

Operators $\text{Max } L$ and $\text{Min } U$ and duals of Boolean posets

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When working with lattices, we use the lattice operations $\wedge$ and $\vee$, which for couples of entries x, y yield their infimum and supremum, respectively. When working with posets, the results of these operations are not defined in some cases. Instead of these operations, in posets one usually uses the operators of the lower cone $L(x, y)$ and the upper cone $U(x, y)$, respectively, see e.g. [3], [5] and [6]. Similarly as in lattices, in posets we have $z \leq x, y$ for all $z \in L(x, y)$ and $x, y \leq w$ for all $w \in U(x, y)$, respectively. However, in the case when $\inf(x, y)$ or $\sup(x, y)$ exists, it may happen that $\inf(x, y) \neq L(x, y)$ and $\sup(x, y) \neq U(x, y)$. This is the reason why we prefer to use the operators $\text{Max } L(x, y)$ and $\text{Min } U(x, y)$ denoting the set of all maximal elements of the lower cone of x and y and the set of all minimal elements in the upper cone of x and y , respectively.

Then we have that in case $\inf(x, y)$ exists, it coincides with $\text{Max } L(x, y)$, and in case $\sup(x, y)$ exists, it coincides with $\text{Min } U(x, y)$, respectively. Hence, the operators $\text{Max } L(x, y)$ and $\text{Min } U(x, y)$ are closer to lattice operations than the operators $L(x, y)$ and $U(x, y)$ and this advantage is used in some constructions (cf. e.g. [2]) where congruences in posets are defined by using these operators. We axiomatize the operators $\text{Max } L$ and $\text{Min } U$ in bounded posets. Obviously, this axiomatization differs much from that of lattice operations since the operators in question do not share such nice properties as e.g. associativity.

When working with Boolean algebras, one often uses the term operation symmetric difference. In the second part of the paper we introduce the operator of symmetric difference and the Sheffer operator also for complemented posets. It is well-known that in Boolean algebras the Sheffer operation turns out to be very useful since by means of this single operation all the operations of the Boolean algebra can be generated. We axiomatize the Sheffer operator on complemented posets and we show how this Sheffer operator can induce the partial order relation, the complementation operation as well as the afore mentioned operators $\text{Max } L$ and $\text{Min } U$.

It is well-known that there is a natural one-to-one correspondence between Boolean algebras and unitary Boolean rings, see e.g. [1]. This correspondence is realized by using the operation of symmetric difference which serves as the addition operation of the unitary Boolean ring. In the last part of the paper we establish a similar construction for Boolean posets by using the operator of symmetric difference.

First we recall some more or less familiar concepts and then we formulate several results needed for our study.

Let $P = (P, \leq)$ be a poset, $a, b \in P$ and $A, B \subseteq P$. We say that P satisfies the *Ascending Chain Condition* (ACC, for short) or the *Descending Chain Condition* (DCC, for short) if P has no infinite ascending or descending chains, respectively. Let $\text{Max } A$ and $\text{Min } A$ denote the set of all maximal and minimal elements of A , respectively. If P satisfies the ACC or the DCC then for every non-empty subset A of P the set $\text{Max } A$ or $\text{Min } A$ is non-empty, respectively. We define

$$A \leq B \text{ if } x \leq y \text{ for all } x \in A \text{ and all } y \in B,$$
$$L(A) := \{x \in P \mid x \leq A\},$$
$$U(A) := \{x \in P \mid A \leq x\}.$$

Hence, by the binary operator $\text{Max } L(x, y)$ and $\text{Min } U(x, y)$ we mean the mapping assigning to each $(x, y) \in P^2$ the set of all maximal elements in the lower cone $L(x, y)$ of x and y and the set of all minimal elements in the upper cone $U(x, y)$ of x and y , respectively. Of course, if the poset in question is a lattice then $\text{Max } L(x, y) = x \wedge y$ and $\text{Min } U(x, y) = x \vee y$. Thus also in the more general case of a poset, $\text{Max } L(x, y)$ and $\text{Min } U(x, y)$ should play the role of $x \wedge y$ and $x \vee y$, respectively.

