

Symmetric Lattice Identities

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Symmetric identities

Definition

Let $FL(n)$ be the free lattice generated by n elements. A lattice term $f \in FL(n)$ is **symmetric** when its value does not change when we permute its variables, i.e, if $f_{\sigma}(x_1, \dots, x_n) := f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n)$ holds for every permutation $\sigma \in S_n$.

The symmetric terms form a sublattice of $FL(n)$, denoted by $SL(n)$.

Definition

A lattice identity is **symmetric** if both sides are symmetric terms.

Example: this symmetric identity holds only in trivial lattices:

$$x \wedge y \approx x \vee y.$$

Symmetrizing a term or an identity

Definition

The **meet symmetrization** of a term t is $\bigwedge_{\pi} t = \bigwedge_{\sigma \in S_n} t_{\sigma}$, and its **join symmetrization** is $\bigvee_{\pi} t = \bigvee_{\sigma \in S_n} t_{\sigma}$,

Example

Symmetrizing the distributive identity $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z)$$

$$z \wedge (y \vee x) = (z \wedge y) \vee (z \wedge x)$$

Meet symmetrization:

$$\begin{aligned} x \wedge y \wedge z \wedge (x \vee y) \wedge (x \vee z) \wedge (y \vee z) &= ((x \wedge y) \vee (x \wedge z)) \wedge \\ &\quad ((y \wedge x) \vee (y \wedge z)) \wedge \\ &\quad ((z \wedge y) \vee (z \wedge x)) \end{aligned}$$

Example

By absorption law, the left hand side is equivalent to $x \wedge y \wedge z$. For the right side:

$$\begin{aligned} ((x \wedge y) \vee (x \wedge z)) \wedge ((y \wedge x) \vee (y \wedge z)) \wedge ((z \wedge y) \vee (z \wedge x)) &\geq \\ (x \wedge y) \wedge (x \wedge z) \wedge (y \wedge z) &= x \wedge y \wedge z \end{aligned}$$

Since $x \wedge y \leq x$ and $x \wedge z \leq x$ then $(x \wedge y) \vee (x \wedge z) \leq x$. (similarly for y and z . Thus,

$$((x \wedge y) \vee (x \wedge z)) \wedge ((y \wedge x) \vee (y \wedge z)) \wedge ((z \wedge y) \vee (z \wedge x)) \leq x \wedge y \wedge z$$

So we got $x \wedge y \wedge z = x \wedge y \wedge z$.

Example

Join Symmetrization:

$$(x \wedge (y \vee z)) \vee (y \wedge (x \vee z)) \vee (z \wedge (x \vee y)) = \\ ((x \wedge y) \vee (x \wedge z)) \vee ((y \wedge x) \vee (y \wedge z)) \vee ((z \wedge y) \vee (z \wedge x))$$

The right hand side will be simplified: $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$.

The identity above holds in distributive lattices since it is derived from the distributive law.

Question

Problem (1)

Is it true that any lattice identity is equivalent to either its meet or join symmetrization?

Problem (2)

Is it true that any lattice identity is equivalent to a lattice identity of the same arity?

Problem (3)

Is it true that any lattice identity is equivalent to a set of lattice identities of the same arity?

Problem (4)

Is it true that any lattice identity is equivalent to a set of lattice identities of any arity?

Symmetric identities

Proposition

A lattice is distributive iff it satisfies the equality

$$\bigvee_{\pi} (x \wedge y) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z) \wedge (y \vee z) = \bigwedge_{\pi} (x \vee y)$$

Proposition

A lattice is modular iff it satisfies the inequality

$$\bigwedge_{\pi} (x \vee (z \wedge (x \vee y))) \geq \bigvee_{\pi} (x \wedge (z \vee (x \wedge y)))$$

(will be discussed later)

Definitions

Definition (Symmetric variety)

A lattice variety is *symmetric* if it can be defined by symmetric identities.

