

Representations of distributive residuated lattice expansions (II)

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The team



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Outline

- DqRAs and representability (slight reframing from yesterday)
- Finite and bounded representability
- Nested sums of DqRAs
- Frames
- Games for frames
- Consequences for representability

Residuated lattice expansions

$\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \backslash, / \rangle$ is a **residuated lattice** (RL) if $\langle A, \wedge, \vee \rangle$ is a lattice and $\langle A, \cdot, 1 \rangle$ is a monoid such that:

$$a \cdot b \leq c \iff b \leq a \backslash c \iff a \leq c / b.$$

Definition (cf. Galatos et al. 2007)

Let $\mathbf{A}$ be a residuated lattice and $0 \in A$. Then $\langle A, \wedge, \vee, \cdot, 1, 0, \backslash, / \rangle$ is an **FL-algebra**. Define

$$\sim a = a \backslash 0 \quad \text{and} \quad -a = 0 / a.$$

If $\sim -a = a = -\sim a$ for all $a \in A$, then $\langle A, \wedge, \vee, \cdot, 1, 0, \backslash, / \rangle$ is an **InFL-algebra**.

NB: can axiomatize InFL-algebras with $\langle A, \wedge, \vee, \cdot, 1, \sim, - \rangle$.

Quasi relation algebras

Definition (Jipsen & Galatos 2013)

A **quasi relation algebra** (qRA) is an InFL-algebra with a unary operation $\neg$ satisfying:

$$\neg(a \vee b) = \neg a \wedge \neg b$$

$$\neg\neg a = a$$

$$\neg(a \cdot b) = \sim(-\neg b \cdot -\neg a)$$

A qRA with a distributive lattice reduct will be called a DqRA.

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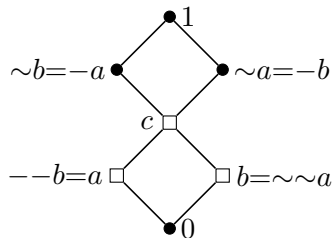
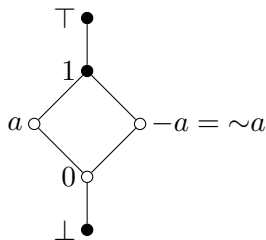
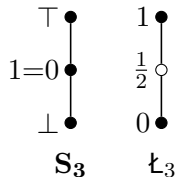
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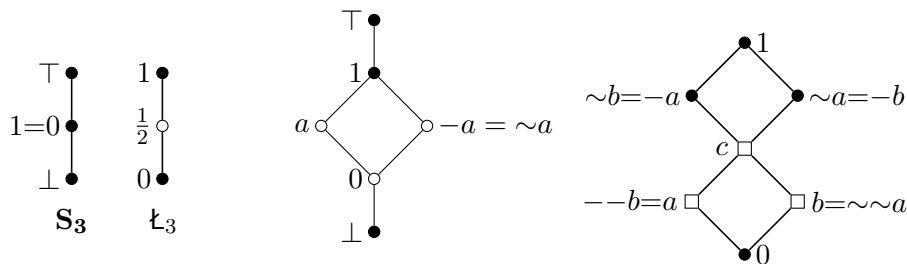
A qRA with a distributive lattice reduct will be called a DqRA.

Goal: find representations of DqRAs as algebras of binary relations.

Examples of DqRAs



Examples of DqRAs



All algebras up to size eight:

<https://github.com/jipsen/>

Distributive-quasi-relation-algebras-and-DInFL/blob/main/DInFL1.pdf

Representable DqRAs

Let $(X, \leq)$ be a poset and $\leq \subseteq E$ an equivalence relation on X .
For $(x, y), (w, z) \in E$ define:

$$(x, y) \preceq (w, z) \iff w \leq x \text{ \& } y \leq z$$

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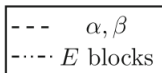
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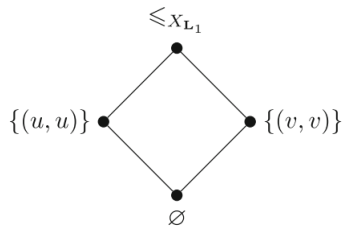
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$\mathbf{X}_{L_1}$



