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Exploration of non-classical negation axioms

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Motivation

- Wesley Holliday, *A fundamental non-classical logic*, Logics 1(1) 36–79 (2023) doi:10.3390/logics1010004
- *fundamental (propositional) logic* (natural deduction): $\wedge, \vee, \neg$
- not requiring **distributivity** of $\wedge, \vee$ / **pseudocomplementation** of $\neg$
(as in *quantum logic* (orthomodular lattices))
- weakening motivated by natural language argumentation
absurdities occurring with modalities ‘might’/‘must’
... but those also happen with **orthomodularity**
- $\rightsquigarrow$ *orthologic* (orthocomplemented lattices)
Holliday drops **excluded middle**, $x \vee \neg x \approx 1$
(as in *intuitionistic logic* (Heyting lattices))
‘to find a neutral base logic’
- $\rightsquigarrow$ *fundamental logic*
(weakly pseudocomplemented bounded lattices)

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Locating fundamental logic in the literature

Holliday collects (other) negation properties...

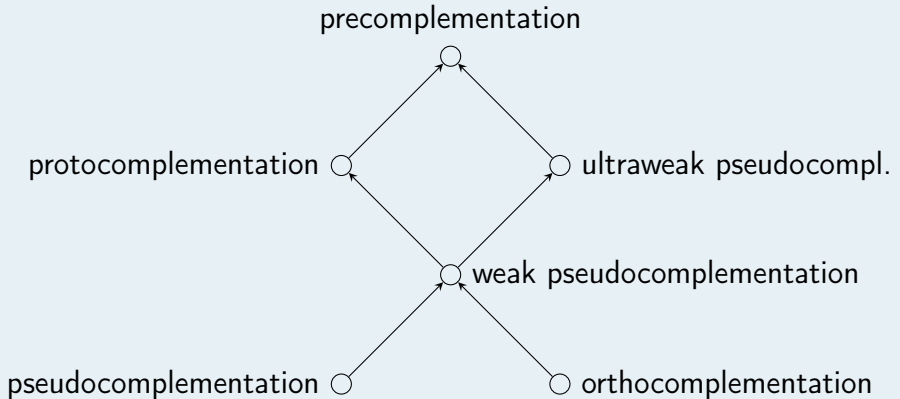
- pseudocomplementation
- orthocomplementation
- complementation
- semicomplementation
- precomplementation
- proto-complementation
- weak pseudocomplementation
- ultraweak pseudocomplementation

...defined by conjunctions of more basic properties

Locating fundamental logic in the literature

Holliday presents implications. . .

(dually drawn after Holliday, Fig. 6, p. 45)



...Are these all? Which implications between basic axioms?

Definitions

Lattice with negation (fundamental implicit assumption)

bounded lattice $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$ where $\neg: L \longrightarrow L$ any unary op.

Following Holliday's paper: negation $\neg$ is a...

semicomplementation $x\bar{x} \approx 0$

dual semicomplementation $x + \bar{x} \approx 1$

complementation semicomplementation & dual semicomplementation

precomplementation antitone & $\bar{1} \approx 0$

protocomplementation antitone semicomplementation & $\bar{0} \approx 1$

ultraweak pseudocomplementation precomplementation & $x \lesssim \bar{\bar{x}}$

weak pseudocomplementation antitone semicomplementation & $x \lesssim \bar{\bar{x}}$

pseudocomplementation maximal ($xy \approx 0 \rightarrow y \lesssim \bar{x}$) semicmpl.

orthocomplementation antitone involutive ($\bar{\bar{x}} \approx x$) complementation

Implied properties

Holliday's definitions are minimalistic, but...

Additional properties

(reproduced after Holliday, Tbl. 1, p. 45)

	$y \lesssim x \rightarrow \bar{x} \lesssim \bar{y}$	$\bar{1} \approx 0$	$\bar{0} \approx 1$	$x\bar{x} \approx 0$	$x \lesssim \bar{\bar{x}}$	$xy \approx 0 \rightarrow y \lesssim \bar{x}$	$\bar{\bar{x}} \lesssim x$
pre	×	×					
proto	×	×	×	×			
uwk pseudo	×	×	×		×		
wk pseudo	×	×	×	×	×		
pseudo	×	×	×	×	×	×	
ortho	×	×	×	×	×		×

Which are the (actual/basic) implications?

Which ones do I need to prove, which ones follow?