Definition (Symmetric variety generated by a lattice)

The *symmetric variety generated by a lattice L* is the class of all lattices satisfying every symmetric identity that holds in L .

Definition (Ternary symmetric variety generated by a lattice)

The *ternary symmetric variety generated by a lattice L* is the class of all lattices satisfying every symmetric identity in three variables that holds in L .

We are interested in studying ternary symmetric variety generated by a lattice

Medians and Inner Medians

Definition

If L is a lattice, then a symmetric ternary monotone $f : L^3 \rightarrow L$ function that is a majority function is called a **lattice median**.

A median is an **inner median** if it is a term function. Otherwise, it is an **outer median**.

Definition

The medians of L form a lattice $\text{OutMed}(L)$, called the **outer median lattice**, in which the inner medians form a sublattice $\text{InnMed}(L)$, called the **inner median lattice**.

Lower medians and upper medians

Definition

The **lower median** is the inner median

$m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$, its dual is the **upper median**
 $M(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$.

Note that in distributive, and only in distributive, lattices

$$m(x, y, z) = M(x, y, z).$$

Proposition

For any median f , $m \leq f \leq M$ holds. Consequently, $\text{OutMed } L$ is trivial iff L is distributive.

Theorem

$|\text{OutMed}(L)| \leq 2$ iff $\text{OutMed}(L) = \text{InnMed}(L)$

Permitted Interval

Definition

Let L be a lattice. For each triple $(l_1, l_2, l_3) \in L^3$, we call the interval

$$[m(l_1, l_2, l_3), M(l_1, l_2, l_3)]$$

by the **permitted interval** of this triple, and we will denote it by $PI(l_1, l_2, l_3)$.

The poset $\mathcal{T}_L$ (called the **T-poset** of L) is defined the following way: its underlying set is the set of three-element subsets of L having nontrivial permitted interval. The order on $\mathcal{T}_L$ is defined by

$$\{l_1, l_2, l_3\} \leq \{k_1, k_2, k_3\} \Leftrightarrow \exists \sigma \in S_{\{1,2,3\}} : \forall 1 \leq i \leq 3 : l_i \leq k_{\sigma(i)}.$$

Proposition

The outer medians of L correspond to the monotone maps $f : \mathcal{T}_L \rightarrow L$ satisfying $f(l_1, l_2, l_3) \in PI(l_1, l_2, l_3)$ for all $(l_1, l_2, l_3) \in \mathcal{T}_L$.

The induced cut

Proposition

In $SL(3)$, the single atom is $\bigvee_{\pi}(x \wedge y)$, the single coatom is $\bigwedge_{\pi}(x \vee y)$, and $SL(3)$ is atomic and coatomic.

Definition

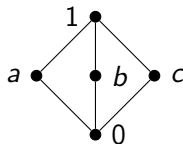
We denote by $SL(3)^*$ the lattice obtained from $SL(3)$ by deleting its smallest ($\bigwedge_{\pi} x$) and largest ($\bigvee_{\pi} x$) elements.

Definition

Let L be a lattice. Then the kernel of the natural homomorphism from $SL(3)^*$ to $\text{InnMed}(L)$ is called the **cut induced by L** .

The cut encodes the the same information as the set of ternary symmetric identities satisfied by L .

The cut induced by M_3



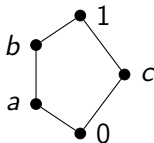
Proposition

M_3 has five medians (depending on the image of (a, b, c)), out of these two are inner $((a, b, c) \mapsto 0$ and $(a, b, c) \mapsto 1$).

Proposition

In the cut induced by M_3 , the smaller class has no largest element and the larger class has no smallest element.

The cut induced by N_5



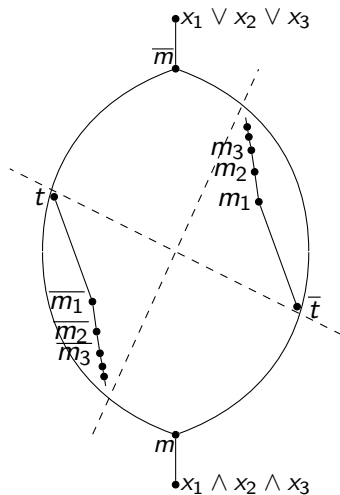
Proposition

N_5 has two medians (depending on the image of (a, b, c) , which can be a or b), both are inner medians.