Theorem (C., Robinson 2025)

Let $\mathbf{X} = \langle X, \leq \rangle$ be a poset and E an equivalence relation on X with $\leq \subseteq E$. Let $\alpha : X \rightarrow X$ be an order automorphism of $\mathbf{X}$ where $\alpha \subseteq E$. For $R \in \text{Up}(E, \leq)$, define $\sim R = R^{c\smile}$; α and $\neg R = \alpha ; R^{c\smile}$. Further let $\beta : X \rightarrow X$ be a self-inverse dual order automorphism such that $\beta \subseteq E$, $\beta = \alpha ; \beta ; \alpha$, then defining $\neg R = \alpha ; \beta ; R^c ; \beta$ we get that

$$\langle \text{Up}(E, \leq), \cap, \cup, ;, 1, -, \sim, \neg \rangle$$

is a distributive quasi relation algebra.

Algebras of the form above are called **equivalence DqRAs**.

If $E = X^2$ it is a **full DqRA**.

Classes denoted EDqRA and FDqRA.

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A DqRA $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \sim, -, \neg \rangle$ is **representable** if $\mathbf{A} \in \text{ISP}(\text{FDqRA})$ or, equivalently, $\mathbf{A} \in \text{IS}(\text{EDqRA})$.

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Definition

A DqRA $\mathbf{A}$ has a **bounded representation** if it has a greatest element, $\top$, and there exists a structure $\langle X, \leq, E, \alpha, \beta \rangle$ and an embedding $\varphi : A \hookrightarrow \text{Up}(E, \leq)$ such that $\varphi(\top) = E$.

Finite representability can fail

Theorem

Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \sim, -, \neg \rangle$ be a DqRA. If there exists $a \in A$ such that $0 < a < 1$ and $a \cdot a \leq 0$, then $\mathbf{A}$ is not finitely representable.

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Proof.

Suppose $\mathbf{A}$ is representable over $(X, \leq)$ and E with $|\alpha| = n \in \mathbb{Z}^+$. Then there is an embedding $h : A \rightarrow \text{Up}(E, \preceq)$ such that $h(0) = \alpha$; $(\leq^c)^\sim$ and $h(1) = \leq$.

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Theorem

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Let $\mathbf{K}$ and $\mathbf{L}$ be DqRAs. If $1_{\mathbf{K}}$ satisfies certain conditions, then we can create a new DqRA by replacing $1_{\mathbf{K}}$ by $\mathbf{L}$. We denote this new algebra by $\mathbf{K}[\mathbf{L}]$.

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$\mathbf{S}_3$ satisfies the conditions required by the above theorem.

Therefore:

Corollary

Let $\mathbf{A}$ be a DqRA. Then $\mathbf{S}_3[\mathbf{A}]$ is a DqRA. Moreover, its lattice reduct is order-isomorphic to $\{\perp\} \oplus \mathbf{A} \oplus \{\top\}$.

Nested sums and representability

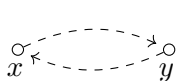
Theorem (C.,Morton,Robinson 2026)

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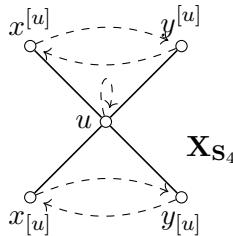
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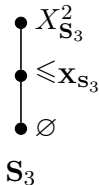
$\mathbf{X}_{\mathbf{S}_3}$



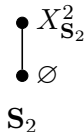
$\mathbf{X}_{\mathbf{S}_2}$



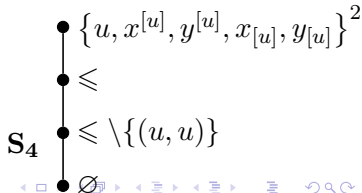
$\mathbf{X}_{\mathbf{S}_4}$



$\mathbf{S}_3$



$\mathbf{S}_2$



$\mathbf{S}_4$

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Proposition

Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \sim, -, \neg \rangle$ be a DqRA. If $S_3[\mathbf{A}]$ is representable, then $\mathbf{A}$ is representable.