Most properties 'are' equations

Examples

semicomplementation $x\bar{x} \approx 0$

extensionality of double negation $x \lesssim \bar{\bar{x}}$ equivalent to $x\bar{\bar{x}} \approx x$

... I prefer **equations** because they **define varieties**

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... except for

Antitonicity: $y \lesssim x \rightarrow \bar{x} \lesssim \bar{y}$

Maximality property of pseudocomplementation: $xy \approx 0 \rightarrow y \lesssim \bar{x}$

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equivalent to $\bar{x} \lesssim \overline{xy}$ equivalent to $\bar{x} \cdot \overline{xy} \approx \bar{x}$ (known?, trivial?)

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Theorem (Raymond Balbes, Alfred Horn, *Stone lattices*, Duke Math. J. **37** 537–545 (1970) doi:10.1215/S0012-7094-70-03768-3)

$\mathbf{L} = \langle L; \cdot, \bar{\cdot}, 0 \rangle$ is a pseudocomplemented meet-semilattice with zero
iff $\mathbf{L}$ is a meet-semilattice with zero & $\mathbf{L} \models x\bar{0} \approx x, \bar{\bar{0}} \approx 0, \overline{xy} \approx x\bar{y}$

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Maximality property of pseudocomplementation: $xy \approx 0 \rightarrow y \lesssim \bar{x}$

Corollary (derived from Balbes & Horn, 1970)

$\mathbf{L} = \langle L; \cdot, +, 0, 1, \bar{} \rangle$ is a pseudocomplemented bounded lattice

iff $\mathbf{L}$ is a bounded lattice & $\mathbf{L} \models \bar{0} \approx 1, \bar{1} \approx 0, x\overline{xy} \approx x\bar{y}$

I consider the following 23 basic properties

- $x\bar{x} \approx 0$
- $x + \bar{x} \approx 1$
- $x \approx \bar{\bar{x}}$
- $x \lesssim \bar{\bar{x}}$
- $\bar{1} \approx 0$
- $\bar{0} \approx 1$
- $\bar{x} \lesssim \overline{xy}$
- $\bar{x} \approx x$
- $\bar{x} \approx 0$
- $\bar{x} \approx 1$
- $xy \approx 0 \rightarrow y \lesssim \bar{x}$
- $x\bar{y} \approx x\overline{xy}$
- antitone
- const swap
- pre
- uwk pseudo
- wk pseudo
- proto
- ortho
- pseudo
- semi
- dual semi
- compl

Which (basic) implications hold?

Similar explorations of negation properties

- Frithjof Dau, *Implications of properties concerning complementation in finite lattices*, in Contributions to general algebra **12** 145–154, Heyn, Klagenfurt (2000)

including de Morgan laws, orthomodularity, dual orthomodularity, modularity, distributivity,

- Léonard Kwuida, Christian Pech, Heiko Reppe, *Generalizations of Boolean algebras. An attribute exploration*, Mathematica Slovaca **56**(2) 145–165 (2006)

including de Morgan laws, orthomodularity, dual orthomodularity, pseudocomplemented lattices, weakly and dually weakly complemented lattices, always $\bar{0} \approx 1$, $\bar{1} \approx 0$.

Implications and closure systems

Formal implication $A \twoheadrightarrow B$ on X

a pair (A, B) of subsets $A, B \subseteq X$, written as $A \twoheadrightarrow B$

Subset $M \subseteq X$ respects $A \twoheadrightarrow B$

$M \models A \twoheadrightarrow B$ iff (provided that $A \subseteq M$, then also $B \subseteq M$),
i.e., $A \not\subseteq M$ or $B \subseteq M$

Galois connection between $\mathfrak{P}(X) \times \mathfrak{P}(X)$ and $\mathfrak{P}(X)$ induced by $\models$

implication systems $\mathfrak{P}(X) \times \mathfrak{P}(X) \supseteq \mathcal{I} \mapsto \text{Mod } \mathcal{I} \subseteq \mathfrak{P}(X)$
set of models, always a closure system
each closure system $\mathcal{C} = \text{Mod}\{A \twoheadrightarrow c(A) \mid A \subseteq X\}$
sets of subsets $\mathfrak{P}(X) \supseteq \mathcal{M} \mapsto \text{Imp } \mathcal{M} \subseteq \mathfrak{P}(X) \times \mathfrak{P}(X)$
set of valid implications, always an implication theory
(implication systems closed under Armstrong rules)

On the exploration method

- given: target closure system $\mathcal{C} \subseteq \mathfrak{P}(X)$, X finite
 - $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \dots \mathcal{L}_n \subseteq \dots \mathcal{C} \subseteq \dots \mathcal{U}_n \subseteq \dots \mathcal{U}_1 \subseteq \mathcal{U}_0$
- (want different desc.)