Proposition

In the cut induced by N_5 , the smaller class has largest element $t := \bigwedge_{\pi} (x \vee (z \wedge (x \vee y)))$ and the larger class has smallest element $\bar{t}$.

The two cuts of $SL(3)$

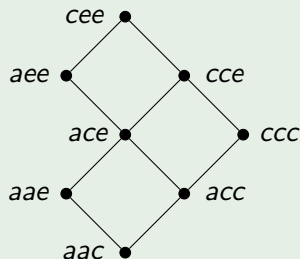
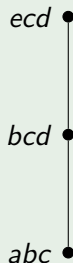
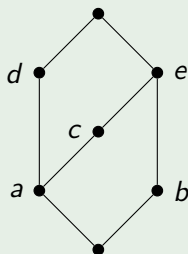


From the N_5 -cut to symmetric identity for modularity

- Suppose $s_1 \geq s_2$ is a symmetric 3-variable lattice identity that characterizes modularity.
- As it does not hold in N_5 , s_1 must be in the lower half of the N_5 -cut and s_2 in the upper half
- Hence, $s_1 \leq t$ and $s_2 \geq \bar{t}$ in $SL(3)$.
- Since $s_1 \geq s_2$ holds in modular lattices, then $t \geq s_1 \geq s_2 \geq \bar{t}$ also hold in modular lattices.
- But $t \geq \bar{t}$ is not true in N_5 by the definition of the N_5 -cut.
- Therefore, if **any** symmetric 3-variable identity characterizes modularity, then $t \geq \bar{t}$ also does so.
- It is enough to check whether $t \geq \bar{t}$ holds for modular lattices. The answer is yes.

Example – Finding Inner Medians

Example

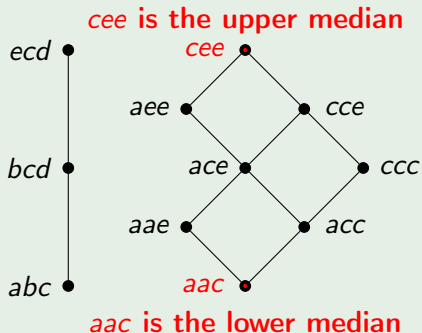
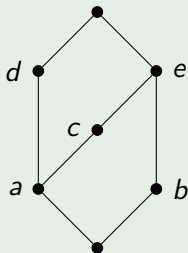


The permitted intervals of the elements of the T-poset are:

$$PI(abc) = \{a, c\}, PI(bcd) = \{ace\}, PI(ecd) = \{c, e\}$$

Example – Finding Inner Medians

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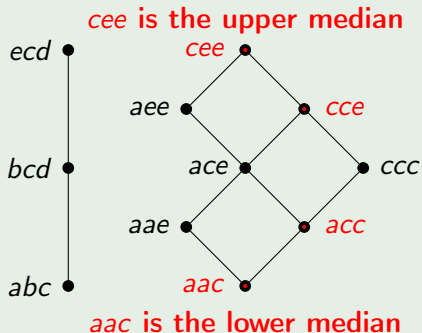
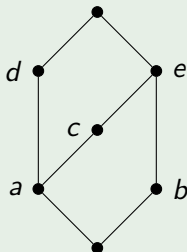


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Example – Finding Inner Medians

Example



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Reductions

Proposition

- *The cut induced by a lattice L is the intersection of the cuts induced by its 3-generated sublattices.*
- *If L is a subdirect product $L \leq_{sd} \prod_i L_i$, then the cut induced by L is the intersection of the cuts induced by the L_i .*

Hence, we are interested mostly in the 3-generated subdirectly irreducible lattices.

