

Monoidal Categories of Effect Algebras

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Outline

1 Preliminaries

- Effect algebras and related structures
- Tensor products
- Multicategories and representability

2 Main results

- Representability of multicategories of quantum structures
- Representability induces a monoidal category
- Chain of (multi)adjunctions

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1. Preliminaries

What is an effect algebra?

Definition

By an **effect algebra** we mean a partial algebra $(E; \oplus, 0, 1)$ with special elements $0, 1 \in E$ called the *zero* and the *unit*, and a partially defined binary operation $\oplus$ satisfying:

- 1 commutative law: $a \oplus b = b \oplus a$
- 2 associative law: $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
- 3 orthosupplement law: $\forall a \in E \exists ! a' \in E: a \oplus a' = 1$;
 a' is called the *orthosupplement* of a
- 4 zero-one law: if $a \oplus 1$ is defined, then $a = 0$

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We write $a \perp b$ whenever $a \oplus b$ is defined.

Homomorphisms and bihomomorphisms

Definition

By a **homomorphism** between effect algebras E and F we mean a map $\varphi: E \rightarrow F$ satisfying:

- 1 if $a \perp b$, then $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$
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- **objects**: effect algebras
- **morphisms**: homomorphisms of effect algebras

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By a **bihomomorphism** $\beta: E \times F \rightarrow G$ we mean a map satisfying:

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 $\beta(a, b_1 \oplus b_2) = \beta(a, b_1) \oplus \beta(a, b_2)$ for all $a \in E$ and $b_1 \perp b_2 \in F$
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Related structures

GEA

Generalized effect algebras

“effect algebras without unit 1”

PCM

Partial commutative monoids

*“generalized effect algebras
without the cancellation law:*

$a \oplus b = a \oplus c$ implies $b = c$ ”

BPCM

Bounded partial commutative monoids

*“partial commutative monoids
with unit 1”,*

or equivalently,

*“effect algebras where
orthosupplements are not unique”*

AbPOG

Abelian po-groups

and

AbPOG_u

Abelian po-groups with strong unit u

*“every element is under $n \cdot u$ for
sufficiently large $n \in \mathbb{N}$ ”*

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**All these structures together
with naturally defined homo-
morphisms form categories!**

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Tensor products

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Definition (of tensor product)

Let A and B be objects of $\mathbf{C}$. By a tensor product of A and B we mean an object $A \otimes B$ together with a bimorphism $\otimes: A \times B \rightarrow A \otimes B$ with the following **universal property**:

For every bimorphism $\beta: A \times B \rightarrow X$ there exists a unique morphism $\varphi: A \otimes B \rightarrow X$ such that $\varphi \circ \otimes = \beta$ (i.e., $\varphi(a \otimes b) = \beta(a, b)$ for all $a \in A$ and $b \in B$).

In other words, the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{\otimes} & A \otimes B \\ & \searrow \beta & \downarrow \varphi \\ & & X \end{array}$$

Tensor products

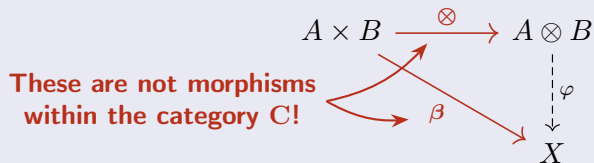
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From categories to multicategories

A multicategory $\mathbf{C}$ consists of the following data:

- a collection C_0 of objects
- a collection C_1 of unary morphisms $A \rightarrow B$
- a collection C_2 of bimorphisms $(A_1, A_2) \rightarrow B$
- a collection C_3 of trimorphisms $(A_1, A_2, A_3) \rightarrow B$
- $\vdots$
- a collection C_n of n -morphisms $(A_1, A_2, \dots, A_n) \rightarrow B$
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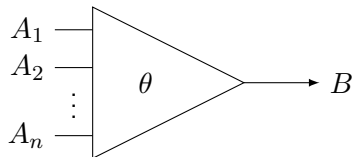
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From categories to multicategories