The poset P is called *bounded* if it has a bottom element 0 and a top element 1 . In this case we sometimes write $(P, \leq, 0, 1)$.

It is evident that for any subset A of a bounded poset $(P, \leq, 0, 1)$ the sets $L(A)$, $U(A)$, $\text{Max } L(A)$ and $\text{Min } U(A)$ are non-empty since $0 \in L(A)$ and $1 \in U(A)$.

Two elements a and b of a bounded poset $(P, \leq, 0, 1)$ are called *complements* of each other if $L(a, b) = 0$ and $U(a, b) = 1$.

A unary operation $'$ on a poset $(P, \leq)$ is called

- *antitone* if $x, y \in P$ and $x \leq y$ together imply $y' \leq x'$,
- an *involution* if it satisfies the identity $(x')' \approx x$.

By a *binary operator* φ on a set P we mean a mapping from P^2 to 2^P , i.e. to every couple of elements x, y of P there is assigned a subset $\varphi(x, y)$ of P . In the following such an operator is often extended from P^2 to $(2^P)^2$, i.e. to every couple of subsets A, B of P there is assigned a subset $\varphi(A, B)$ of P . In this case, elements of P are identified with the corresponding one-element subsets of P .

Axiomatization of the operators Max L and Min U

In this section we investigate the operators Max L and Min U on a poset. The aim of this section is to axiomatize these operators.

Definition 1

An *operator structure* is an ordered quintuple $(P, \sqcup, \sqcap, 0, 1)$ where $\sqcup$ and $\sqcap$ are binary operators of P and $0, 1 \in P$ and the following conditions are satisfied:

- i) $x \sqcup x \approx x$ and $x \sqcap x \approx x$,
- ii) $x \sqcup y \approx y \sqcup x$ and $x \sqcap y \approx y \sqcap x$,
- iii) $0 \sqcup x \approx x$ and $x \sqcap 1 \approx x$,
- iv) $x \sqcup ((x \sqcup y) \sqcup z) \approx 0 \sqcup ((x \sqcup y) \sqcup z)$ and $x \sqcap ((x \sqcap y) \sqcap z) \approx ((x \sqcap y) \sqcap z) \sqcap 1$,
- v) $(x \sqcap y) \sqcup y \approx y$ and $x \sqcap (x \sqcup y) \approx x$,
- vi) $z \in x \sqcup y$ if and only if $x \sqcup z = y \sqcup z = z$ and if $u \in P$ and $x \sqcup u = y \sqcup u = u \sqcap z = u$ together imply $u = z$; $z \in x \sqcap y$ if and only if $z \sqcap x = z \sqcap y = z$ and if $u \in P$ and $u \sqcap x = u \sqcap y = z \sqcup u = u$ together imply $u = z$.

In what follows, we show that these operator structures can be used for an axiomatization of the operators $\text{Max } L$ and $\text{Min } U$.

Theorem 2

Let $P = (P, \leq, 0, 1)$ be a bounded poset satisfying the ACC and the DCC. Then $\mathbb{A}(P) := (P, \text{Min } U, \text{Max } L, 0, 1)$ is an operator structure.

Also conversely, we show how to obtain a bounded poset with the operators $\text{Max } L$ and $\text{Min } U$ from a given operator structure.

Theorem 3

Let $A = (P, \sqcup, \sqcap, 0, 1)$ be an operator structure. Define a binary relation $\leq$ on P by $x \leq y$ if $x \sqcup y = y$ ($x, y \in P$). Then $\mathbb{P}(A) := (P, \leq, 0, 1)$ is a bounded poset satisfying the following conditions:

- a) $x \leq y$ if and only if $x \sqcap y = x$,*
- b) $x \sqcup y \approx \text{Min } U(x, y)$,*
- c) $x \sqcap y \approx \text{Max } L(x, y)$.*

That the correspondence described in Theorems 2 and 3 is nearly one-to-one is proved in the next theorem.