Conjecture

All 3-generated subdirectly irreducible lattices induce distinct cuts.

Proposition

All nondistributive modular lattices induce the same cut, but M_3 is the only 3-generated subdirectly irreducible among them.

Computing medians

We have computed the number of medians and inner medians for each lattice of cardinality less than 8.

lattice	Number of medians	number of inner medians	reducible /irreducible	number of generators
1	1	1	irreducible	1
2	1	1	irreducible	2
3	1	1	reducible	3
4	1	1	reducible	4
2^2	1	1	reducible	2
5	1	1	reducible	5
$1 \oplus 2^2$	1	1	reducible	3
N_5	2	2	irreducible	3
M_3	5	2	irreducible	3

Table: Lattices with elements < 6

6 elements lattices

Name	Lattice	$\mathcal{T}_L$	OutMed L	InnMed L
M_3		$\{abc\}$	M_3	C_2
N_5		$\{abc\}$	C_2	C_2
M_4		$\{abc, abd, acd, bcd\}$	M_4^4	C_2
A_1		$\{abd, bcd, abc, acd\}$	$3 \times A_1^2$	C_2^2
A_2		$\{abc, acd, abd, bcd\}$	C_3^2	C_2
A_3		$\{abc, abd, bcd\}$		C_2
L_4		$\{abc, acd, bcd\}$	C_3^2	C_3
L_5		$\{abc, acd, bcd\}$	C_3^2	C_3

Note that: L_4 and L_5 are 3-generated irreducible lattices.

6 elements

B_1		$\{abc\}$	M_3	C_2
B_2		$\{abc\}$	M_3	C_2
B_3		$\{abc\}$	N_5	C_2
B_4		$\{abc\}$	N_5	C_2

7-elements lattices

lattice	Number of medians	Number of inner medians	Reducible/irreducible	Number of generators
N01	1296	2	reducible	4
N02	5	3	irreducible	3
N03	64	4	reducible	5
N04	9	3	reducible	4
N05	5	2	reducible	5
N06	9	2	reducible	5
N07	3	2	reducible	4
N08	3	3	reducible	4
N09	1	1	reducible	4
N10	9	3	reducible	4
N11	8	2	reducible	5
N12	1	1	reducible	5
N13	2	2	reducible	5
N14	1	1	reducible	5
N15	12	2	reducible	3
N16	6642	4	irreducible	4
N17	990	4	irreducible	4
N18	65	2	irreducible	4
N19	172	2	irreducible	4
N20	180	3	irreducible	4

7-elements lattices

SD01	282475249	2	irreducible	5
SD02	384104	4	irreducible	5
SD03	4464	4	reducible	5
SD04	3008	4	irreducible	5
SD05	134	4	irreducible	4
SD06	36	2	reducible	3
SD07	64	2	reducible	5
SD08	124	2	reducible	5
SD09	8	4	irreducible	3
SD10	1	1	reducible	4
SD11	2	2	reducible	5
SD12	5	2	reducible	5
SD13	1	1	reducible	7

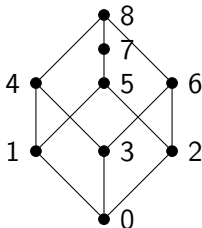


Figure: The lattice $L = 2^{2+}$

This lattice is 3-generated, subdirectly irreducible, contains N_5 , and has 2 innermedians (the lower and the upper medians). $N_5 \subseteq L$. $L \notin \mathcal{V}(N_5)$. There is an identity that holds in N_5 but not in L . But, they satisfy the same symmetric identities.

Question

Problem (1)

Is it true that any lattice identity is equivalent to either its meet or join symmetrization? NO

Problem (2)

Is it true that any lattice identity is equivalent to a lattice identity of the same arity? NO

Problem (3)

Is it true that any lattice identity is equivalent to a set of lattice identities of the same arity? NO

Problem (4)

Is it true that any lattice identity is equivalent to a set of lattice identities of any arity?

Thanks for listening!

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