Quasi-equational bases for directed paths

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For a finite relational structure P, let $Q(P)$ and be the class of all relational structures isomorphic to an induced substructure of a power of P. The class $Q(P)$ is axiomatizable by *quasi-identities*, i.e., sentences of the form

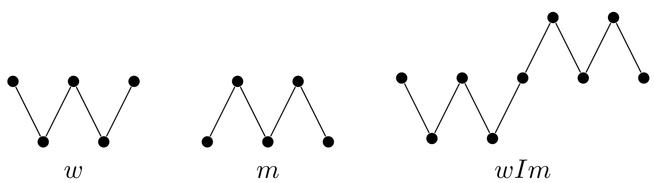
$$(\forall \bar{x})\ [\varphi_1(\bar{x}) \wedge \cdots \wedge \varphi_n(\bar{x}) \Rightarrow \varphi(\bar{x})],$$

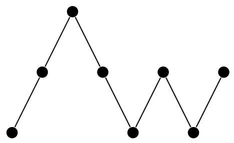
where φ_i, φ are atomic formulas. In digraphs these are of the form $x \to y$ (there is an edge from x to y) and $x = y$. Then P is *finitely q-based* if $Q(P)$ is axiomatizable by a finite number of quasi-identities. We study the following question.

Problem 1 *Which finite relational structures are finitely q-based?*

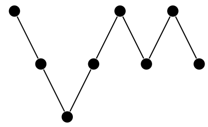
Related questions for finite algebras have stimulated research in universal algebra for decades. Perhaps the most prominent result in this direction is McKenzie's solution to Tarski's problem.[3]. Deep results were also obtained for quasivarieties, see e.g. [2, 4]. However the results for relational structures are rare. As an example we recall Pultr's and Nešetřil's theorem stating that the only finitely q-based finite simple graphs are disjoint unions of complete bipartite graphs [1].

In the hope of discovering a general pattern, we decided to look closer at a class of digraphs that, on one side, behave essentially different than simple graphs and, on the other, are not too complicated. We focus on *directed paths*, that is, digraphs obtained from a path in a simple graph by choosing a direction for each of its edges. One may visualize directed paths by Hasse-like diagrams where arrows are edges whose direction is always upwards.





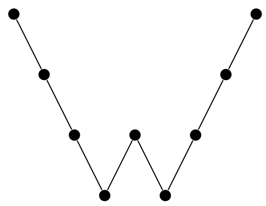
Λw



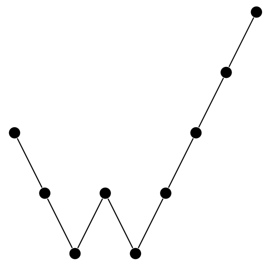
Vm

Our most general resulthas a rather technical formulation. But, during the talk, we will be mainly focused on proof technique that allows answering question in Problem 1, at least in some cases. Namely the lack of a finite q-base may be witnessed by the existence of an infinite family of so called critical digraphs.

Then is not difficult to show that if an directed path P contains one of wIm, Λw or Vm (see the picture above), then P is not finitely q-based. Let us consider directed paths that contain w or m but $\{wIm, \Lambda w, Vm\} \cap Q(P) = \emptyset$. We call them valleys and hills. Critical digraphs relative to valleys and hills may be very complex. As a not complicated examples, consider the following valleys

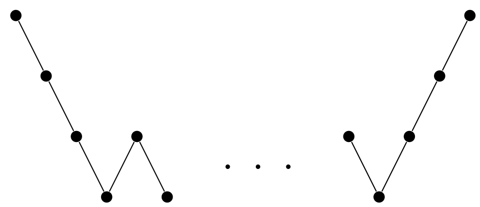


W_{33}



W_{24}

Then W_{33} is finitely q-based while W_{24} is not. The latter fact is witnessed by the depicted infinite family of critical digraphs relative to W_{24}:



References

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[3] Ralph McKenzie. Tarski's finite basis problem is undecidable. *Int. J. Algebra Comput.*, 6(1):49–104, 1996.

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