Theorem 4

- i) If $P = (P, \leq, 0, 1)$ is a bounded poset satisfying the ACC and the DCC then $\mathbb{P}(\mathbb{A}(P)) = P$.
- ii) If $A = (P, \sqcup, \sqcap, 0, 1)$ is a finite operator structure and $\mathbb{A}(\mathbb{P}(A)) = (P, \sqcup_1, \sqcap_1, 0, 1)$ then $x \sqcup_1 y = x \sqcup y$ and $x \sqcap_1 y = x \sqcap y$ for all $x, y \in P$.

Contrary to the case of the lattice operations $\wedge$ and $\vee$, the binary operators $\text{Max } L$ and $\text{Min } U$ need not be associative, i.e. they need not satisfy the identities

$$\begin{aligned}\text{Max } L(\text{Max } L(x, y), z) &\approx \text{Max } L(x, \text{Max } L(y, z)), \\ \text{Min } U(\text{Min } U(x, y), z) &\approx \text{Min } U(x, \text{Min } U(y, z)).\end{aligned}$$

Consider the poset depicted in Fig. 1

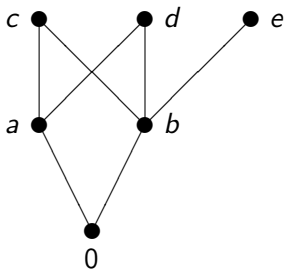


Figure 1. Poset

Then

$$\begin{aligned}
 \text{Max } L(\text{Max } L(c, d), e) &= \text{Max } L(a, b, e) = 0 \neq b = \text{Max } L(c, b) \\
 &= \text{Max } L(c, \text{Max } L(d, e)).
 \end{aligned}$$

We continue our investigations by considering another operator on a complemented poset, namely the so-called *symmetric difference* which we will introduce by means of the operators $\text{Max } L$ and $\text{Min } U$.

Let $(P, \leq, ', 0, 1)$ be a complemented poset. For every subset A of P put $A' := \{a' \mid a \in A\}$. In a complemented lattice $(L, \vee, \wedge, ', 0, 1)$ the so-called *symmetric difference* $+$ is usually defined as follows (see e.g. [1]):

$$x + y := (x' \wedge y) \vee (x \wedge y')$$

for all $x, y \in L$. A similar binary operator can be introduced in the complemented poset $(P, \leq, ', 0, 1)$ by

$$x + y := \text{Min } U(\text{Max } L(x', y), \text{Max } L(x, y'))$$

for all $x, y \in P$. We extend this definition from elements x, y of P to all subsets A, B of P by defining

$$A + B := \text{Min } U(\text{Max } L(A', B), \text{Max } L(A, B')).$$

Although the result of $x + y$ need not be an element of P , but may be a subset of P , the corresponding operator on complemented posets shares some familiarly known properties of the usual symmetric difference in lattices. In particular we have

$$\begin{array}{lll} x + x \approx 0, & x + y \approx y + x, & x + 0 \approx x, \\ x + 1 \approx x', & (x + 1) + 1 \approx x, & x + x' \approx 1, \\ x + y \approx x' + y', & x + y' \approx x' + y. & \end{array}$$

Recall that a poset $(P, \leq)$ is called *distributive* if it satisfies one of the following equivalent conditions (see [5] and [6]):

$$\begin{aligned}L(U(x, y), z) &\approx LU(L(x, z), L(y, z)), \\U(L(x, y), z) &\approx UL(U(x, z), U(y, z)), \\L(U(x, y), A) &\approx LU(L(x, A), L(y, A)), \\U(L(x, y), A) &\approx UL(U(x, A), U(y, A))\end{aligned}$$

where A runs through all subsets of P . A poset is called *Boolean* if it is distributive and complemented.

