot c$  and  $c \cdot a \leq c \cdot b$

For a pomonoid  $\mathbf{P}$  and  $U, V \in \text{Up}(P, \leq)$  define

$$U \bullet V = \uparrow \{ a \cdot b \mid a \in U, b \in V \}.$$

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## Theorem (cf. Galatos 2003)

Let  $\mathbf{P}$  be a pomonoid. Then

$$\mathcal{U}(\mathbf{P}) = \langle \text{Up}(P, \leq), \cap, \cup, \bullet, \uparrow \{1\}, \setminus, / \rangle$$

is a distributive residuated lattice.

# Representing DRLs with binary relations

Consider  $\sigma : \mathcal{U}(\mathbf{P}) \rightarrow \langle \text{Up}(P^\partial \times P), \cap, \cup, ;, \leq, \backslash, / \rangle$  defined by

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## Definition (GJKO 2007, GJL 2023)

An *involutive partially ordered monoid* (or *ipo-monoid*) is a structure  $\mathbf{P} = \langle P, \leq, \cdot, 1, ^-, \sim \rangle$  s.t.  $\langle P, \leq \rangle$  is a poset,  $\langle P, \cdot, 1 \rangle$  is a monoid, and for all  $x, y \in P$ :

$$x \leq y \quad \text{iff} \quad x \cdot y^\sim \leq 1^- \quad \text{iff} \quad y^- \cdot x \leq 1^-.$$

# DInFL-algebras from ipo-monoids

Let  $\mathbf{P} = \langle P, \leq, \cdot, 1, ^-, \sim \rangle$  be an ipo-monoid and  $U \in \text{Up}(P, \leq)$ .

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## Theorem

*Let  $\mathbf{P} = \langle P, \leq, \cdot, 1, -, \sim \rangle$  be an ipo-monoid. Then*

$$\mathcal{D}(\mathbf{P}) = \langle \text{Up}(P, \leq), \cap, \cup, \bullet, \uparrow 1, -, \sim \rangle$$

*is a DInFL-algebra. Moreover,  $\mathcal{D}(\mathbf{P})$  is cyclic iff  $\mathbf{P}$  is cyclic.*

## Another nice property of ipo-monoids

Recall  $\sigma : \mathcal{U}(\mathbf{P}) \rightarrow \langle \mathbf{Up}(P^\partial \times P), \cap, \cup, ;, \leq, \backslash, / \rangle$  defined by

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If  $\mathbf{P} = \langle P, \leq, \cdot, 1, -, \sim \rangle$  is an ipo-monoid, then it satisfies the condition above.

Proof:  $w = x \backslash y$  (since ipo-monoids are residuated).

## Theorem

*Let  $\mathbf{P} = \langle P, \leq, \cdot, 1, -, \sim \rangle$  be an ipo-monoid. Then the algebra of binary relations  $\langle \text{Up}(P^2, \preceq), \cap, \cup, ;, \leq, -, \sim \rangle$  is a DInFL-algebra.*

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Define  $\alpha : P \rightarrow P$  by  $\alpha(x) = x^{\sim\sim}$ . Then  $\alpha$  is an order isomorphism. □

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NB: we can't yet show that  $\sigma : \text{Up}(P, \leq) \rightarrow \text{Up}(P^2, \preceq)$  is a DInFL-algebra embedding.

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**Theorem (C., Robinson 2026)**

*Let  $\mathbf{A}$  be a DInFL-algebra. If there exists a pregroup  $\mathbf{P}$  such that  $\mathbf{A} \approx \mathcal{D}(\mathbf{P})$ , then  $\mathbf{A}$  is representable.*

### Theorem (C., Robinson 2025)

Let  $\mathbf{X} = \langle X, \leq \rangle$  be a poset and  $\alpha : X \rightarrow X$  an order automorphism of  $\mathbf{X}$ . Set  $1 = \leq$  and for  $R \in \text{Up}(X^2, \leq)$ , define  $\sim R = R^{c\sim} ; \alpha$  and  $-R = \alpha ; R^{c\sim}$ . Further, if  $\beta : X \rightarrow X$  is a self-inverse dual order automorphism of  $\mathbf{X}$  such that  $\beta = \alpha ; \beta ; \alpha$ , then defining  $\neg R = \alpha ; \beta ; R^c ; \beta$  we get that

$$\langle \text{Up}(X^2, \leq), \cap, \cup, ;, 1, -, \sim, \neg \rangle$$

is a distributive quasi relation algebra.

Algebras of the form described above are **full** DqRAs. Classes denoted by FDqRA.

### Definition (C., Robinson 2025)

A DqRA  $\mathbf{B} = \langle B, \wedge, \vee, \cdot, 1, -, \sim, \neg \rangle$  is *representable* if  $\mathbf{B} \in \text{ISP}(\text{FDqRA})$ .

