

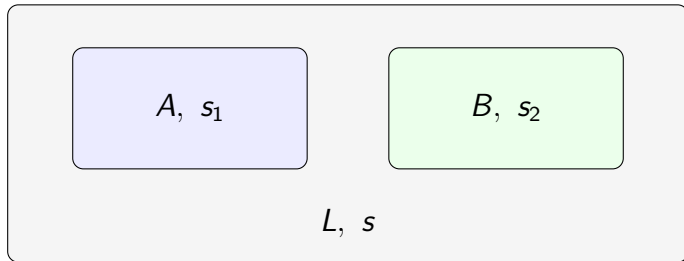
Simultaneous extensions of states on quantum logics

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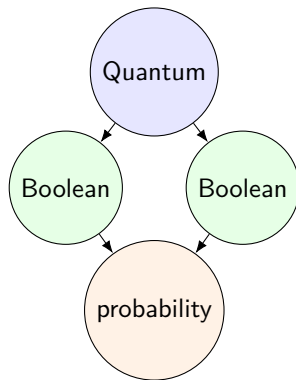
Introduction

We ask when two classical probability assignments, living on two Boolean “islands” inside a quantum logic, can be seen as a restriction of a global state.



Why should we pursue this question?

- Quantum experiments are linked with event structures that are not Boolean.
- Compatible measurements are associated with Boolean subalgebras.
- States are probability measures on the quantum logic.
- The question: when are local probabilities on Boolean subalgebras subordinated to a state on the entire logic?
- The problem is natural also from the point of view of physics.



Quantum logic

Definition

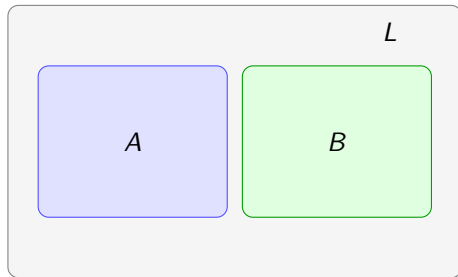
A quantum logic is an orthomodular poset

$$(L, \leq, ', 0, 1)$$

where the orthomodular law is valid: $a \leq b \implies b = a \vee (b \wedge a')$

Boolean sublogics (technical definitions)

A subset $A \subseteq L$ is a Boolean sublogic if it is a Boolean algebra with the operations inherited from L .



Two Boolean islands inside L

A state is a mapping $s : L \rightarrow [0, 1]$ such that

$$s(1) = 1, \quad a \leq b' \Rightarrow s(a \vee b) = s(a) + s(b).$$

- $\mathcal{S}(L)$: the set of all states on L .
- $\mathcal{S}(L)$ is convex and compact (in the pointwise topology), but may be very small or even empty.
- $\mathcal{S}_2(L)$: the set of all two-valued states, i.e. states with values in $\{0, 1\}$.

The extension problem

Let $A, B \subseteq L$ be Boolean sublogics. Let $s_1 \in \mathcal{S}(A)$ and $s_2 \in \mathcal{S}(B)$.

Question

When is there a state $s \in \mathcal{S}(L)$ such that

$$s|_A = s_1, \quad s|_B = s_2?$$

Well-conditioned couple of states

The quadruple (A, B, s_1, s_2) is **well conditioned** when the following conditions are satisfied:

$$a \in A, b \in B, a \geq b \Rightarrow s_1(a) \geq s_2(b),$$

$$c \in A, d \in B, c \leq d \Rightarrow s_1(c) \leq s_2(d).$$

The above two conditions are necessary!

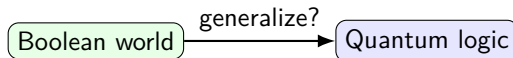
If a simultaneous extension exists, the couple of states must be well conditioned (observe that the conditions imply that if $a \in A \cap B$, then $s_1(a) = s_2(a)$).

Motivation for our result

Theorem (Rao–Rao)

If L is a Boolean algebra, then every well-conditioned pair of states on Boolean subalgebras has a simultaneous extension over L .

The proof is based on the deep extension theorem of Horn and Tarski.



Why the quantum case is not automatically adapted

- Every Quantum logic is a pasting of Boolean algebras...
- ...and Boolean algebras have a lot of two-valued states...
- but Quantum logics may have too few states...
- ... or even no state at all - for these cases there is no chance.

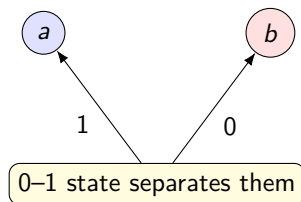
Strategy

Try to find a state-space condition of L that would allow for simultaneous extensions.

