

Some new semidirect-product-type constructions of residuated lattices

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Motivation

- ▶ Our goal is to construct residuated lattices in as much generality as possible.
- ▶ A first target — still an open problem in general — is to find such constructions at least for *finite* residuated lattices.

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- ▶ A first target — still an open problem in general — is to find such constructions at least for *finite* residuated lattices.
- ▶ We take our inspiration from the case of hoops, where the finite case is already fully solved.
- ▶ We then proceed analogously: first via product morphisms, and finally in an even more general, categorical setting.

Outline

Hoops

Construction via product morphisms

A diagrammatic view

Hoops

The first and most special class of algebras for which our construction works is the class of hoops.

Definition

A hoop is an algebra $\mathbf{A} = (A; \cdot, \rightarrow, 1)$ of type $\langle 2, 2, 0 \rangle$ such that $(A; \cdot, 1)$ is a commutative monoid satisfying the identities

$$(H1) \quad x \rightarrow x = 1,$$

$$(H2) \quad (x \cdot y) \rightarrow z = x \rightarrow (y \rightarrow z),$$

$$(H3) \quad x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x).$$

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Equivalently, a hoop is a naturally ordered commutative monoid satisfying the adjointness property

$$x \cdot y \leq z \iff x \leq y \rightarrow z.$$

Filters and congruences

If $\mathbf{A} = (A; \cdot, \rightarrow, 1)$ is a hoop, then a *filter* is a nonempty subset $F \subseteq A$ such that

(F1) if $x \in F$ and $x \leq y$, then $y \in F$,

(F2) if $x, y \in F$, then $x \cdot y \in F$.

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The set of all filters of $\mathbf{A}$ is denoted by $\mathbf{Fil} \mathbf{A}$. Every filter is a subalgebra (written $\mathbf{F}$) and induces the congruence

$$\theta_F = \{ \langle x, y \rangle \in A^2 \mid (x \rightarrow y) \cdot (y \rightarrow x) \in F \},$$

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This correspondence between filters and congruences is a lattice isomorphism: **filters are exactly the kernels of homomorphisms** out of $\mathbf{A}$.

The f -product of hoops

Definition

Let $\mathbf{A} = (A; \cdot, \rightarrow, 1)$ and $\mathbf{B} = (B; \cdot, \rightarrow, 1)$ be hoops. A mapping $f: A \longrightarrow B$ is called a *product morphism* from $\mathbf{A}$ to $\mathbf{B}$ if it satisfies

$$(pM1) \quad f(1) = 1,$$

$$(pM2) \quad f(x) \cdot f(y) = f(x \cdot y) = f(x) \wedge f(y) = f(x \wedge y)$$

for all $x, y \in A$.

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for all $x, y \in A$.

Theorem

If $\mathbf{A}$ and $\mathbf{B}$ are hoops and $f: \mathbf{B} \longrightarrow \mathbf{A}$ is a product morphism, then

$$\mathbf{A} \ltimes_f \mathbf{B} = (\sum_{x \in B} (f(x)], \cdot, \rightarrow, (1, 1)),$$

where

$$(\cdot) \quad (a, x) \cdot (b, y) := (a \cdot b, x \cdot y),$$

$$(\rightarrow) \quad (a, x) \rightarrow (b, y) := (f(x \rightarrow y) \wedge (a \rightarrow b), x \rightarrow y),$$

is a hoop, called the f -product of $\mathbf{A}$ and $\mathbf{B}$.

Associated pairs of product morphisms

Definition

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be hoops. A pair (f, g) is left-associated with respect to $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ if

$$f: \mathbf{B} \longrightarrow \mathbf{A} \quad \text{and} \quad g: \mathbf{C} \longrightarrow \mathbf{A} \ltimes_f \mathbf{B}$$

are product morphisms; we therefore have the hoop $(\mathbf{A} \ltimes_f \mathbf{B}) \ltimes_g \mathbf{C}$.

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are product morphisms; we therefore have the hoop $(\mathbf{A} \ltimes_f \mathbf{B}) \ltimes_g \mathbf{C}$.

Similarly, (f, g) is right-associated with respect to $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ if

$$g: \mathbf{C} \longrightarrow \mathbf{B} \quad \text{and} \quad f: \mathbf{B} \ltimes_g \mathbf{C} \longrightarrow \mathbf{A}$$

are product morphisms; we therefore have the hoop $\mathbf{A} \ltimes_f (\mathbf{B} \ltimes_g \mathbf{C})$.