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 were $C \in \mathcal{L}_n$, then $A \dashrightarrow B \in \text{Imp } \mathcal{L}_n \subseteq \text{Imp}\{C\} \not\subseteq$

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 $\implies S_{n+1} := S_n \cup \{s\}$ with $s' := C$

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- $\mathcal{L}_{n+1} := \mathcal{L}_n \subseteq \mathcal{C} \subseteq \mathcal{U}_n \cap \text{Mod}\{A \dashrightarrow B\} = \text{Mod } \mathcal{I}_{n+1} = \mathcal{U}_{n+1}$
- or $A \dashrightarrow B \notin \text{Imp } \mathcal{C}$, then $\exists C \in \mathcal{C} \setminus \mathcal{L}_n$: $C \not\models A \dashrightarrow B$
 $\implies S_{n+1} := S_n \cup \{s\}$ with $s' := C$
- $\mathcal{L}_{n+1} = \langle \{s' \mid s \in S_n\} \cup \{C\} \rangle_{\langle \mathfrak{P}(X); \cap \rangle} \subseteq \mathcal{C} \subseteq \mathcal{U}_n =: \mathcal{U}_{n+1}$

Exploration algorithm

- continue until for some $n \in \mathbb{N}$: $\mathcal{U}_n = \mathcal{L}_n \subseteq \mathcal{C} \subseteq \mathcal{U}_n$, i.e.,
 $\{O' \mid O \subseteq S_n\} = \mathcal{L}_n = \mathcal{C} = \mathcal{U}_n = \text{Mod } \mathcal{I}_n$.
- ‘new’ implications can be generated algorithmically, for details see Bernhard Ganter, Sergei Obiedkov, *Conceptual Exploration*, xvii+315 p., Springer (2016) doi:10.1007/978-3-662-49291-8
- once Galois connection between S_n and X is known, **implication base** $\mathcal{I}_n$ can also be determined without (stepwise) exploration, has to be proved

Our concrete case

- X the set of 23 negation properties
- $\mathcal{C}$ the Galois closed sets of the Galois connection between all bounded lattices with negation & satisfaction of properties in X

The Galois connection between S_n and X

Property	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}	S_{19}	S_{20}	S_{21}	S_{22}	S_{23}
$a\bar{a} = 0$							x	x			x	x	x				x			x		x	x
$a + \bar{a} = 1$							x	x	x	x			x						x	x	x	x	x
$a = \bar{\bar{a}}$									x				x				x	x	x			x	x
$a \leq \bar{\bar{a}}$					x	x			x	x	x		x	x		x		x		x	x	x	x
$\bar{1} = 0$			x	x		x	x	x	x		x	x	x				x	x	x	x		x	x
$\bar{0} = 1$		x		x	x	x	x	x	x	x	x		x	x	x		x	x	x	x	x	x	x
$\bar{a}\bar{a}b = \bar{a}$	x	x	x	x	x	x	x			x	x	x	x		x			x	x	x	x		x
$\bar{a} = a$																	x						
$\bar{a} = 0$												x											
$\bar{a} = 1$										x													
$ab = 0 \rightarrow b \leq \bar{a}$									x	x	x			x	x			x	x	x	x		x
$a\bar{b} = a\bar{a}b$										x	x	x		x	x	x				x			x
antitone	x	x	x	x	x	x	x			x	x	x	x		x			x	x	x	x		x
const swap				x		x	x	x	x		x		x				x	x	x	x		x	x
precpl			x	x		x	x				x	x	x					x	x	x			x
uweak pseudocpl						x					x		x					x		x			x
wk pseudocpl											x		x							x			x
proto							x				x		x							x			x
ortho													x										x
pseudocpl											x									x			x
semicpl							x	x			x	x	x				x			x		x	x
dual semicpl							x	x	x	x			x						x	x	x	x	x
compl							x	x					x							x		x	x

Still need the implications between properties in X !