Examples of Boolean posets that are not lattices are as follows:

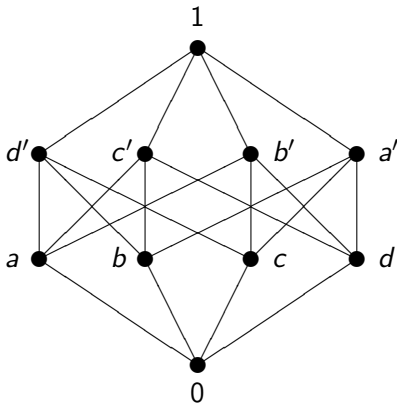


Figure 3. Boolean poset

The table for the symmetric difference is as follows:

+	0	a	b	c	d	a'	b'	c'	d'	1
0	0	a	b	c	d	a'	b'	c'	d'	1
a	a	0	$c'd'$	$b'd'$	$b'c'$	1	$a'b'$	$a'c'$	$a'd'$	a'
b	b	$c'd'$	0	$a'd'$	$a'c'$	$a'b'$	1	$b'c'$	$b'd'$	b'
c	c	$b'd'$	$a'd'$	0	$a'b'$	$a'c'$	$b'c'$	1	$c'd'$	c'
d	d	$b'c'$	$a'c'$	$a'b'$	0	$a'd'$	$b'd'$	$c'd'$	1	d'
a'	a'	1	$a'b'$	$a'c'$	$a'd'$	0	$c'd'$	$b'd'$	$b'c'$	a
b'	b'	$a'b'$	1	$b'c'$	$b'd'$	$c'd'$	0	$a'd'$	$a'c'$	b
c'	c'	$a'c'$	$b'c'$	1	$c'd'$	$b'd'$	$a'd'$	0	$a'b'$	c
d'	d'	$a'd'$	$b'd'$	$c'd'$	1	$b'c'$	$a'c'$	$a'b'$	0	d
1	1	a'	b'	c'	d'	a	b	c	d	0

Another example of a Boolean poset:

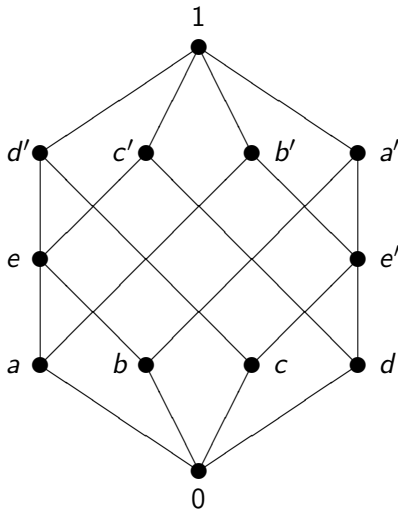


Figure 4. Boolean poset

As mentioned in the introduction, an important term operation in Boolean algebras is the so-called *Sheffer operation* (alias *Sheffer stroke*). It is usually denoted by $|$ and defined by $x|y := x' \wedge y'$. We extend this concept to complemented posets.

Definition 5

A *Sheffer structure* is an ordered pair $(P, |)$ where P is a non-empty set and $|$ is a binary operator on P satisfying the following conditions (we abbreviate $x|x$ by x' for $x \in P$ and for subsets A of P , $A|A$ by A'):

- i) $x|y \approx y|x$,
- ii) $(x|x)|(x|x) \approx x$,
- iii) $(x|y)|(x|x) \approx x$,
- iv) $(x'|(y'|z')')|z' \approx (x'|(y'|z')')|(x|x')$,
- v) $x|x' \in P$ and $x|x' \approx y|y'$,
- vi) $(x|x')'|x \approx x|x'$,
- vii) $z \in x|y$ if and only if $z'|x = z'|y = z$ and if $u'|x = u'|y = u$ and $z'|u' = z$ together imply $z = u$.

The natural correspondence between a complemented poset and a Sheffer structure is described by the next two theorems.

Theorem 6

Let $(P, \leq, ', 0, 1)$ be a complemented poset satisfying the ACC and the DCC and define a binary operator $|$ on P by $x|y := \text{Max } L(x', y')$ for all $x, y \in P$. Then $(P, |)$ is a Sheffer structure.

Now we show how a complemented poset together with the operators $\text{Max } L$ and $\text{Min } U$ can be recovered back from a Sheffer structure.