Proof.

$\mathbf{A} \hookrightarrow S_3[\mathbf{A}]$



Frames for DqRAs

Definition

A **DqRA frame** is a tuple $\mathbb{W} = (W, I, \preccurlyeq, \circ, \sim, -, \neg)$ with $I \subseteq W$, a partial order $\preccurlyeq$ on W , a binary set-operation $\circ : W \times W \rightarrow \wp(W)$ and unary functions $\sim$, $-$ and $\neg$ s.t. the following hold:

- i $x \preccurlyeq y \iff y \in I \circ x \iff y \in x \circ I$
- ii $x \preccurlyeq y$ and $x \in I \implies y \in I$
- iii $x \preccurlyeq y$ and $x \in u \circ v \implies y \in u \circ v$
- iv $(x \circ y) \circ z = x \circ (y \circ z)$
- v $z^\sim \in x \circ y \iff y^- \in z \circ x$
- vi $x^{\sim-} \preccurlyeq x$ and $x^{-\sim} \preccurlyeq x$
- vii $x^{\neg\neg} = x$.
- viii $x \preccurlyeq y \implies y^\neg \preccurlyeq x^\neg$.
- ix $z^- \in x \circ y \iff z^\neg \in y^{\sim\neg} \circ x^{\sim\neg}$.

DqRAs and DqRA frames

For a DqRA $\mathbf{A}$, denote by $\mathbf{A}_+$ its dual frame, whose elements are the join-irreducibles of $\mathbf{A}$. The ordering on $\mathbf{A}_+$ is $\preceq = \supseteq$. Also,

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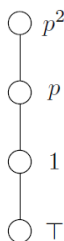
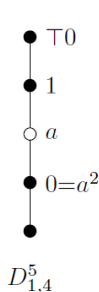
The map $\kappa : J(\mathbf{A}) \rightarrow M(\mathbf{A})$, $\kappa(j) = \bigvee \{a \in A \mid j \not\leq a\}$, is used to define the unary operation on $\mathbf{A}_+$:

$$a^\sim = \sim \kappa(a), a^- = -\kappa(a), a^\neg = \neg \kappa(a)$$

Theorem (C., Jipsen, Robinson 2026+)

There is a one-to-one correspondence between finite DqRAs and finite DqRA frames. Moreover $(\mathbf{A}_+)^+ \cong \mathbf{A}$ and $(\mathbb{W}^+)_+ \cong \mathbb{W}$.

The algebra $D_{1,4}^5$ is isomorphic to $S_3[\mathfrak{L}_3]$.



$\circ$	p^2	p	1	$\top$
p^2	$\{p^2\}$	$\{p^2\}$	$\{p^2\}$	W
p	$\{p^2\}$	$\{p^2\}$	$\{p, p^2\}$	W
1	$\{p^2\}$	$\{p, p^2\}$	$\{1, p, p^2\}$	W
$\top$	W	W	W	W

$\circ_{\max}$	p^2	p	1	$\top$
p^2	$\{p^2\}$	$\emptyset$	$\emptyset$	$W \setminus \{p^2\}$
p	$\emptyset$	$\emptyset$	$\{p\}$	$\emptyset$
1	$\emptyset$	$\{p\}$	$\{1\}$	$\emptyset$
$\top$	$W \setminus \{p^2\}$	$\emptyset$	$\emptyset$	$\emptyset$



Summary of game theoretic approach

Let $\mathbf{A} = \langle A, \wedge, \vee, \cdot, 1, \sim, -, \neg \rangle$ be a finite DqRA.