Easy results needed for accepting implications

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$, $\neg$ involutive $\implies \models$ extensional

$$a \leq a = \overline{\overline{a}} \text{ for } a \in L.$$

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$, $\models$ extensional, $\neg$ antitone $\implies \overline{0} \approx 1$

$$0 \leq \overline{1} \implies 1 \leq \overline{\overline{1}} \leq \overline{0} \leq 1.$$

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$, $\models$ extensional, $\overline{1} = 0 \implies \overline{0} \approx 1$

$$1 \leq \overline{\overline{1}} = \overline{0} \leq 1.$$

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$, $\neg$ involutive $\overline{0} \approx 1 \implies \overline{1} \approx 0$

$$\overline{1} = \overline{\overline{0}} = 0.$$

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$, $xy \approx 0 \rightarrow y \lesssim \overline{x} \implies \overline{0} \approx 1$

$$0 \cdot 1 = 0 \implies 1 \leq \overline{0} \leq 1.$$

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$, $\neg$ semicompl $\implies \overline{1} \approx 0$

$$\overline{1} = 1 \cdot \overline{1} = 0.$$

Less trivial implications

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$, $x\overline{xy} \approx x\overline{y}$, $\overline{1} \approx 0 \implies x\overline{x} \approx 0$ ($\neg$ semicompl)

$$a \cdot \overline{a} = a \cdot \overline{a \cdot 1} = a \cdot \overline{1} = a \cdot 0 = 0 \quad \text{for any } a \in L.$$

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$,

$$x\overline{x} \approx 0, x\overline{xy} \approx x\overline{y} \implies \overline{x} \approx \overline{x \overline{xy}} \iff \overline{x} \lesssim \overline{xy} \quad (\neg \text{ antitone})$$

$$\overline{a} \overline{ab} = \overline{a} \overline{a \overline{ab}} = \overline{a \overline{0} b} = \overline{a \overline{0}} = \overline{a \overline{a \overline{a}}} = \overline{a \overline{a}} = \overline{a} \quad \text{for any } a, b \in L.$$

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$, $\neg$ extensional, $\neg$ antitone $\overline{1} \approx 0$, $x + \overline{x} \approx 1$
 $\implies x\overline{x} \approx 0$ ($\neg$ compl)

$$1 = ab + \overline{ab} \leq a + \overline{ab} \leq 1,$$

thus for $b = \overline{a}$: $1 = a + \overline{a \overline{a}} = \overline{a \overline{a}}$ because $a \leq \overline{a} \leq \overline{a \overline{a}}$ from $a \overline{a} \leq \overline{a}$;

$$\text{hence } 0 \leq a \overline{a} \leq \overline{\overline{a \overline{a}}} = \overline{1} = 0 \quad \text{for any } a \in L.$$

Even less trivial implications

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle, x + \bar{x} \approx 0, x\bar{x}y \approx x\bar{y} \implies$

- (a) $\bar{x} \geq x\bar{x} \approx x\bar{1}, \text{ i.e., } \bar{x} + \bar{1}x \approx \bar{x}$
- (b) $\bar{x}\bar{x} \approx 1$
- (c) $x \leq \bar{\bar{x}}$ ($\bar{\bar{}}$ extensional)
- (d) $\bar{}$ antitone

Proof

- (a) $\bar{a} \geq a\bar{a} = a\bar{a} \cdot 1 = a\bar{1}.$
- (b) For $b \in L$ let $a := \bar{1}b$; clearly $\bar{1}a = \bar{1}\bar{1}b = \bar{1}b = a.$
Thus, $1 = \bar{a} + a = \bar{a} + \bar{1}a \stackrel{(a)}{=} \bar{a} = \bar{\bar{1}b} \stackrel{(a)}{=} \bar{\bar{b}} \quad (\text{for any } b \in L).$
- (c) $b = b \cdot 1 \stackrel{(b)}{=} b \cdot \bar{\bar{b}} = \bar{\bar{b}} \leq \bar{b} \quad \text{for any } b \in L.$
- (d) For $b \leq a \in L$ let $x := ab, y := \bar{a}.$
 $1 \stackrel{(b)}{=} \bar{x}\bar{x} = \overline{abab} = \overline{ab\bar{a}} = \overline{\bar{a}ab} = \overline{y\bar{x}}, \text{ thus}$
 $\bar{a} = y = y \cdot 1 = y \cdot \overline{y\bar{x}} = y\bar{x} = \bar{a}\bar{ab} \leq \bar{ab} = \bar{b} \quad (\text{antitonicity}).$

One more

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$, $\neg$ involutive, antitone semicompl, $\bar{0} \approx 1$
 $\implies x + \bar{x} \approx 1$ (dual semicompl), thus (compl)

For $a, b \in L$ $\bar{a} \cdot \overline{b + \bar{a}} \stackrel{\text{abs}}{=} \bar{a}(b + \bar{a})\overline{b + \bar{a}} = \bar{a} \cdot 0 = 0.$

also: $\overline{a + \bar{a}} \leq \bar{a}$ from $a \leq a + \bar{a}$

for $b = a$: $0 = \bar{a} \cdot \overline{a + \bar{a}} = \overline{a + \bar{a}}$

by involution: $a + \bar{a} = \overline{\overline{a + \bar{a}}} = \bar{0} = 1.$

One more

Lemma: $\mathbf{L} = \langle L; \cdot, +, 0, 1, \neg \rangle$, $\neg$ involutive, antitone semicompl, $\bar{0} \approx 1$
 $\implies x + \bar{x} \approx 1$ (dual semicompl), thus (compl)

