

A Colax Colimit Construction of Biresiduation Algebras

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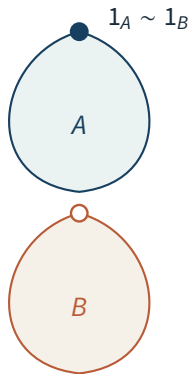
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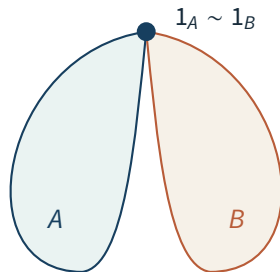
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Gluing is common and useful in algebra

Ordinal sum



Horizontal sum



Applications: BL-chains as ordinal sums of Wajsberg hoops; orthomodular lattices/ortholattices as horizontal sums...

Integral biresiduation algebras

An **integral biresiduation algebra** is

$$\mathbf{A} = (A, \leq, 1, \backslash, /),$$

where 1 is the greatest element and, for all $x, y, z \in A$,

$$y \leq x \backslash z \Leftrightarrow x \leq z / y,$$

$$x \backslash (y / z) = (x \backslash y) / z,$$

$$1 \backslash x = x = x / 1, \quad x \backslash 1 = 1 = 1 / x,$$

and $x \leq y$ implies

$$z \backslash x \leq z \backslash y, \quad x / z \leq y / z.$$

Antitonicity in the other argument follows from these laws.

Underlying order satisfies:

$$x \leq y \Leftrightarrow x \backslash y = 1 \Leftrightarrow y / x = 1.$$

Division orders

Division orders.

$$x \sqsubseteq_L y \iff y = a \backslash x,$$

$$x \sqsubseteq_R y \iff y = x / a,$$

$$x \sqsubseteq_J y \iff y = (a \backslash x) / b$$

for suitable $a, b \in A$.

The three relations are orders contained in the natural order $\leq$.

A biresiduation algebra is called **duo** if $\sqsubseteq_L = \sqsubseteq_R$. We then use simply $\sqsubseteq$.

Nuclei and upsets

A closure operator γ is a **nucleus** if

$$\begin{aligned}\gamma(x \setminus y) &\leq \gamma(x) \setminus \gamma(y), \\ \gamma(y/x) &\leq \gamma(y) / \gamma(x).\end{aligned}$$

In the duo case, the principal nucleus is

$$\gamma_a(x) = (a/x) \setminus a = a/(x \setminus a).$$

For the image of the principal nucleus we write

$$\mathbf{A}_a = \gamma_a[A]$$

Nuclear images are $\sqsubseteq$ -upsets

In the duo case:

Images of nuclei are upsets in $\sqsubseteq$.

Principal nuclear images are exactly principal upsets in $\sqsubseteq$, i.e., $\mathbf{A}_a = \{x \in A \mid a \sqsubseteq x\}$.

Tanaka and locally Tanaka

A duo biresiduation algebra is **Tanaka** if $\sqsubseteq = \leq$ or equivalently

$$x/(y \setminus x) = (y/x) \setminus y. \quad (T)$$

A duo biresiduation algebra is called **locally Tanaka** if

$$\gamma_z(x)/(\gamma_z(y) \setminus \gamma_z(x)) = (\gamma_z(y)/\gamma_z(x)) \setminus \gamma_z(y) \quad (LT)$$

Locally Tanaka yields join semilattice

If a locally Tanaka algebra satisfies

$$\gamma_x(\gamma_y(x)) = \gamma_y(\gamma_x(y)). \quad (J)$$

then $(A, \sqsubseteq)$ is a join semilattice with $x \sqcup y = \gamma_x(\gamma_y(x))$.

Divisible biresiduation algebras

A biresiduation algebra **A** is **divisible** if

$$\begin{aligned}(x \backslash y) \backslash (x \backslash z) &= (y \backslash x) \backslash (y \backslash z), \\ (z / x) / (y / x) &= (z / y) / (x / y).\end{aligned}\tag{H}$$

- ▶ Every duo divisible biresiduation algebra is locally Tanaka and satisfies (J).
- ▶ Every Tanaka is locally Tanaka.
- ▶ Every principal nuclear image of a locally Tanaka is **bounded prelinear Tanaka**
– (J) does not matter.