- Define a game $\Gamma(\mathbf{A})$ to be played on the algebra $\mathbf{A}$ between $\forall$ (Abelard) and $\exists$ (Heloïse). $\forall$ is the challenger, and $\exists$ is the builder.
- The games are played on **networks**.
- Prove that $\mathbf{A}$ is representable iff $\exists$ has a winning strategy in the game $\Gamma(\mathbf{A})$.
- Proof: build $\langle X, \leq, E, \alpha, \beta \rangle$ from a network and vice versa.
- Consider the dual frame $\mathbf{A}_+$ of $\mathbf{A}$ and define a game $\mathcal{G}(\mathbf{A}_+)$ to be played on $\mathbf{A}_+$.
- Prove that $\exists$ has a winning strategy in $\mathcal{G}(\mathbf{A}_+)$ if and only if $\exists$ has a winning strategy in $\Gamma(\mathbf{A})$.
- For $\mathbf{A}$, play $\mathcal{G}(\mathbf{A}_+)$ to find representations.

Network for an algebra

Definition

Let $\mathbf{A} = \langle A, \vee, \wedge, \cdot, 1, \sim, -, \neg \rangle$ be a DqRA. A **network** for $\mathbf{A}$ is a tuple $\mathcal{N} = \langle N, \lambda, \alpha, \beta \rangle$ where N is the set of **nodes**, $\lambda : N^2 \rightarrow \mathcal{P}(A)$ is a **labelling function** such that for all $x \in N$ we have $1 \in \lambda(x, x)$, and $\alpha, \beta \subseteq N^2$.

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$\mathcal{N} = \langle N, \lambda, \alpha, \beta \rangle$ is a **prenetwork** of $\mathcal{N}' = \langle N', \lambda', \alpha', \beta' \rangle$ (or $\mathcal{N}'$ is an **extension** of $\mathcal{N}$), denoted $\mathcal{N} \subseteq \mathcal{N}'$, if $N \subseteq N'$, for all $x, y \in N$ we have $\lambda(x, y) \subseteq \lambda'(x, y)$, $\alpha \subseteq \alpha'$, and $\beta \subseteq \beta'$.

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For $S \subseteq A$, define $-S = \{-a \mid a \in S\}$ and $\neg S = \{\neg a \mid a \in S\}$.

A network $\mathcal{N}$ is **consistent** if the following conditions hold for all $x, y, z, v, w \in N$:

- 1 If $(x, z) \in \alpha$ then $\lambda(x, y) \cap -\lambda(y, z) = \emptyset$.
- 2 If $(x, z) \in \alpha, (z, v), (w, y) \in \beta$ then $\lambda(x, y) \cap \neg\lambda(v, w) = \emptyset$.

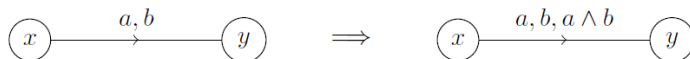
Game on the algebra

Initialisation move: $\forall$ picks $a, b \in A$ such that $a \not\leq b$ and $\exists$ must return a network $\mathcal{N}_0 = \langle N_0, \lambda_0, \alpha_0, \beta_0 \rangle$ such that there exist some $x, y, z \in N_0$ such that $a \in \lambda_0(x, y)$, $\sim b \in \lambda_0(y, z)$ and $(x, z) \in \alpha_0$.

Game on the algebra

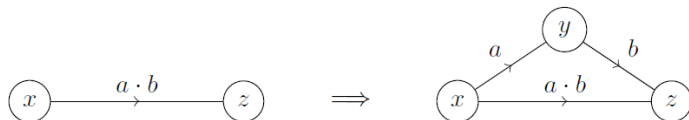
Initialisation move: $\forall$ picks $a, b \in A$ such that $a \not\leq b$ and $\exists$ must return a network $\mathcal{N}_0 = \langle N_0, \lambda_0, \alpha_0, \beta_0 \rangle$ such that there exist some $x, y, z \in N_0$ such that $a \in \lambda_0(x, y)$, $\sim b \in \lambda_0(y, z)$ and $(x, z) \in \alpha_0$.