For $a, b \in L$ $\bar{a} \cdot \overline{b + \bar{a}} \stackrel{\text{abs}}{=} \bar{a}(b + \bar{a})\overline{b + \bar{a}} = \bar{a} \cdot 0 = 0.$

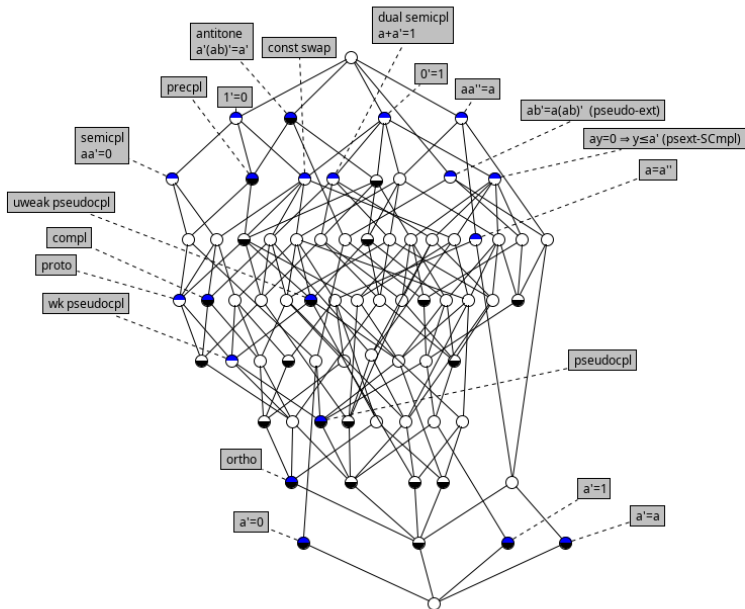
also: $\overline{a + \bar{a}} \leq \bar{a}$ from $a \leq a + \bar{a}$

for $b = a$: $0 = \bar{a} \cdot \overline{a + \bar{a}} = \overline{a + \bar{a}}$

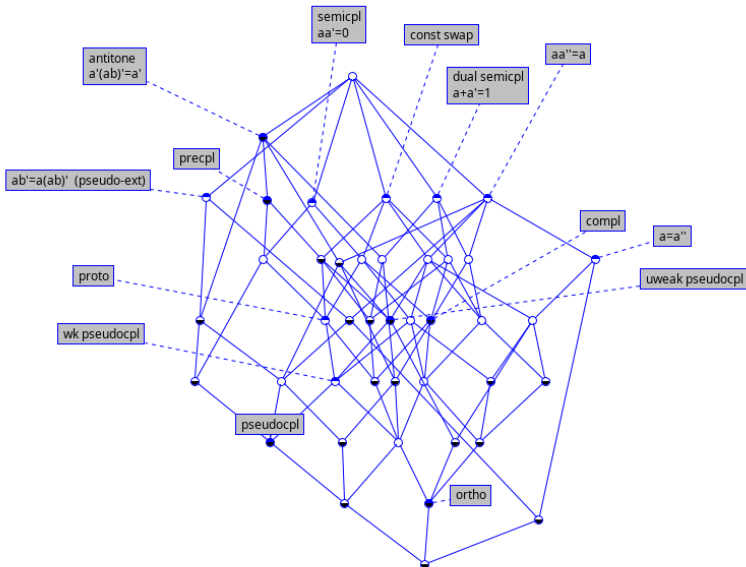
by involution: $a + \bar{a} = \overline{\overline{a + \bar{a}}} = \bar{0} = 1.$

There are some more implications to prove, but those are definitional or trivial.

The lattice



A reduced variant (fewer properties)



Conclusions / future work

What we did

- separated 23 negation properties in bounded lattices
- with 23 example lattices (not shown in the slides)
- proved a basis for implicative logic
- gave a lattice representation

A few conclusions

- impl entailed by **Holliday's Tbl 1** are **verifiable** from the lattice
- **Holliday's Figure 1** is an order, **not a lattice diagram**
(proto $\vee$ uwk pseudo = constant swapping precompl < precompl)

Future work

- better lattice representation needed (**nested line diagram**)
- perhaps enriching the picture with some **dual properties**?

Questions?