# Ortho pregroups and DqRAs

## Definition (Jipsen & Galatos 2013)

A **quasi relation algebra** (qRA) is an InFL-algebra with a unary operation  $\neg$  satisfying:

$$\neg(a \vee b) = \neg a \wedge \neg b$$

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## Example

If  $\mathbf{G} = \langle G, \cdot, {}^{-1}, e \rangle$  is a group and  $\text{id} : G \rightarrow G$  the identity function, then  $\langle G, =, \cdot, e, {}^{-1}, {}^{-1}, \text{id} \rangle$  is an ortho pregroup.

# More examples of ortho pregroups

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Now let  $f^\neg(x) = -f(-x)$ . This gives us an ortho pregroup.

# Representing DqRAs with binary relations

Theorem (C., Robinson 2026)

Let  $\mathbf{P} = \langle P, \leq, \cdot, 1, \ell, r, \neg \rangle$  be an ortho pregroup. Define  $\neg U = \{ a^\neg \mid a \notin U \}$  for  $U \in \text{Up}(P, \leq)$ . Then

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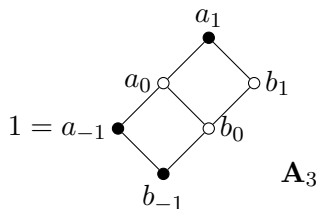
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## Proof.

Define  $\beta : P \rightarrow P$  by  $\beta(x) = x^\neg$  and use the embedding  $\sigma$ . □

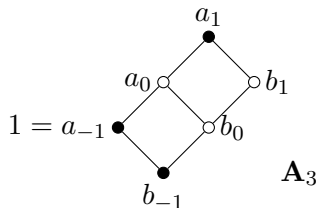
# Examples from products of groups



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*Let  $n \geq 3$ . Then the DqRA  $\mathbf{A}_n$  is representable.*

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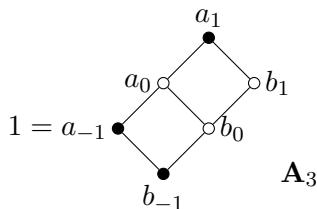
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## Proof.

For  $n = 3$ , consider the orthopregroup  $\mathbf{Z}_7 = \langle \mathbb{Z}_7, =, +, 0, -, -, - \rangle$ . We get  $\psi : \mathbf{A}_3 \hookrightarrow \mathcal{Q}(\mathbf{Z}_7)$ .

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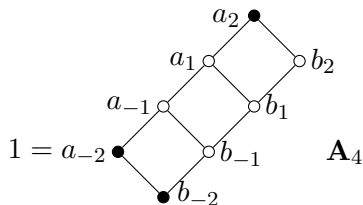
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$\psi(b_{-1}) = \emptyset$ ,  $\psi(a_{-1}) = \{0\}$ ,  $\psi(b_0) = \{1, 2, 4\}$ ,  $\psi(b_1) = \mathbb{Z}_7 \setminus \{0\}$   $\square$

## Examples from products of groups cont...

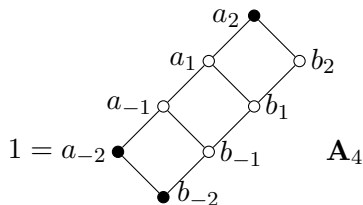


Proof.

Sketch: For  $n \geq 4$ , consider the ortho pregroup

$$\mathbf{Z}_{n-2} \times \mathbf{Z}_7 = \langle \mathbb{Z}_{n-2} \times \mathbb{Z}_7, =, +, (0, 0), -, -, - \rangle.$$

# Examples from products of groups cont...



Proof.

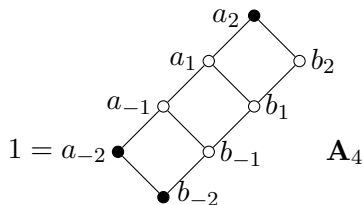
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$$T = \{(0, 1), (0, 2), (0, 4)\} \cup \{(m, \ell) \mid 1 \leq m \leq n-3, \ell \in \{3, 5, 6\}\}.$$

Use  $T$  to define the images of the  $b_i$  under an embedding  $\psi : A_n \rightarrow \text{Up}(\mathbb{Z}_{n-2} \times \mathbb{Z}_7, =)$ .

# Examples from products of groups cont...



Proof.

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$\psi : A_n \rightarrow \text{Up}(\mathbb{Z}_{n-2} \times \mathbb{Z}_7, =)$ . Then, let

$$\psi(a_i) = \psi(b_i) \cup \{(0,0)\}.$$



# Open problems/future work

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- The condition required for  $\sigma$  to preserve meets is also satisfied by lattice-ordered monoids. What do we know about the residuated lattice  $\langle \text{Up}(P, \leq), \cap, \cup, \bullet, \uparrow\{1\}, \backslash, / \rangle$  when  $\mathbf{P}$  is an  $\ell$ -monoid?

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