Meet: $\forall$ picks some $x, y \in N_i$ with some $a, b \in \lambda_i(x, y)$. $\exists$ must return a network $\mathcal{N}_{i+1} = \langle N_{i+1}, \lambda_{i+1}, \alpha_{i+1}, \beta_{i+1} \rangle \supseteq \mathcal{N}_i$ with $a \wedge b \in \lambda_{i+1}(x, y)$.

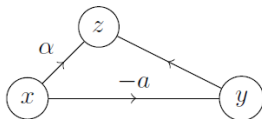


Some more rules of the game

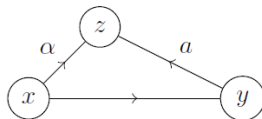
Witness: $\forall$ picks some $x, z \in N_i$ with some $a \cdot b \in \lambda_i(x, z)$. $\exists$ must return a network $\mathcal{N}_{i+1} = \langle N_{i+1}, \lambda_{i+1}, \alpha_{i+1}, \beta_{i+1} \rangle \supseteq \mathcal{N}_i$ where there exists a $y \in N_{i+1}$ with $a \in \lambda_{i+1}(x, y)$ and $b \in \lambda_{i+1}(y, z)$.



Involution-Choice: $\forall$ picks some $a \in A$ as well as some $x, y, z \in N_i$ with $(x, z) \in \alpha_i$. $\exists$ must return a network $\mathcal{N}_{i+1} = \langle N_{i+1}, \lambda_{i+1}, \alpha_{i+1}, \beta_{i+1} \rangle \supseteq \mathcal{N}_i$ with either $a \in \lambda_{i+1}(y, z)$ or $-a \in \lambda_{i+1}(x, y)$.



or



Winning strategy

The game $\Gamma_n(\mathbf{A})$ is played by Abelard ($\forall$) and Heloïse ($\exists$) over rounds $0, 1, \dots, n$ for some $0 < n < \omega$.

At the i th round, $\forall$ challenges $\exists$ to produce a network $\mathcal{N}_i$ such that $\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \dots \subseteq \mathcal{N}_n$. If at any point during the game $\exists$ introduces an inconsistent network, then $\forall$ wins. If the networks remain consistent throughout, then $\exists$ wins.

The infinite version of the game, denoted $\Gamma(\mathbf{A})$, has the same gameplay, however, it only ends if $\exists$ introduces an inconsistent network and $\forall$ wins. If the game can continue indefinitely, then $\exists$ is said to have a winning strategy.

Dominant network

A network $\mathcal{N} = \langle N, \lambda, \alpha, \beta \rangle$ is **dominant** if

- i If $a \in \lambda(x, y)$ and $a \leq b$ then $b \in \lambda(x, y)$.
- ii If $a, b \in \lambda(x, y)$ then $a \wedge b \in \lambda(x, y)$.
- iii If $a \vee b \in \lambda(x, y)$ then $a \in \lambda(x, y)$ or $b \in \lambda(x, y)$.
- iv If $a \in \lambda(x, y)$ and $b \in \lambda(y, z)$ then $a \cdot b \in \lambda(x, z)$.
- v If $a \cdot b \in \lambda(x, z)$, then there exists $y \in N$ such that $a \in \lambda(x, y)$ and $b \in \lambda(y, z)$.
- vi If $(x, z) \in \alpha$ then there exists $y \in N$ s.t. $a \in \lambda(y, z)$ or $\neg a \in \lambda(x, y)$.
- vii If $(x, z) \in \alpha$, $(z, v) \in \beta$ and $(w, y) \in \beta$, then $\neg a \in \lambda(x, y)$ or $a \in \lambda(v, w)$.
- viii For all $x, y, z \in N$ and $\gamma \in \{\alpha, \beta\}$ with $(x, y) \in \gamma$ and $(x, z) \in \gamma$, we have $1 \in \lambda(y, z)$ and $1 \in \lambda(z, y)$.
- ix For all $x \in N$ and $\gamma \in \{\alpha, \beta\}$, there exist $y, z \in N$ such that $(y, x) \in \gamma$ and $(x, z) \in \gamma$.