Theorem 7

Let $(P, |)$ be a Sheffer structure, put $x' := x|x$ for all $x \in P$, define a binary relation $\leq$ on P by $x \leq y$ if $x'|y' = x$ and put $0 := x|x'$ and $1 := 0'$ ($x, y \in P$). Then $P := (P, \leq, ', 0, 1)$ is a complemented poset, $x|y \approx \text{Max } L(x', y')$ and $(x|y)' \approx \text{Min } U(x, y)$.

It was established already by G. Boole and formally described by G. Birkhoff [1] that Boolean algebras and unitary Boolean rings are in a natural one-to-one correspondence, in other words, unitary Boolean rings are structures being dual to Boolean algebras. One may ask whether there exists also a structure being dual to a Boolean poset. We construct such a structure by using the operators mentioned in the previous sections and we show that one can get back a Boolean poset from its dual. We start with the following definition.

Definition 8

A *dual of a Boolean poset* is a structure $(P, +, \cdot, 0, 1)$ where $+$ and $\cdot$ are binary “operators” on P , i.e. they are defined also for subsets of P , and where $0, 1 \in P$ and the following conditions for all $x, y, z \in P$ and all $A, B \subseteq P$ are satisfied:

- i) $x \cdot x = x, x \cdot y = y \cdot x, x \cdot 0 = 0, x \cdot 1 = x,$
 $(x \cdot (y \cdot z)) \cdot z = (x \cdot (y \cdot z)) \cdot 1,$
- ii) $(x + 1) + 1 = x,$
- iii) $x \cdot y = x$ implies $(x + 1) \cdot (y + 1) = y + 1,$
- iv) $x \cdot (x + 1) = 0,$
- v) $((x + 1) \cdot (y + 1) + 1) \cdot z = ((x \cdot z + 1) \cdot (y \cdot z + 1) + 1) \cdot 1,$
- vi) $x \in A \cdot B$ if and only if $x \cdot y = x$ for all $y \in A \cup B$ and if $x \cdot z = x$ and $z \cdot y = z$ for all $y \in A \cup B$ together imply $z = x$.

How a dual of a Boolean poset can be derived from a given Boolean poset is illuminated by the next result.

Theorem 9

Let $P = (P, \leq, ', 0, 1)$ be a Boolean poset satisfying the ACC and the DCC and define binary operators $+$ and $\cdot$ on P by

$$\begin{aligned}a + b &:= \text{Min } U(\text{Max } L(a', b), \text{Max } L(a, b')), \\a \cdot b &:= \text{Max } L(a, b)\end{aligned}$$

for all $a, b \in P$ and

$$\begin{aligned}A + B &:= \text{Min } U(\text{Max } L(A', B), \text{Max } L(A, B')), \\A \cdot B &:= \text{Max } L(A, B)\end{aligned}$$

for all $A, B \subseteq P$. Then $\mathbb{D}(P) := (P, +, \cdot, 0, 1)$ is a dual of a Boolean poset.

For finite duals of Boolean posets we can prove the following.

Theorem 10

Let $D = (P, +, \cdot, 0, 1)$ be a finite dual of a Boolean poset. Define a binary relation $\leq$ on P by $x \leq y$ if $x \cdot y = x$, and a unary operation $'$ on P by $x' := x + 1$ ($x, y \in P$). Then $\mathbb{B}(D) := (P, \leq, ', 0, 1)$ is a Boolean poset.

The next theorem shows that the correspondence between Boolean posets and their duals is nearly one-to-one.

Theorem 11

- i If $P = (P, \leq, ', 0, 1)$ is a finite Boolean poset then $\mathbb{B}(\mathbb{D}(P)) = P$.
- ii If $D = (P, +, \cdot, 0, 1)$ is a finite dual of a Boolean poset and $\mathbb{D}(\mathbb{B}(D)) = (P, \oplus, \odot, 0, 1)$ then $(P, \odot, 0, 1) = (P, \cdot, 0, 1)$.

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The end!

Thanks for your attention!!