Mimicking the Stone technique

Assume:

$$a \not\leq b \implies \exists s \in \mathcal{S}_2(L) : s(a) = 1, s(b) = 0.$$



(application of main idea behind Stone representation theorem for Boolean algebras to quantum logics)

Main theorem

Theorem

Let L be a quantum logic. If its two-valued states separate every pair $a \not\leq b$, then:

- ① *L is set-representable;*
- ② *L allows simultaneous extensions of states.*

A set-representable logic may be far from Boolean.

Theorem (Kalmbach-Harding-Navara)

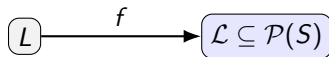
Each lattice can be lattice embedded in a set-representable quantum logic.

Stone representation for logics (Which logics are set-representable?)

Let $S = \mathcal{S}_2(L)$ and define

$$f(a) = \{s \in \mathcal{S}_2(L) : s(a) = 1\}.$$

- The separation assumption makes f injective and order-reflecting.
- Complements become set complements.
- Orthogonal joins become disjoint unions.



set representation

Proof idea: extend to a larger Boolean algebra

Once L is represented as a set-logic $\mathcal{L} \subseteq \mathcal{P}(S)$:

- ① Main point: It can be proved that A is a **Boolean subalgebra** exactly when it is closed under **the formation of intersections, unions and complements!**
- ② A and B are Boolean subalgebras of $\mathcal{P}(S)$.
- ③ Rao–Rao gives a state extension over $\mathcal{P}(S)$.
- ④ Restrict that extension back to $L = \mathcal{L}$.

$$A, B \subseteq L \subseteq \mathcal{P}(S) \implies s_{\mathcal{P}(S)}|_L = s.$$

Consequence: compatibility regular logics

Call L **compatibility regular** if

$$a \wedge b = 0 \iff a \leq b'.$$

The compatibility regular logics form a special class - a and b are compatible exactly when $a \wedge b$ exists in L .

Theorem

If L is compatibility regular, then L allows for simultaneous extensions of states.

The point: The compatibility regularity implies a set representation (result of Pták and Navara), so the main theorem applies.

Trying to find a necessary and sufficient condition

Proposition

Suppose $a \not\leq b$, and no state $s \in \mathcal{S}(L)$ satisfies

$$s(a) = 1, \quad s(b) = 0.$$

Then L does not allow simultaneous extensions of states.

0–1 distinguishing logics

A weaker condition replaces two-valued states:

$$a \not\leq b \implies \exists s \in \mathcal{S}(L) : s(a) = 1, s(b) = 0.$$

(s does not have to attain only 0 and 1).

Open question

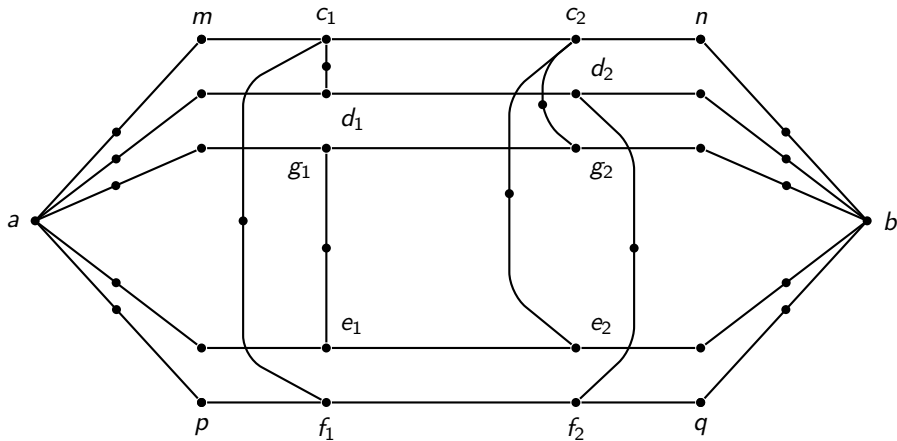
Is this weaker condition enough to guarantee simultaneous extensions?

Observation ($A=B$ works):

Proposition

Let A be a Boolean subalgebra of L . If $\mathcal{S}(L)$ is 0–1 distinguishing, then every state on A extends to a state on L .

Test example



This is 0 – 1 distinguishing but **not set-representable**.

Thank you for your attention!

Acknowledgements

This research was supported by the GACR (project No. 25-20013L) and the SGS (project No. SGS26/073/OHK3/1T/13).