Dominant network continued

- ✳ For all $x, y, z, w \in N$ such that $(x, z) \in \alpha$, $(z, w) \in \beta$ and $(w, y) \in \alpha$, we have $(x, y) \in \beta$.
- ✳_i For all $x, y \in N$ such that $(x, y) \in \beta$, there exist $z, w \in N$ such that $(x, z) \in \alpha$, $(z, w) \in \beta$ and $(w, y) \in \alpha$.
- ✳_{ii} For all $x, y, z, w \in N$ with $(x, z) \in \alpha$, $(y, w) \in \alpha$ and $1 \in \lambda(x, y)$, we have $1 \in \lambda(z, w)$.
- ✳_{iii} If $(x, z) \in \beta$, $(y, w) \in \beta$ and $1 \in \lambda(x, y)$, then $1 \in \lambda(w, z)$.
- ✳_{iv} If $(x, y) \in \beta$ then $(y, x) \in \beta$.

Winning strategy iff representable

Theorem

$\exists$ has winning strategy in the game $\Gamma(\mathbf{A})$ if and only if $\mathbf{A}$ has a bounded representation.

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- Show that if there exists a representation that from it one can build a (dominant) network that can be played as the first move (and all subsequent moves).

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Proof sketch.

- Show that if there exists a representation that from it one can build a (dominant) network that can be played as the first move (and all subsequent moves).
- Given a winning strategy, we quotient the network by:

$$x \equiv y \quad \text{iff} \quad 1 \in \lambda(x, y) \text{ and } 1 \in \lambda(y, x).$$

Order equivalence classes by: $[x] \leq [y] \quad \text{iff} \quad 1 \in \lambda(x, y).$



Network for a frame

Definition

A *frame network* for a frame $\mathbb{W} = \langle W, I, \preceq, \circ, -, \sim, \neg \rangle$ is a tuple $\mathcal{N} = \langle N, \lambda, \alpha, \beta \rangle$, where N is a set of nodes, $\lambda : N^2 \rightarrow W$ is a **labelling function**, and $\alpha, \beta : N \rightarrow N$ are partial functions. $\mathcal{N}$ is *consistent*, if for all $x, y, z \in N$, we have

- i $\beta(\beta(x)) = x$ if $\beta(x)$ is defined,
- ii $\beta(x) = \alpha(\beta(\alpha(x)))$ if $\alpha(\beta(\alpha(x)))$ is defined,
- iii $\lambda(x, x) \in I$,
- iv $\lambda(x, y) \in I \ \& \ \lambda(y, x) \in I \Rightarrow x = y$,
- v $\lambda(x, y) \in I \Leftrightarrow \lambda(\alpha(x), \alpha(y)) \in I$ if $\alpha(x), \alpha(y)$ are defined,
- vi $\lambda(x, y) \in I \Leftrightarrow \lambda(\beta(y), \beta(x)) \in I$ if $\beta(x), \beta(y)$ are defined,
- vii $\lambda(x, y) = \lambda(y, \alpha(x))^-$ if $\alpha(x)$ is defined,
- viii $\lambda(x, y) = \lambda(\beta(\alpha(x)), \beta(y))^\neg$ if $\beta(\alpha(x)), \beta(y)$ are defined,
- ix $\lambda(x, z) \in \lambda(x, y) \circ \lambda(y, z)$.

The game $\mathcal{G}(\mathbb{W})$

Let $\mathcal{N} = \langle N, \lambda, \alpha, \beta \rangle$ and $\mathcal{N}' = \langle N', \lambda', \alpha', \beta' \rangle$ be two frame networks for $\mathbb{W}$. Again, we define the notion of a prenetwork and an extension by saying $\mathcal{N} \subseteq \mathcal{N}'$ if, and only if,

- ❶ $N \subseteq N'$,
- ❷ $\lambda = \lambda' \upharpoonright_{N^2}$,
- ❸ $\alpha(x) = \alpha'(x)$ for all $x \in N$ if $\alpha(x)$ is defined, and
- ❹ $\beta(x) = \beta'(x)$ for all $x \in N$ if $\beta(x)$ is defined.