We depict n -morphisms as

$$\theta: (A_1, A_2, \dots, A_n) \rightarrow B$$

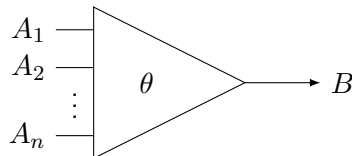
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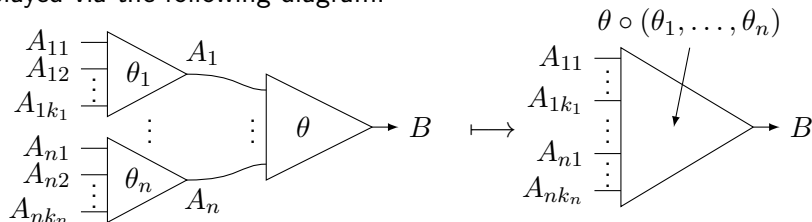
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Again, we have a **composition rule**:

$$(\theta, \theta_1, \dots, \theta_n) \mapsto \theta \circ (\theta_1, \dots, \theta_n),$$

displayed via the following diagram:



From categories to multicategories

- The composition is **associative**:

$$\begin{aligned}\theta \circ (\theta_1 \circ (\theta_{11}, \dots, \theta_{1k_1}), \dots, \theta_n \circ (\theta_{n1}, \dots, \theta_{nk_n})) \\ = \theta \circ (\theta_1, \dots, \theta_n) \circ (\theta_{11}, \dots, \theta_{1k_1}, \dots, \theta_{n1}, \dots, \theta_{nk_n})\end{aligned}$$

- The **composition with identities** leaves multimorphisms unchanged:

$$\theta \circ (\text{id}_{A_1}, \dots, \text{id}_{A_n}) = \theta = \text{id}_B \circ \theta$$

for all $\theta: (A_1, \dots, A_n) \rightarrow B$; as depicted in the following diagram:

$$\begin{array}{ccccc}(A_1, \dots, A_n) & & (A_1, \dots, A_n) & & (A_1, \dots, A_n) \\ \text{id}_{A_1} \downarrow \quad \dots \quad \downarrow \text{id}_{A_n} & & \downarrow \theta & & \downarrow \theta \\ (A_1, \dots, A_n) & = & B & = & B \\ \downarrow \theta & & & & \downarrow \text{id}_B \\ B & & & & B\end{array}$$

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From categories to multicategories

All categories **EA**, **GEA**, **PCM**, **BPCM**, **AbPOG** and **AbPOG_u** can be naturally extended to multicategories, for which we use the same notation.

Generalization of tensor product – representability of a multicategory

Definition (of binary tensor product)

Let A_1 and A_2 be objects of a multicategory $\mathbf{C}$. By a tensor product of A_1 and A_2 we mean an object $A_1 \otimes A_2$ together with a *bimorphism* $\tau_{A_1, A_2}: (A_1, A_2) \rightarrow A_1 \otimes A_2$ with the following **universal property**:

For every *bimorphism* $\beta: (A_1, A_2) \rightarrow X$ there exists a unique morphism $\varphi: A_1 \otimes A_2 \rightarrow X$ such that $\varphi \circ \tau_{A_1, A_2} = \beta$; as indicated in the following diagram:

$$\begin{array}{ccc} (A_1, A_2) & \xrightarrow{\tau_{A_1, A_2}} & A_1 \otimes A_2 \\ & \searrow \beta & \downarrow \varphi \\ & & X \end{array}$$

Generalization of tensor product – representability of a multicategory

Definition (of tensor product of a sequence of objects)

Let $A_1, \dots, A_n$ be objects of a multicategory $\mathbf{C}$. By a tensor product of $A_1, \dots, A_n$ we mean an object $\bigotimes_{i=1}^n A_i$ for which there exists a *universal n -morphism* $\tau_{A_1, \dots, A_n} : (A_1, \dots, A_n) \rightarrow \bigotimes A_i$ with the following **universal property**:

For every n -morphism $\gamma : (A_1, \dots, A_n) \rightarrow X$ there exists a unique morphism $\varphi : \bigotimes A_i \rightarrow X$ such that $\varphi \circ \tau_{A_1, \dots, A_n} = \gamma$; as indicated in the following diagram:

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A multicategory $\mathbf{C}$, for which every sequence of objects (including the empty sequence $()$) admits a tensor product and universal morphisms are closed under composition is called **representable**.