Associativity up to isomorphism

Theorem

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be hoops. There is a bijective correspondence between left-associated pairs (f, g) and right-associated pairs $(\bar{f}, \bar{g})$ with respect to $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ such that

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$$(\mathbf{A} \ltimes_f \mathbf{B}) \ltimes_g \mathbf{C} \cong \mathbf{A} \ltimes_{\bar{f}} (\mathbf{B} \ltimes_{\bar{g}} \mathbf{C}).$$

The decomposition theorem

For a finite hoop $\mathbf{A}$ and $F \in \mathbf{Fil} \mathbf{A}$, write t_X, l_X for the top and least element of a class $X \in \mathbf{A}/F$, and $l := l_F$. A short computation shows $t_X \wedge t_Y = t_{X \wedge Y}$, $t_X \rightarrow t_Y = t_{X \rightarrow Y}$, and $t_X \cdot l = l_X$ for all $X, Y \in \mathbf{A}/F$.

The decomposition theorem

For a finite hoop $\mathbf{A}$ and $F \in \mathbf{Fil} \mathbf{A}$, write t_X, l_X for the top and least element of a class $X \in \mathbf{A}/F$, and $I := I_F$. A short computation shows $t_X \wedge t_Y = t_{X \wedge Y}$, $t_X \rightarrow t_Y = t_{X \rightarrow Y}$, and $t_X \cdot I = l_X$ for all $X, Y \in \mathbf{A}/F$.

Theorem

If $\mathbf{A}$ is a finite hoop and $F \in \mathbf{Fil} \mathbf{A}$, then $\psi: \mathbf{A}/F \longrightarrow \mathbf{F}$ defined by $\psi(X) = t_X \vee I$ is a product morphism and, moreover, $\mathbf{A} \cong \mathbf{F} \ltimes_{\psi} (\mathbf{A}/F)$.

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The isomorphism is given by $\Omega(a) = (a \vee I, a/F)$, with $\Omega^{-1}(a, X) = a \wedge t_X$.

Finite hoops as products of MV-chains

A hoop $\mathbf{A}$ is *irreducible* if $\mathbf{A} \cong \mathbf{B} \times_f \mathbf{C}$ implies that $\mathbf{B}$ or $\mathbf{C}$ is trivial.

Theorem

A finite hoop is irreducible if and only if it is simple.

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Corollary

For every finite hoop $\mathbf{A}$ there exist finite MV-chains $\mathbf{M}_{i_1}, \dots, \mathbf{M}_{i_n}$ and product morphisms $f_1, \dots, f_{n-1}$ such that

$$\mathbf{A} \cong \mathbf{M}_{i_1} \times_{f_1} (\mathbf{M}_{i_2} \times_{f_2} (\dots \times_{f_{n-2}} (\mathbf{M}_{i_{n-1}} \times_{f_{n-1}} \mathbf{M}_{i_n}) \dots)),$$

and, by associativity, also with the opposite bracketing.

Remark: connection to poset sums

For *finite* algebras, the same conclusion is known through a different, older route:

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- ▶ Jipsen–Montagna (2009) generalised this — as a generalisation of *ordinal sums* — to all finite GBL-algebras, hence to all finite hoops:

$$\mathbf{A} \cong \bigsqcup_{i \in P} \mathbf{A}_i, \quad \bigsqcup_{i \in P} \mathbf{A}_i = \left\{ a \in \prod_{i \in P} \mathbf{A}_i : a_j < e \Rightarrow a_k = 0_k \text{ for all } j < k \right\}.$$

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What is new in our corollary is a *binary, associative* product $\bowtie_f$ producing the same decomposition — and it is exactly this binary associativity that survives once divisibility is dropped.

A. Di Nola, A. Lettieri, *Finite BL-algebras*, Discrete Mathematics **269** (2003), 93–112.

P. Jipsen, F. Montagna, *The Blok–Ferreirim theorem for normal GBL-algebras and its applications*, Algebra Universalis **60** (2009), 381–404.

Product morphisms for residuated lattices

Definition

A mapping $\phi: B \rightarrow A$ between integral residuated lattices is a product morphism if

$$\phi(1) = 1, \quad \phi(x) \wedge \phi(y) = \phi(x \wedge y), \quad \phi(x) \cdot \phi(y) \leq \phi(x \cdot y).$$

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Theorem

For a product morphism $\phi: \mathbf{B} \rightarrow \mathbf{A}$ the algebra

$$\mathbf{A} \ltimes_{\phi} \mathbf{B} = (\sum_{x \in B} (\phi(x)], \wedge, \vee, \cdot, \backslash, /, (1, 1)),$$

with componentwise $\wedge, \vee, \cdot$ and

$$(a, x) \bullet (b, y) = ((a \bullet b) \wedge \phi(x \bullet y), x \bullet y) \quad \text{for } \bullet \in \{\backslash, /\},$$

is a residuated lattice.