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Define the game $\mathcal{G}(\mathbb{W})$: $\forall$ challenges $\exists$ to build networks $\mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \dots$ over any number of moves by demanding witnesses.

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If at any point $\forall$ is unable to demand a witness (i.e., the network is *saturated*), $\exists$ wins. If $\exists$ is unable to return a network, $\forall$ wins. If the game continues indefinitely, $\exists$ is the winner.

The game $\mathcal{G}(\mathbb{W})$

Definition

Let $\mathbb{W} = \langle W, \preccurlyeq, \circ, -, \sim, \neg \rangle$ be a DqRA frame. Define $\circ^{\max} : W \times W \rightarrow \wp(W)$ by saying that $p \in q \circ^{\max} r$ if, and only if,

- i $p \in q \circ r$,
- ii for all $q', r' \in W$, if $q \preccurlyeq q'$, $r \preccurlyeq r'$ and $p \in q' \circ r'$, then $q' = q$ and $r' = r$.

Initialisation move: At the initial (zeroth) move, $\forall$ picks a $p \in W$ and $\exists$ must return a network $\mathcal{N}_0 = \langle N_0, \lambda_0, \alpha_0, \beta_0 \rangle$ with some $x, y \in N_0$ such that $\lambda_0(x, y) = p$.

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Note: since $\forall$ can pick any $p \in W$, for $\exists$ to have a winning strategy, she must have a winning strategy for all $p \in W$.

The game $\mathcal{G}(\mathbb{W})$

Composition Witness: $\forall$ picks $x, y \in N_{i-1}$ and some $p, q \in W$ such that $\lambda_{i-1}(x, y) \in p \circ^{\max} q$ and there is no $z \in N_{i-1}$ such that $\lambda_{i-1}(x, z) = p, \lambda_{i-1}(z, y) = q$. $\exists$ returns a network $\mathcal{N}_i = \langle N_i, \lambda_i, \alpha_i, \beta_i \rangle$ with $N_i = N_{i+1} \uplus \{z\}$, and $\lambda_i(x, z) = p, \lambda_i(z, y) = q$.

Incoming Morphism Witness: $\forall$ picks $\gamma \in \{\alpha, \beta\}$ and an $y \in N_{i-1}$ such that there is no $x \in N_{i-1}$ with $\gamma_{i-1}(x) = y$. $\exists$ returns a network $\mathcal{N}_i = \langle N_i, \lambda_i, \alpha_i, \beta_i \rangle$ with $N_i = N_{i+1} \uplus \{x\}$ such that $\gamma_i(x) = y$.

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Connection between $\Gamma(\mathbf{A})$ and $\mathcal{G}(\mathbf{A}_+)$

Theorem

$\exists$ has a winning strategy in $\mathcal{G}(\mathbf{A}_+)$ if and only if $\exists$ has a winning strategy in $\Gamma(\mathbf{A})$.

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Proof sketch.

- Assume $\exists$ has a winning strategy in $\mathcal{G}(\mathbf{A}_+)$. Then for $a \not\leq b$ we can build a consistent dominant network $\mathcal{N}_{a,b}$ (and hence $\exists$ has a winning strategy in $\Gamma(\mathbf{A})$).

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- If $\exists$ has a winning strategy in $\Gamma(\mathbf{A})$ then we can build a consistent saturated network for $\mathbf{A}_+$.



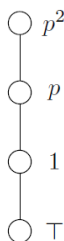
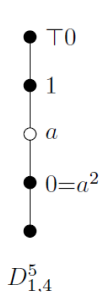
Summary of connection between $\Gamma(\mathbf{A})$ and $\mathcal{G}(\mathbf{A}_+)$

Corollary

The following are equivalent for a finite DqRA $\mathbf{A}$.