Generalization of tensor product – representability of a multicategory

The word “**representable**” comes from the fact that there is a bijection

$$\mathbf{C}(A_1, \dots, A_n; X) \cong \mathbf{C}\left(\bigotimes A_i; X\right)$$

between n -morphisms $A_1, \dots, A_n \rightarrow X$ and unary morphisms $\bigotimes A_i \rightarrow X$ natural in X , that is, every n -morphism γ can be uniquely represented by a unary morphism from the tensor product of the domain sequence of γ . This bijection is given by the assignment

$$\begin{array}{ccc} (A_1, \dots, A_n) & \xrightarrow{\tau_{A_1, \dots, A_n}} & \bigotimes A_i \\ & \searrow \gamma & \downarrow \varphi \\ & & X \end{array}$$

2. Main Results

Representability of multicategories of quantum structures

Theorem

*The multicategories **EA**, **GEA**, **PCM** and **BPCM** are representable.*

Sketch of the proof.

- First we show that every pair of objects A and B in the aforementioned multicategories **admits a tensor product** $A \otimes B$.
- For a given sequence of objects $(A_1, \dots, A_n)$, we **construct their tensor product** via binary tensor products as

$$\bigotimes_{i=1}^n A_i := (((A_1 \otimes A_2) \otimes A_3) \otimes \cdots \otimes A_n),$$

and show that they, together with inductively defined universal n -morphisms $\tau_{A_1, \dots, A_n}$, satisfy the desired **universal property**.



Representability implies monoidal structure

For a multicategory $\mathbf{C}$, let $\underline{\mathbf{C}}$ denote the restriction of $\mathbf{C}$ onto classes $\mathbf{C}_0$ of objects and $\mathbf{C}_1$ of unary morphisms. Thus, $\underline{\mathbf{C}}$ is a category – called the **underlying category** of the multicategory $\mathbf{C}$.

Theorem (C. Hermida)

Every representable multicategory $\mathbf{C}$ induces a monoidal structure on its underlying category $\underline{\mathbf{C}}$ via the binary tensor product $\otimes$.

While C. Hermida observes that the coherence axioms of a monoidal category are a consequence of the universal property of tensors, the original text omits the explicit calculations. These were explicitly carried out and verified in this work.

Representability implies monoidal structure

What was the strategy? We illustrate this with the proof that Mac Lane's pentagon

$$\begin{array}{ccccc} & & (A \otimes B) \otimes (C \otimes D) & & \\ & \nearrow^{\alpha_{A,B,C \otimes D}} & & \nwarrow_{\alpha_{A \otimes B,C,D}} & \\ A \otimes (B \otimes (C \otimes D)) & & \text{("Mac Lane's pentagon")} & & ((A \otimes B) \otimes C) \otimes D \\ \downarrow \text{id}_A \otimes \alpha_{BCD} & & & & \uparrow \alpha_{ABC} \otimes \text{id}_D \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha_{A,B \otimes C,D}} & & & (A \otimes (B \otimes C)) \otimes D \end{array}$$

commutes.

Representability implies monoidal structure

The definition of the associators: $\alpha_{ABC}: A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ is the unique morphism fitting into

$$\begin{array}{ccccc} & & (A, B \otimes C) & \xrightarrow{\tau_{A, B \otimes C}} & A \otimes (B \otimes C) \\ & \nearrow^{(\text{id}_A, \tau_{B, C})} & & & \downarrow \alpha_{ABC} \\ (A, B, C) & & & & \\ & \searrow_{(\tau_{A, B}, \text{id}_C)} & (A \otimes B, C) & \xrightarrow{\tau_{A \otimes B, C}} & (A \otimes B) \otimes C \end{array}$$

Representability implies monoidal structure

$$\begin{array}{ccc} & (A \otimes B) \otimes (C \otimes D) & \\ \alpha_{A,B,C \otimes D} \nearrow & & \searrow \alpha_{A \otimes B,C,D} \\ A \otimes (B \otimes (C \otimes D)) & & ((A \otimes B) \otimes C) \otimes D \\ \text{id}_A \otimes \alpha_{BCD} \downarrow & & \uparrow \alpha_{ABC} \otimes \text{id}_D \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha_{A,B \otimes C,D}} & (A \otimes (B \otimes C)) \otimes D \end{array}$$