Associativity for the product-morphism product

Associated pairs and their transfer formulas are defined exactly as for hoops, with f, g replaced by product morphisms ϕ, ψ between (arbitrary) integral residuated lattices.

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Theorem

There is a bijective correspondence between left-associated pairs (ϕ, ψ) and right-associated pairs $(\overline{\phi}, \overline{\psi})$ of product morphisms with respect to $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ such that

$$(\mathbf{A} \ltimes_{\phi} \mathbf{B}) \ltimes_{\psi} \mathbf{C} \cong \mathbf{A} \ltimes_{\overline{\phi}} (\mathbf{B} \ltimes_{\overline{\psi}} \mathbf{C}).$$

The ordinal sum as a diagram

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We work in the class **K** of **complete** integral residuated lattices; completeness guarantees that every **A** $\in$ **K** has a least element 0, which we use freely.

Remark

*Everything below works equally well if **K** is taken to be the class of quantales, or a similarly complete class of structures.*

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Přenosil's observation: the poset sum of a family $(\mathbf{A}_i)_{i \in I}$ can be recast as a **diagram** over the index poset $(I, \leq)$, with connecting maps given exactly by the least product morphisms.

Poset diagrams via least product morphisms

For $\mathbf{C}, \mathbf{D} \in \mathbf{K}$, recall the *least product morphism* $0_{\mathbf{CD}}: C \longrightarrow D$,

$$0_{\mathbf{CD}}(1) = 1, \quad 0_{\mathbf{CD}}(x) = 0_{\mathbf{D}} \text{ for } x \neq 1.$$

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Definition

For a poset $(I, \leq)$ and nontrivial algebras $(\mathbf{A}_i)_{i \in I}$ from $\mathbf{K}$, the *poset diagram* is the functor $D: I \longrightarrow \mathbf{K}$ given by

$$D(i) = \mathbf{A}_i, \quad D(i \leq j) = \begin{cases} \text{id}_{\mathbf{A}_i}, & i = j, \\ 0_{\mathbf{A}_i \mathbf{A}_j}, & i < j. \end{cases}$$

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This is well defined: for $i < j < k$ one checks directly that $0_{\mathbf{A}_j \mathbf{A}_k} \circ 0_{\mathbf{A}_i \mathbf{A}_j} = 0_{\mathbf{A}_i \mathbf{A}_k}$. The poset sum is then

$$\biguplus_{i \in I} \mathbf{A}_i = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} \mathbf{A}_i : a_j \leq D(i \leq j)(a_i) \text{ for all } i \leq j \right\}.$$

Further generalizations

The previous notions generalize in two steps:

1. *Maximally*: replace id and 0 by arbitrary product morphisms, replace the poset by an arbitrary small category, and let the diagram be *lax* (composition need only satisfy an inequality, not an equality) — i.e. pass to a lax functor on an arbitrary small category.

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2. *Economically*: several arrows between two algebras can equivalently be replaced by their meet, so a single arrow between any two algebras suffices — the lax functor is an unnecessarily large scaffold, and what remains is exactly adequate for our representations.

This leads to the following notion, which we call a *fundamental diagram*.

Beyond posets: duo-morphisms

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A duo-morphism $\alpha: A \longleftrightarrow B$ is a pair $\alpha = (\alpha_1, \alpha_2)$ of product morphisms $\alpha_1: A \rightarrow B$, $\alpha_2: B \rightarrow A$.

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We write $\alpha^{op}: B \longleftrightarrow A$, $\alpha^{op} = (\alpha_2, \alpha_1)$.

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For $a \in |A|$, $b \in |B|$ we use the compact notation

$$a\alpha := \alpha_1(a) \in B, \quad \alpha b := \alpha_2(b) \in |A|.$$

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Composition of duo-morphisms: for $\alpha: A \longleftrightarrow B$ and $\beta: B \longleftrightarrow C$ we get $\alpha\beta: A \longleftrightarrow C$.