- i $\exists$ has a winning strategy for $\Gamma(\mathbf{A})$.*
- ii $\exists$ has a winning strategy for $\mathcal{G}(\mathbf{A}_+)$.*
- iii There exists a bounded representation of $\mathbf{A}$.*

The algebra $D_{1,4}^5$ is isomorphic to $S_3[\mathfrak{L}_3]$.



$\circ$	p^2	p	1	$\top$
p^2	$\{p^2\}$	$\{p^2\}$	$\{p^2\}$	W
p	$\{p^2\}$	$\{p^2\}$	$\{p, p^2\}$	W
1	$\{p^2\}$	$\{p, p^2\}$	$\{1, p, p^2\}$	W
$\top$	W	W	W	W

$\circ_{\max}$	p^2	p	1	$\top$
p^2	$\{p^2\}$	$\emptyset$	$\emptyset$	$W \setminus \{p^2\}$
p	$\emptyset$	$\emptyset$	$\{p\}$	$\emptyset$
1	$\emptyset$	$\{p\}$	$\{1\}$	$\emptyset$
$\top$	$W \setminus \{p^2\}$	$\emptyset$	$\emptyset$	$\emptyset$

A consistent saturated network for $D_{1,4}^5$

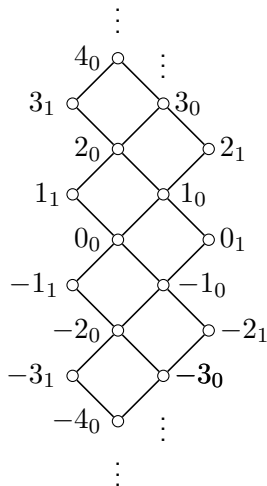
Define a labelled graph with morphisms $\mathcal{N} = \langle \mathbb{Z}_0 \uplus \mathbb{Z}_1, \lambda, \alpha, \beta \rangle$ where for $i, j \in \{0, 1\}$ and $x, y \in \mathbb{Z}$ we have:

$$\lambda(x_i, y_j) = \begin{cases} \top & \text{if } x > y \text{ and } i = j, \\ \top & \text{if } x \geq y \text{ and } i \neq j, \\ \top & \text{if } x + 1 = y \text{ and } i = j = 1, \\ 1 & \text{if } x = y \text{ and } i = j = 0, \\ p & \text{if } x + 1 = y \text{ and } i = j = 0, \\ p^2 & \text{if } x + 1 < y \text{ and } i = j, \\ p^2 & \text{if } x < y \text{ and } i \neq j, \\ p^2 & \text{if } x = y \text{ and } i = j = 1, \end{cases}$$

$$\alpha(x_i) = (x + 1)_i,$$

$$\beta(x_i) = (-x)_i.$$

The poset used to represent $D_{1,4}^5$



$$\alpha(x_i) = (x+1)_i \quad \beta(x_i) = (-x)_i$$

Theorem

The representability of finite DqRAs is undecidable.

Further results

Theorem

The representability of finite DqRAs is undecidable.

Proof.

Take a DqRA that is term-equivalent to a relation algebra. Show that $\exists$ has a winning strategy in our game iff $\exists$ has a winning strategy in the relation algebra game. It is known that the representability problem for finite relation algebras is undecidable. □

See Chapter 18 of Hirsch & Hodkinson (2002) for why the representability of a finite relation algebra can be translated into an undecidable tiling problem.

Ongoing work and open problems

- Can we extend the representability of $\mathcal{L}_3$ to any finite MV-chain?

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- Can we extend the representability of $\mathfrak{L}_3$ to any finite MV-chain? Likely “yes”, but work still to be done to confirm.
- Is the class of representable DqRAs (**RDqRA**) closed under homomorphic images?
 - Some initial results from our Masters student Sarah Opperman: if **A** is finitely representable then any homomorphic image of **A** is representable.
 - If **RDqRA** closed under $\mathbb{H}$, then **RDqRA** forms a variety.
 - If finite MV-chains all representable and **RDqRA** closed under $\mathbb{H}$, then all MV-algebras representable in this way.
 - Maybe: **RDqRA** is not closed under $\mathbb{H}$, but MV-chains are?

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