Representability implies monoidal structure

$$\begin{array}{ccc} & (A \otimes B) \otimes (C \otimes D) & \\ \alpha_{A,B,C \otimes D} \nearrow & & \searrow \alpha_{A \otimes B,C,D} \\ A \otimes (B \otimes (C \otimes D)) & & ((A \otimes B) \otimes C) \otimes D \end{array}$$

$$\begin{array}{ccc} A \otimes (B \otimes (C \otimes D)) & & ((A \otimes B) \otimes C) \otimes D \\ \text{id}_A \otimes \alpha_{BCD} \downarrow & & \uparrow \alpha_{ABC} \otimes \text{id}_D \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha_{A,B \otimes C,D}} & (A \otimes (B \otimes C)) \otimes D \end{array}$$

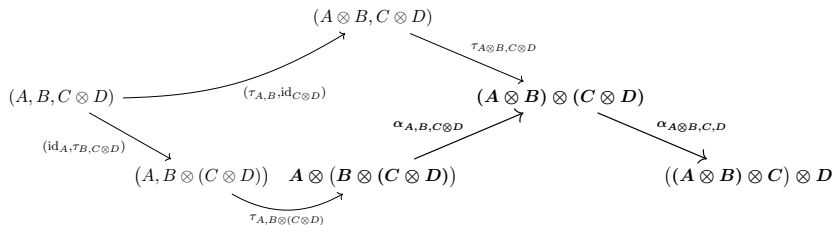
Representability implies monoidal structure

The upper half of the Mac Lane's pentagon:

$$\begin{array}{ccc} & (A \otimes B) \otimes (C \otimes D) & \\ \alpha_{A, B, C \otimes D} \nearrow & & \nwarrow \alpha_{A \otimes B, C, D} \\ A \otimes (B \otimes (C \otimes D)) & & ((A \otimes B) \otimes C) \otimes D \end{array}$$

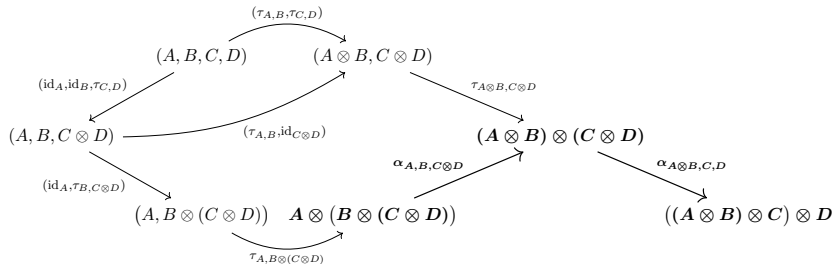
Representability implies monoidal structure

The upper half of the Mac Lane's pentagon:



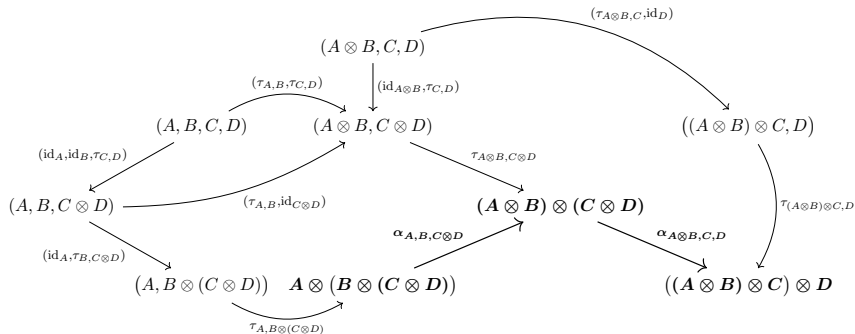
Representability implies monoidal structure

The upper half of the Mac Lane's pentagon:



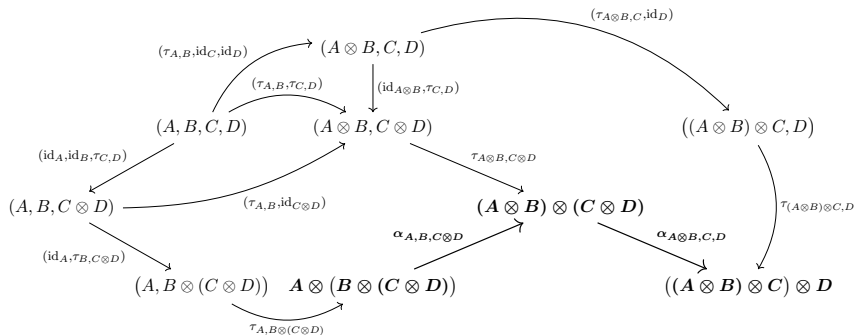
Representability implies monoidal structure

The upper half of the Mac Lane's pentagon:



Representability implies monoidal structure

The upper half of the Mac Lane's pentagon:



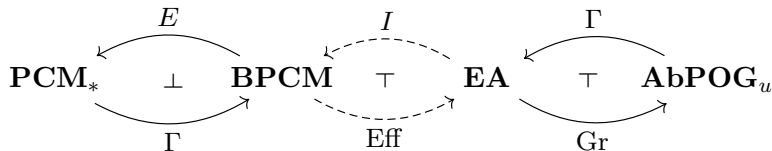
Representability implies monoidal structure

The lower half of the Mac Lane's pentagon:

$$\begin{array}{ccccc}
 (A \otimes B, C, D) & \xrightarrow{(\tau_{A \otimes B, C}, \text{id}_D)} & ((A \otimes B) \otimes C, D) & & \\
 \uparrow (\tau_{A, B}, \text{id}_C, \text{id}_D) & \nearrow (\text{id}_A, \tau_{B, C \otimes D}) & & \searrow \tau_{(A \otimes B) \otimes C, D} & \\
 (A, B, C \otimes D) & (A, B \otimes (C \otimes D)) & & & \\
 \nearrow (\text{id}_A, \text{id}_B, \tau_{C, D}) & \searrow \tau_{A, B \otimes (C \otimes D)} & & & \\
 (A, B, C, D) & & A \otimes (B \otimes (C \otimes D)) & ((A \otimes B) \otimes C) \otimes D & \\
 \downarrow (\text{id}_A, \tau_{B, C}, \text{id}_D) & & \downarrow \text{id}_A \otimes \alpha_{BCD} & & \uparrow \alpha_{ABC} \otimes \text{id}_D \\
 (A, B \otimes C, D) & & A \otimes ((B \otimes C) \otimes D) & (A \otimes (B \otimes C)) \otimes D & \\
 \searrow (\text{id}_A, \tau_{B \otimes C, D}) & \nearrow \tau_{A, (B \otimes C) \otimes D} & \nearrow \alpha_{A, B \otimes C, D} & & \\
 (A, (B \otimes C) \otimes D) & & & & \\
 \downarrow (\tau_{A, B \otimes C}, \text{id}_D) & & & & \\
 (A \otimes (B \otimes C), D) & & & &
 \end{array}$$

Some adjunctions between multicategories of quantum structures

- ① Chain of pointed and bounded structures:

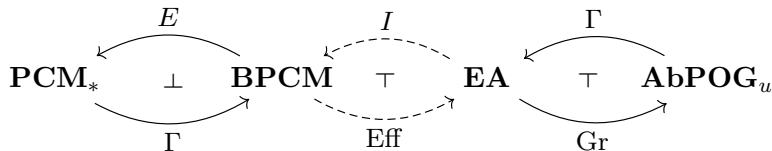


- ② Unbounded structures:

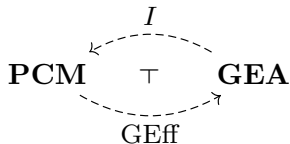


Some adjunctions between multicategories of quantum structures

- ① Chain of pointed and bounded structures:



- ② Unbounded structures:



Conclusion and future research

- **Summary:** This work establishes a multicategorical foundation for further research into the categorical framework of quantum logic.
- **Future work:**
 - Further investigate the adjunctions between functors of multicategories of quantum structures.
 - Explore the use of higher categorical structures (e.g. enriched categories) in the context of quantum logic.
 - Provide a more explicit description of tensor products in the studied multicategories (currently described only via their universal property).

Thank you for your attention!

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