Fundamental diagrams

Definition

A fundamental diagram D over an index set I consists of nodes $(\mathbf{A}_i)_{i \in I}$ and duo-morphisms $(\alpha_{ij}: \mathbf{A}_i \leftrightarrow \mathbf{A}_j)_{i,j \in I}$ such that

- (i) $\alpha_{ii} = (\text{id}_{\mathbf{A}_i}, \text{id}_{\mathbf{A}_i})$ for all $i \in I$,
- (ii) $\alpha_{ij}^{op} = \alpha_{ji}$ for all $i, j \in I$,
- (iii) $\alpha_{ik} \leq \alpha_{ij} \alpha_{jk}$ for all $i, j, k \in I$,

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Here $\leq$ denotes the pointwise order on pairs of product morphisms, componentwise on α_1 and on α_2 .

The sum of a fundamental diagram

Definition

For a fundamental diagram D over I , define

$$\boxtimes D := \left\{ \bar{a} \in \prod_{i \in I} \mathbf{A}_i : a_j \leq a_i \alpha_{ij} \text{ for all } i, j \in I \right\}.$$

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Each node embeds into, and projects from, the sum via a duo-morphism $\pi_i: \mathbf{A}_i \leftrightarrow \boxtimes D$:

$$(a\pi_i)_j := a\alpha_{ij} \quad (j \in I), \quad \pi_i \bar{a} := a_i.$$

Quotients of a fundamental diagram

Definition

Let D be a fundamental diagram over I and $\Theta \in \text{Eq}(I)$. For $i \in I$ let D_i be the sub-diagram of D on the Θ -class of i ,

$$|D_i| = \{\mathbf{A}_j : j \Theta i\},$$

and define the quotient diagram D/Θ by

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The connecting duo-morphisms $\bar{\alpha}_{ij} : \boxtimes D_i \leftrightarrow \boxtimes D_j$ are given, for $a \in \boxtimes D_i$, $b \in \boxtimes D_j$, by

$$(a\bar{\alpha}_{ij})(j') = \bigwedge_{i' \Theta i} a(i')\alpha_{i'j'} \quad (j' \Theta j), \quad (\bar{\alpha}_{ij}b)(i') = \bigwedge_{j' \Theta j} b(j')\alpha_{j'i'} \quad (i' \Theta i).$$

Undoing a quotient: diagrams of diagrams

Definition

Let $(D_j)_{j \in J}$ be fundamental diagrams, D_j over I_j , and suppose $\bar{D}$ is a fundamental diagram over J with nodes $\boxtimes D_j$ ($j \in J$) and connecting duo-morphisms $\bar{\alpha}_{j_1 j_2}: \boxtimes D_{j_1} \leftrightarrow \boxtimes D_{j_2}$. Define the diagram ΣD over $I := \bigsqcup_{j \in J} I_j$ by

$$|\Sigma D| = (\mathbf{A}_{ij})_{i \in I_j, j \in J}, \quad \alpha_{(i_1, j_1)(i_2, j_2)} := \pi_{i_1} \bar{\alpha}_{j_1 j_2} \pi_{i_2}^{op}.$$

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This construction is exactly inverse to the quotient $D \mapsto D/\Theta$ of the previous slide.

Inclusions between algebras

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- ▶ Each $\pi_i: \mathbf{A}_i \leftrightarrow \boxtimes D$ is an inclusion: $\mathbf{A}_i \leq_{\pi_i} \boxtimes D$.
- ▶ If both α and α^{op} are inclusions (equivalently, α_1 and α_2 are mutually inverse bijective isomorphisms), we write $A =_{\alpha} B$ and call α an *equality*.

Basic laws

- ▶ $A =_{\delta} A \boxtimes_{\text{id}_A} A$, $a\delta = a$, $\delta(a, a) = a$;
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The associativity law is exactly the statement that grouping $\{A, B\}$ first, factoring by that grouping, and reconstructing via ΣD gives back the same sum as grouping $\{B, C\}$ first.

Summary and outlook

- ▶ In suitable special classes, the techniques presented here again yield decomposition results (e.g. into simple objects), just as in the case of hoops.
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Summary and outlook

- ▶ In suitable special classes, the techniques presented here again yield decomposition results (e.g. into simple objects), just as in the case of hoops.
- ▶ Such results are not yet developed enough to be worth presenting, and time is short — the focus of this talk has been the new technique itself.
- ▶ Striking, and not yet fully understood: the basic laws exhibit a semilattice-like structure — $\boxtimes$ is idempotent and commutative (up to $\cong$). This alone highlights the qualitative difference between a plain product morphism and a duo-morphism.

Thank you for your attention!

- ▶ M. Botur, *A new representation of finite hoops using a new type of product of structures*, *Studia Logica* **114** (2026), 145–167; arXiv:2511.06730.
- ▶ W. J. Blok, I. M. A. Ferreirim, *On the structure of hoops*, *Algebra Universalis* **43** (2000), 233–257.
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