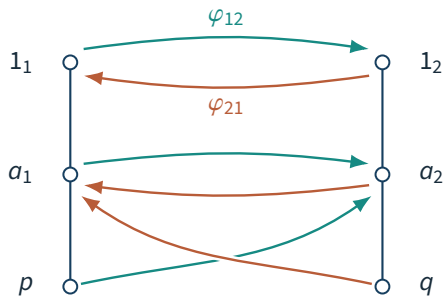
A Tanaka algebra is called **prelinear** if

$$(x \backslash y) \vee (y \backslash x) = 1, \quad (y / x) \vee (x / y) = 1.$$

Example: gluing two Tanaka chains

$$A_1 = [p, 1] \cong L_3$$

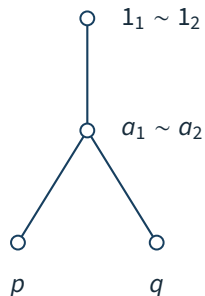
$$A_2 = [q, 1] \cong L_3$$



Glue



$\text{Glue}(\Phi)$



Categorical setting

Ambient category: order-enriched category **Bires**_≤ of integral biresiduation algebras and subhomomorphisms.

A **subhomomorphism** $f : A \rightarrow B$ is monotone, preserves 1, and satisfies

$$\begin{aligned}f(x \backslash y) &\leq f(x) \backslash f(y), \\f(y / x) &\leq f(y) / f(x).\end{aligned}$$

The hom-sets are ordered pointwise:

$$f \leq g \iff f(a) \leq g(a) \text{ for every } a \in A.$$

A complete colax diagram yields gluing

A **complete colax diagram** consists of algebras $(A_i)_{i \in I}$ and a subhomomorphism $\varphi_{ij} : A_i \rightarrow A_j$ for every i, j , such that

$$\varphi_{ii} = \text{id}_{A_i}, \quad \varphi_{ik} \leq \varphi_{jk} \varphi_{ij}.$$

We will write $\Phi = (\mathbf{A}_i, \varphi_{ij})$ for the whole diagram.

The gluing

Start with the disjoint union

$$G_0 = \bigsqcup_{i \in I} A_i = \{(a, i) : a \in A_i\}.$$

Define a preorder by

$$(a, i) \leq_{\Phi} (b, j) \iff \varphi_{ij}(a) \leq_j b,$$

and take the quotient by the symmetric kernel $\sim$ of this preorder:

$$\text{Glue}(\Phi) = \left(\bigsqcup_i A_i \right) / \sim_{\Phi}.$$

Gluing gives a biresiduation algebra

For $a \in A_i$ and $b \in A_j$, define

$$[a, i] \setminus [b, j] = [\varphi_{ij}(a) \setminus_j b, j],$$

$$[b, j] / [a, i] = [b /_j \varphi_{ij}(a), j],$$

$$1 = [1_i, i].$$

Construction theorem

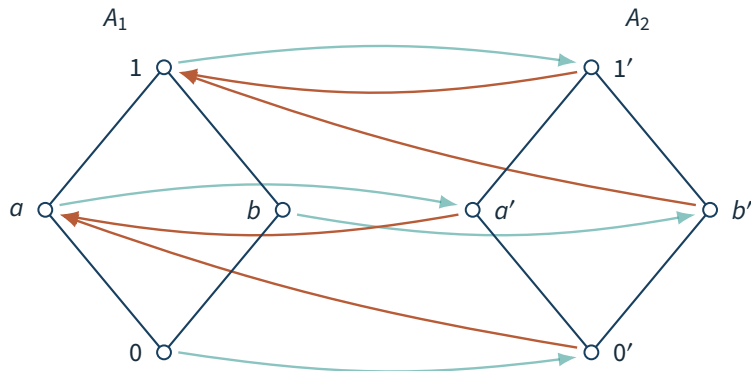
The operations are well defined and make

$$\text{Glue}(\Phi) = (G, \leq, 1, \setminus, /)$$

an integral biresiduation algebra.

Example: gluing two 4-element Tanaka

$$A_1 = \{0, a, b, 1\}, \quad A_2 = \{0', a', b', 1'\}.$$

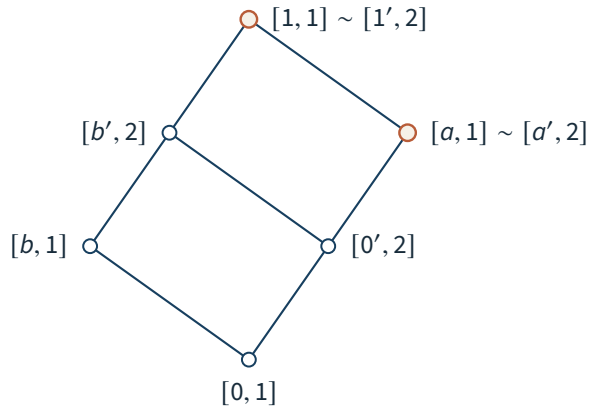


$$\varphi_{12}: x \mapsto x'.$$

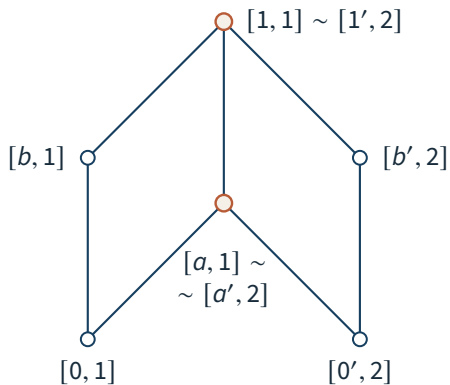
$$\varphi_{21}: \{0', a'\} \mapsto a, \\ \{b', 1'\} \mapsto 1.$$

Obtaining duo divisible

$\leq$



$\sqsubseteq$



The gluing is a colax colimit in $\mathbf{Bires}_{\leq}$

Let $\Phi = ((A_i)_{i \in I}, (\varphi_{ij})_{i,j \in I})$ be a complete colax diagram. A cocone to X is a family of subhomomorphisms $\psi_i : A_i \rightarrow X$ satisfying

$$\psi_i \leq \psi_j \circ \varphi_{ij} \quad (i, j \in I).$$

Definition of the colax colimit

A cocone $(C, (\iota_i)_{i \in I})$ is a **colax colimit** of Φ if:

1. for every cocone $(X, (\psi_i)_i)$ there exists a unique subhomomorphism $u : C \rightarrow X$ such that

$$u \circ \iota_i = \psi_i \quad (i \in I);$$

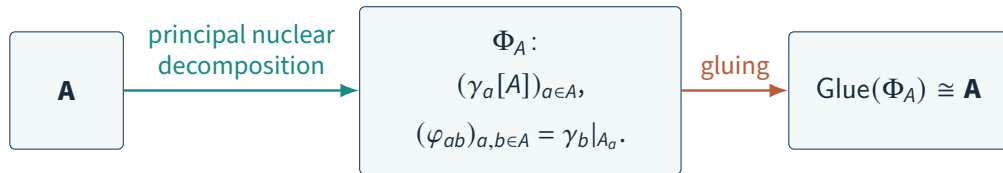
2. for all $u, v : C \rightarrow X$,

$$(u \circ \iota_i \leq v \circ \iota_i \text{ for every } i) \implies u \leq v.$$

For the glued algebra $C = \text{Glue}(\Phi)$, each ι_i is an embedding:

$$\iota_i : A_i \hookrightarrow \text{Glue}(\Phi), \quad a \longmapsto [a, i].$$

Canonical decomposition



- ▶ $\gamma_a[A] = \{x \in A \mid a \sqsubseteq x\}.$
- ▶ The isomorphism is given by $[x, a] \mapsto x.$

Finite case. We may take the maximal principal nuclear images.

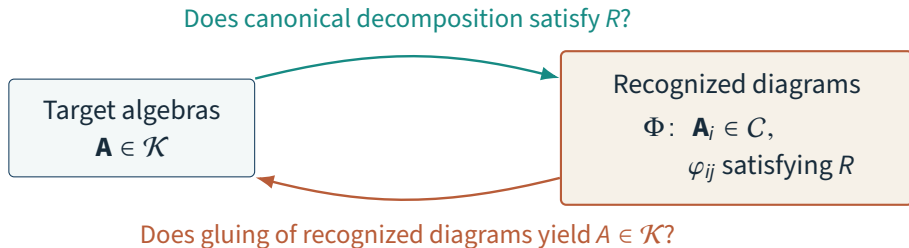
Infinite case. Direct union of maximal directed family of principal nuclear images?
It need not to be a nuclear image – there may be no way of finding the right gluing maps.
If every such direct union is a nuclear image, then $\mathbf{A}$ is **componentwise nuclear** and the component gluing reconstructs $\mathbf{A}.$

If A is locally Tanaka, each A_a is bounded prelinear Tanaka.

Representation and recognition

Let $C \subseteq \mathcal{K}$.

A recognition condition R is an additional condition on the abstract gluing maps $\varphi_{ij} : A_i \rightarrow A_j$.



Together: a representation of the whole class

$$\mathbf{A} \in \mathcal{K} \iff \mathbf{A} \cong \text{Glue}(\Phi) \quad \text{for some } \Phi \text{ with } \mathbf{A}_i \in C \text{ and } \varphi_{ij} \text{ satisfying } R.$$

Recognition cases

Let $\Phi = (\mathbf{A}_i, \varphi_{ij})$ be a complete colax diagram, assume that every $\mathbf{A}_i$ is Tanaka. Then $\text{Glue}(\Phi)$ is automatically **duo and locally Tanaka**.

$$\text{Glue}(\Phi) \text{ satisfies (J)} \iff \varphi_{ji}\varphi_{ij}\varphi_{ji}\varphi_{ij} = \varphi_{ji}\varphi_{ij}$$

$$\text{Glue}(\Phi) \text{ is Tanaka} \iff \varphi_{ij}\varphi_{ji}\varphi_{ij} = \varphi_{ij}$$

$$\text{Glue}(\Phi) \text{ is duo divisible} \iff \Phi \text{ satisfies } \begin{cases} \varphi_{jk}(\varphi_{ij}(x) \setminus y) \setminus (\varphi_{ik}(x) \setminus z) = \varphi_{ik}(\varphi_{ji}(y) \setminus x) \setminus (\varphi_{jk}(y) \setminus z) \\ (z / \varphi_{ik}(x)) / \varphi_{jk}(y / \varphi_{ij}(x)) = (z / \varphi_{jk}(y)) / \varphi_{ik}(x / \varphi_{ji}(y)) \end{cases}$$

Representation

$$\text{finite } \mathbf{A} \in \mathcal{K} \iff \mathbf{A} \cong \text{Glue}(\Phi)$$

for some finite diagram $\Phi = (\mathbf{A}_i, \varphi_{ij})$ of finite bounded prelinear Tanaka algebras satisfying R .

Recognition cases

Let $\Phi = (\mathbf{A}_i, \varphi_{ij})$ be a complete colax diagram, assume that every $\mathbf{A}_i$ is Tanaka. Then $\text{Glue}(\Phi)$ is automatically **duo and locally Tanaka**.

$$\text{Glue}(\Phi) \text{ satisfies (J)} \iff \varphi_{ji}\varphi_{ij}\varphi_{ji}\varphi_{ij} = \varphi_{ji}\varphi_{ij}$$

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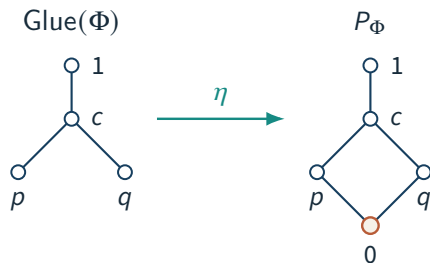
Representation

$$\text{componentwise nuclear } \mathbf{A} \in \mathcal{K} \iff \mathbf{A} \cong \text{Glue}(\Phi)$$

for some **component presentation** $\Phi = (\mathbf{A}_i, \varphi_{ij})$ of **$\sqsubseteq$ -downward directed prelinear Tanaka** algebras satisfying R .

Outlook: a product-like construction

$$P_{\Phi} = \{(x, y) \in A_1 \times A_2 : y \leq \varphi_{12}(x), x \leq \varphi_{21}(y)\}.$$



$$e_1(x) = (x, \varphi_{12}(x)), \quad e_2(y) = (\varphi_{21}(y), y).$$

There is an order embedding

$$\eta : \text{Glue}(\Phi) \hookrightarrow P_{\Phi}, \quad \eta([x, i]) = e_i(x).$$

Categorical question: if $\text{Glue}(\Phi)$ is the colax colimit, is P_{Φ} a corresponding **colax limit**?

Outlook: the product as a conuclear image

In the example above, let $\bar{\mathbf{A}}_1$ and $\bar{\mathbf{A}}_2$ be the corresponding MV-algebras. Define

$$\delta(x, y) = (x \wedge \varphi_{21}(y), y \wedge \varphi_{12}(x)).$$

A direct check gives

$$\delta \text{ is a conucleus, } \quad \text{Fix}(\delta) = \delta(\bar{\mathbf{A}}_1 \times \bar{\mathbf{A}}_2) = P_\Phi,$$

and P_Φ becomes a commutative integral residuated lattice in which η is a BCK embedding.

Possible general picture?

For a finite complete colax diagram Φ , after extending the diagram to suitable residuated envelopes, consider

$$(\delta_\Phi(x))_k := \bigwedge_{i \in I} \varphi_{ik}(x_i).$$

Does $P_\Phi = \text{Fix}(\delta_\Phi)$ give a conuclear residuated envelope of $\text{Glue}(\Phi)$?

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